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MECHANICS APPLIED TO ENGINEERING

VOL II

BY THE SAME AUTHOR

MECHANICS APPLIED TO ENGINEERING

Volume I. With 741 Illustrations and numerous Examples.

MECHANICS APPLIED TO ENGINEERING

VOLUME II

CHIEFLY WORKED EXAMPLES

BY

JOHN GOODMAN, Wh.Sch., M.Inst.C.E.

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WITH 198 ILLUSTRATIONS, 711 EXAMPLES AND NOTES

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PREFACE

THE first volume of " Mechanics Applied to Engineering " has been before the public for over thirty years, and judging by the number of new editions and reprints which have been called for, also the great number of complimentary letters received from readers in all parts of the world, the book has been helpful to both students and designers, but many have expressed regret that it contained so few worked-out examples. The object of the second volume is to remedy this defect.

The arrangement of the chapters in Vol. II. corresponds to that in Vol. I., the worked-out problems will be found to illustrate the principles and methods described in Vol. I. with frequent cross-references.

Several new articles have been inserted in Vol. II. dealing with points which have been developed since the last revision of Vol. I.

The examples are mostly questions which have been set in various examinations. The calculations have in most cases been made on an ordinary slide rule, which fortunately does not allow of a large number of meaningless digits being used; this is regarded as an important point for engineering work, since some seem to have very little idea of the number of significant figures which are justified in any given case. For example, the buckling load of a strut is sometimes expressed in pounds down to single units or even to decimals of a pound, and then the result is divided by a factor of safety (or factor of ignorance) of, say, 6 to get the working load. Such a procedure is a clear proof that the worker has no conception of the many disturbing causes which may affect the result to such an extent that the buckling load is probably not known to within 20 per cent. when working conditions are taken into account.

This book is not intended for advanced students or for those possessing a knowledge of higher mathematics. Such

readers are generously catered for in many advanced treatises. But there does appear to be a gap between very elementary text-books and the more advanced treatises: this book aims at filling that gap.

In the Appendix will be found tables giving the powers of numbers from 1 to 30 to powers from 1 to 4, advancing in both cases by 0·1, together with simple methods of getting powers of numbers beyond the range of the tables. It is believed that these will save time and will be useful to those who have got rusty or perhaps have never known how to use either logarithms or a log-log slide rule. It may be thought that there are very few of such amongst educated engineers, but the Author receives a great number of letters from people, even from prominent members of engineering institutions, asking what is the meaning of, say, $13^{1\cdot43}$, and how is it worked out?

These tables have been prepared by Mr. L. A. Barry, B.Sc., who has taken extraordinary pains to secure accuracy, only those who have constructed such tables have any conception of the prodigious amount of labour involved. The Author wishes to tender his sincere thanks to Mr. Barry.

The Author gave up teaching some years ago and realised that he was getting out of touch with the most recent work, he therefore felt that it was desirable that the fresh ideas of a younger man should be incorporated into this book. He has been very fortunate in getting Mr. H. W. Swift, M.A., B.Sc., to assist him in the preparation of this volume and in the revision of Vol. I.: his sincere thanks are due to Mr. Swift for very valuable assistance.

The author wishes to thank Sir Thomas Hudson Beare, B.Sc., F.R.S.E., for permission to make use of certain questions (duly acknowledged in every case) set in the Edinburgh University Examinations. His thanks are also due to Mr. J. A. Chippindale for making the drawings and generally assisting.

BEAMSLEY,
SKIPTON,
January, 1927.

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MECHANICS APPLIED TO ENGINEERING

CHAPTER I

INTRODUCTORY

1. *A body weighing* 483 *lb. is lifted from rest by a force of* F *lb. which varies continuously. When the body has risen through the distance* x *feet the force in pounds for the several values of* x *is as follows :*

x	0	1	2	3	4
F	810	810	810	795	750

Find approximately the speed in each position and the time taken to reach that position.

During the first two feet of rise the acceleration is uniform :

$$f_a = \frac{Fg}{W} = \frac{(810-483)32{\cdot}2}{483} = 21{\cdot}8 \text{ f.p.s.p.s.}$$

After rising 1 ft. :

$$V_1^2 = 2 \,.\, f_a \,.\, S = 2 \times 21{\cdot}8 \times 1 = 43{\cdot}6 \quad \therefore \; V_1 = 6{\cdot}6 \text{ f.p.s.}$$

Time taken to rise 1 ft. : $t_1 = \frac{V_1}{f_a} = \frac{6{\cdot}6}{21{\cdot}8} = 0{\cdot}3$ sec.

After rising 2 ft. : $V_2^2 = 2 \times 21{\cdot}8 \times 2 = 87{\cdot}2 \quad V_2 = 9{\cdot}3$ f.p.s.

$$t_2 = \frac{9{\cdot}3}{21{\cdot}8} = 0{\cdot}43 \text{ sec.}$$

During the third foot the acceleration decreases from 21·8 f.p.s.p.s. to

$$f_a = \frac{(795-483)32\cdot2}{483} = 20\cdot7 \text{ f.p.s.p.s.}$$

Mean value during this period is 21·25 f.p.s.p.s.

$$V_3^2 = V_2^2 + 2 \cdot f_a(\text{mean}) \cdot S = 87\cdot2 + 2 \times 21\cdot25 \times 1 = 129\cdot7$$

$$V_3 = 11\cdot4 \text{ f.p.s.}$$

The time is approximately : $t_3 = t_2 + \frac{V_3 - V_2}{f_a(\text{mean})} = 0\cdot53$ sec.

During the fourth period the acceleration falls from 20·7 to 17·8 f.p.s.p.s.

$$V_4^2 = 168\cdot2 \quad V_4 = 13\cdot0 \text{ f.p.s.} \quad t_4 = \mathbf{0\cdot62 \text{ sec.}}$$

2. *A projectile is shot vertically upwards at* 1,000 *f.p.s., what height will it climb during the tenth sec. ? How high will it be at the end of the fifteenth sec. ?*

Neglect air resistance.

Its velocity will be a mean at 9·5 sec.

Height traversed in 10th sec. $= 1{,}000 - 32\cdot2 \times 9\cdot5 = 694$ ft.

Height at end of 15 sec.

$$15 \times 1{,}000 - \tfrac{1}{2}gt^2 = 15{,}000 - 3{,}620 = \mathbf{11{,}380 \text{ ft.}}$$

3. *If a rifle barrel has three complete turns of rifling in its length of* 30 *in., find the angular velocity of spin of a bullet which leaves the muzzle at* 2,000 *ft. per sec.*

The bullet makes one complete rev. in 10 in. advance.

$$\text{R.p.s.} = \frac{2{,}000 \times 12}{10} = 2{,}400 = 2\pi \times 2{,}400 = \mathbf{15{,}100 \text{ rads.p.s.}}$$

If a ring were fitted into a circumferential groove on the bullet the stress due to centrifugal force would be (see p. 202, Vol. I.) $\frac{V^2}{10}$ (nearly). Where V is the circumferential velocity in f.p.s. Let the bullet be 0·4 in. dia., then the circumferential velocity would be $\frac{0\cdot4 \times 3\cdot14 \times 2{,}400}{12} = 251$ f.p.s., and the stress $= \frac{251^2}{10} = \mathbf{6{,}300 \text{ lb. sq. in.}}$

In some forms of shell such a ring (having a soldered joint) is fitted and holds the safety pin in position. When the shell is fired the ring breaks and releases the safety pin.

4. *The speed of a motor car is* 15 *m.p.h. at a given instant, and* 42 *m.p.h. after* 1 *min.* 15 *sec. What was the acceleration if constant?*

Increase of speed = 27 m.p.h. = 39·6 f.p.s.

$$\text{Acceleration} = \frac{v_2 - v_1}{t} = \frac{39{\cdot}6}{75} = \mathbf{0{\cdot}528\ f.p.s.p.s.}$$

5. *A body is projected downwards with a velocity* 8 *f.p.s. Find its velocity after falling* 110 *ft.*

$$V^2 = 8^2 + 2 \times 32{\cdot}2 \times 110 \quad V = \mathbf{84{\cdot}5\ f.p.s.}$$

6. *A train travelling at* 60 *miles per hr. is pulled up by brakes which retard it* 6·2 *f.p.s.p.s. How far will it run before it stops? And how long will it take?*

60 m.p.h. = 88 f.p.s. It will stop in $\frac{88}{6{\cdot}2} = 14{\cdot}2$ sec.

$S = \frac{1}{2} f_a t^2 = 3{\cdot}1 \times 14{\cdot}2^2 = 625$, and will pull up in **208 yds.**

7. *A steam turbine is running at* 3,000 *r.p.m. when steam is suddenly shut off. It slows down and stops in* 5 *min. Assuming that the retardation is uniform, find the number of revolutions it makes before stoppage.*

Adopt the minute as unit of time, and the revolution as unit of angle. If A be the retardation in r.p.m.p.m., and n the speed in r.p.m. at any time, t from stopping.

$n = A \,.\, t$ and $n^2 = 2 \,.\, A \,.\, N$ where N is the no. of rev. made.

Hence $n = \frac{2N}{t}$ and $N = \frac{nt}{2} = \frac{3000}{2} \times 5 = \mathbf{7{,}500\ revs.}$

8. *An electric train can pick up its running speed of* 40 *m.p.h. by uniform acceleration in* 20 *sec. With brakes of type* A *a total retardation of* 3 *f.p.s.p.s. is obtained, with brakes of type* B 5 *f.p.s.p.s. Find the percentage saving in time between stations* 1 *mile apart due to the use of brakes of type* B.

40 m.p.h. = 58·7 f.p.s.

Uniform acceleration in starting $= \frac{58{\cdot}7}{20} = 2{\cdot}94$ f.p.s.p.s.

Distance covered during acceleration

$$= \tfrac{1}{2} f_a t^2 = \tfrac{1}{2} \times 2{\cdot}94 \times 20^2 = 587 \text{ ft.}$$

With brakes of type A, retardation = 3 f.p.s.p.s.

$$\text{Time of retardation} = \frac{58{\cdot}7}{3} = 19{\cdot}6 \text{ sec.}$$

Distance covered in retardation $= \frac{1}{2} \times 3 \times (19{\cdot}6)^2 = 576$ ft.
Total time between stations 5,280 ft. apart is :

$$20 + 19{\cdot}6 + \frac{5280 - 587 - 576}{58{\cdot}7} = 109{\cdot}8 \text{ sec.}$$

With brakes of type B retardation = 5 f.p.s p.s.

$$\text{Hence time of retardation} = \frac{58{\cdot}7}{5} = 11{\cdot}7 \text{ sec.}$$

Distance covered during retardation

$$= \frac{1}{2} \times 5 \times (11{\cdot}7)^2 = 342 \text{ ft.}$$

Total time between stations

$$= 20 + 11{\cdot}7 + \frac{5280 - 587 - 342}{58{\cdot}7} = 105{\cdot}8 \text{ sec.}$$

$$\text{Hence saving} = \frac{4{\cdot}0 \times 100}{109{\cdot}8} = \mathbf{3{\cdot}64 \text{ per cent.}}$$

9. *If in the rifle mentioned in question 3 it is assumed that the force exerted by the propellant is constant during discharge of the bullet, estimate the time taken by the bullet in its passage through the barrel.*

If V is the muzzle velocity in f.p.s., and S the length of the muzzle in feet,

$$V = f_a t \qquad V^2 = 2 f_a S \qquad V = \frac{2S}{t} \quad \text{or} \quad t = \frac{2S}{V}$$

$$\text{Time of discharge} = \frac{2 \times 30}{12 \times 2{,}000} = \mathbf{\frac{1}{400} \text{ sec.}}$$

10. *The velocity of an electric train at various times during its acceleration is given in the following table. Deduce the acceleration-time graph and the distance-time graph.*

Time secs.	0	2	4	6	8	10	12	14	16	18	20
Velocity ft. sec.	0	7·85	15·5	22·7	29·5	35·5	40·5	44·7	47·6	49·5	50·0

If a curve is drawn showing values of velocity plotted

against time, then the slope of this curve will to a certain scale measure the change of velocity with time, or the acceleration of the train. The slope cannot as a rule be found accurately by graphical methods, but it can be deduced by calculating differences. Thus :

Time secs.	Velocity ft. per sec.	Change of velocity in 2 sec.	Acceleration ft. sec.^{2}	Time secs.
0	0			
		7·85	3·92	1
2	7·85			
		7·65	3·82	3
4	15·5			
		7·2	3·6	5
6	22·7			
		6·8	3·4	7
8	29·5			
		6·0	3·0	9
10	35·5			
		5·0	2·5	11
12	40·5			
		4·2	2·1	13
14	44·7			
		2·9	1·45	15
16	47·6			
		1·9	0·95	17
18	49·5			
		0·5	0·25	19
20	50·0			

In the same way from the velocity-time curve the distance covered between any two times, t_1 and t_2, is represented to a certain scale by the area under the curve between the ordinates t_1 and t_2.

If a curve is plotted from the data given so that 1 in. of the base represents 4 sec. and 1 in. of the velocity scale represents 20 f.p.s., it will be found that the total area under this curve between the ordinates corresponding to 0 sec and 20 sec. is 7·97 sq. in.

A square inch on this scale corresponds to a velocity of 20 f.p.s. for 4 sec., *i.e.* to a distance covered of 80 ft.

Hence the distance covered during the 20 sec. of acceleration by the electric train is $7{\cdot}97 \times 80$ ft.$=$**637 ft.**

The result may also be obtained, with more detailed

information of the distance covered at intermediate times, by the tabular method shown here :

Time secs.	Velocity ft. secs.	Mean velocity.	Distance in 2 secs.	Total distance.
0	0			
		3·92	6·84	
2	7·85			6·84
		11·67	23·34	
4	15·5			30·2
		19·1	38·2	
6	22·7			68·4
		26·1	52·2	
8	29·5			120·6
		32·5	65·0	
10	35·5			185·6
		38·0	76·0	
12	40·5			261·6
		42·6	85·2	
14	44·7			346·8
		46·15	92·3	
16	47·6			439·1
		46·55	93·1	
18	49·5			532·2
		49·75	99·5	
20	50·0			631·7

From these figures the distance-time graph can be plotted. The total distance as obtained by the two methods agrees within 1 per cent.

11. *Three tons of coal are discharged in* 5 *sec. at a uniform rate from a hopper into a lorry* 12 *ft. below.*

If the lorry, apart from wheels and axles, weighs 2 *tons unloaded, what is the load on the springs just before delivery ceases ?*

The springs have to carry the dead load also to supply the force necessary to destroy the momentum of the coal.

Momentum destroyed per sec.

$$= \frac{3}{5 \times g} \sqrt{2g \times 12} \text{ ton units.}$$

Hence load in tons on the springs at end of delivery

$$= 2 + 3 + \frac{3}{5g} \sqrt{24g} = \mathbf{5{\cdot}52 \text{ tons.}}$$

12. *Two men and a boat-swing together weigh* 350 *lb. If the tie rods are level at the top of each swing, find the maximum tension in them.*

At any time the tension in the rods is :

$$T=350+\frac{350}{g}\times\frac{v^2}{l}.$$

This has its maximum value at the lowest point of the swing, when $v^2=2gl$. Hence $T_{max}=350\times3=$**1050 lb.**

13. *The net accelerating force which is exerted on a car weighing* $\frac{1}{2}$ *ton varies according to the following table. If the car is moving initially with a velocity of* 2 *f.p.s, find its velocity after* 10 *sec.*

Time secs. .	0	1	2	3	4	5	7	7	8	9	10
Force lb. . .	100	90	80	70	60	50	40	30	20	10	0

The *time* average force is 50 lb. Hence the increase of momentum due to this average force acting for a period of 10 sec. is :

$$\frac{W}{g}(V-2)=50\times10 \qquad V=2+\frac{50\times10\times32{\cdot}2}{1120}=$$**16·4 f.p.s.**

14. *A train weighing* 300 *tons starts from rest and attains a speed of* 60 *m.p.h. in* 10 *min. If the rate of increase is the same throughout, find* (i) *the acceleration*, (ii) *the draw-bar pull, and* (iii) *the horse-power of the engine.*

It attains a speed of 0·1 m.p.h. in 1 sec.

(i) Or $\frac{0{\cdot}1\times5280}{3600}=0{\cdot}146$ f.p.s.p.s.

(ii) $F=\frac{W}{g}f_a=\frac{300}{32{\cdot}2}\times0{\cdot}146=1{\cdot}36$ tons pull.

(iii) Horse-power exerted at 60 m.p.h. $=\frac{1{\cdot}36\times2{,}240\times5{,}280}{33{,}000}=$**487.**

15. *A weight is suspended from a spring balance hung from the roof of a mine cage. When the cage is at rest at the bottom of the shaft the balance reads* $4\frac{1}{2}$ *lb. Shortly after winding commences the balance reads* $9\frac{1}{4}$ *lb. The factor of safety of the winding-rope is* 10 *reckoned on the*

dead weight of the loaded cage, what is the actual factor of safety under the above-mentioned conditions? The rope is 480 *feet long, the dead load is* 8 *tons, section of rope* 0·94 *sq. in. Modulus of elasticity* 5,000 *tons per sq. in. What is the stretch of the rope when winding commences, neglecting the weight of the rope itself?*

$$\text{Factor of safety} = \frac{10 \times 4{\cdot}5}{9{\cdot}25} = 4{\cdot}87.$$

$$\text{Actual load on rope } W_r = 8 \times \frac{9{\cdot}25}{4{\cdot}5} = 16{\cdot}4 \text{ tons.}$$

$$x = \frac{W_r L}{AE} = \frac{16{\cdot}4 \times 480}{0{\cdot}94 \times 5000} = \mathbf{1{\cdot}68\ ft.}$$ See Vol. I., p. 374.

16. *If a man can pull or push a truck weighing* 1 *ton with a force not greater than* 40 *lb., how quickly can he take it to a point* 100 *yd. away and bring it back* (a) *if the start and finish are to be at rest, and* (b) *if the start only is from rest? Friction may be assumed constant at* 10 *lb. per ton. There is no brake.*

During acceleration the effective effort is 30 lb. Hence:

$$30 = \frac{W}{g} \cdot f_a = \frac{2240}{32{\cdot}2} \cdot f_a \quad \therefore f_a = \tfrac{3}{7} \text{ f.p.s.p.s.}$$

During retardation the effective force is 50 lb. Hence:

$$f_r = \tfrac{5}{7} \text{ f.p.s.p.s.}$$

For a single journey of 100 yd. starting and finishing at rest, the times of accelerating and retarding are connected by the relation:

$$f_a t_a = v_{max} = f_r t_r.$$

Hence if t is the time for the single journey:

$$t_a = \tfrac{5}{8}t \quad t_r = \tfrac{3}{8}t \quad \text{also: } S = \tfrac{1}{2} f_a t_a^2 + \tfrac{1}{2} f_r t_r^2$$

$$\therefore 300 = \tfrac{1}{2} \times \tfrac{3}{7}(\tfrac{5}{8}t)^2 + \tfrac{1}{2} \times \tfrac{5}{7}(\tfrac{3}{8}t)^2 \quad \therefore t = 47{\cdot}3 \text{ sec.}$$

(*a*) For a double journey, finishing at rest, the time taken $= 2 \times 47{\cdot}3 = 94{\cdot}6$ sec.

(*b*) If acceleration continues throughout the return journey the time for this is given by:

$$t^2 = \frac{2s}{f_a} = \frac{2 \times 300 \times 7}{3} = 1400 \quad \therefore t = 37{\cdot}4 \text{ sec.}$$

Hence the total time taken for the double journey:

$$= 47{\cdot}3 + 37{\cdot}4 = \mathbf{84{\cdot}7\ sec.}$$

17. *In a winding-plant of a mine the loaded cage weighs* 3·4 *tons, the weight of the rope itself may be neglected. When starting to wind, the cage is uniformly accelerated at* 5 *f.p.s.p.s. for* 8 *sec., then the speed remains constant, and for the last* 8 *sec. before reaching the top the cage is retarded at the rate of* 5 *f.p.s.p.s. Find the time taken in making the journey, and the tensions in the rope at the different periods. The depth of the shaft is* 700 *ft.*

$S=\frac{1}{2}f_a t^2=\frac{1}{2}\times 5\times 8^2=160$ ft. during acceleration, or 320 ft. during acceleration and retardation.

$$V=f_a t=5\times 8=40 \text{ f.p.s.}$$

Cage travels at constant speed for $700-320=380$ ft., or for $\frac{380}{40}=9{\cdot}5$ sec. Total time $9{\cdot}5+2\times 8=25{\cdot}5$ sec.

$$F=\frac{W}{g}f_a=\frac{3{\cdot}4\times 5}{32{\cdot}2}=0{\cdot}53 \text{ ton due to acceleration.}$$

Total tension during acceleration $=3{\cdot}4+0{\cdot}53=3{\cdot}93$ tons.
" " " constant speed $=3{\cdot}4$ tons.
" " " retardation $=3{\cdot}4-0{\cdot}53=$ **2·87 tons.**

18. *A locomotive hauls a train weighing* 100 *tons with a uniform acceleration, such that a speed of* 50 *miles an hour is attained in* 3·6 *min. Find the pull that it would have to exert when at* (a) 30 *miles an hour,* (b) 60 *miles an hour, if the frictional resistances amount to* 10 *lb. per ton, and the air resistance is* 120 *lb. at* 20 *miles an hour; it may be assumed that the latter varies as the square of the speed.*

50 m.p.h. $=73{\cdot}3$ f.p.s.

$$\text{Acceleration } f_a=\frac{73{\cdot}3}{60\times 3{\cdot}6}=0{\cdot}34 \text{ f.p.s.p.s.}$$

(*a*) Resistance $=100\times 10+120(\frac{30}{20})^2=1270$ lb. $=0{\cdot}57$ ton.
Force required to accelerate

$$\frac{W}{g}f_a=\frac{100\times 0{\cdot}34}{32{\cdot}2}=1{\cdot}06 \text{ tons.}$$ Total pull $=1{\cdot}63$ tons.

(*b*) Resistance $=100\times 10+120(\frac{50}{20})^2=2080$ lb. $=0{\cdot}93$ ton.
Force required to accelerate $=1{\cdot}06$ tons.
Total pull = **1·99 tons.**

19. *A locomotive starts from rest and hauls a train on the level. The draw-bar pull is constant at all speeds and amounts to* 1·32 *tons, of which* 0·34 *ton is due to friction. Wind resistance may be neglected. The weight of the train is* 300 *tons.*

Find the speed of the train, also the total distance travelled after 5, 10, 20, 60 secs.

F = 1·32 − 0·34 = 0·98 ton available for acceleration.

$$f_a = \frac{Fg}{W} = \frac{0{\cdot}98 \times 32{\cdot}2}{300} = 0{\cdot}105 \text{ f.p.s.p.s.}$$

$$V = f_a t \qquad S = \frac{f_a t^2}{2}.$$

t (secs.).	V.	S (feet).
5	0·52	1·31
10	1·05	5·25
20	2·10	21·0
60	6·30	189·0

20. *The pull on the draw-bar of a train is constant and is equal to* $\frac{1}{350}$*th of the weight of the train. The brake resistance is equal to* $\frac{1}{15}$*th of the weight of the train. Find the least time in which the train can run a mile including the stopping and starting; the friction of the train is equal to* $\frac{1}{70}$*th of the weight of the train. Find the maximum speed attained.*

When accelerating, effective pull $= W(\frac{1}{350} - \frac{1}{70}) = \frac{W}{52{\cdot}5}$.

Hence $$F = \frac{Wf_a}{g} = \frac{W}{52{\cdot}5}.$$

$$\therefore f_a = \frac{32{\cdot}2}{52{\cdot}5} = 0{\cdot}614 \text{ f.p.s.p.s.}$$

When retarding, effective pull $= W(\frac{1}{15} + \frac{1}{70}) = \frac{W}{12{\cdot}3}$.

Hence in this case $f_r = \frac{32{\cdot}2}{12{\cdot}3} = 2{\cdot}62$ f.p.s.p.s.

Also, if V is the maximum velocity attained:

$$f_a t_a = V = f_r t_r$$

Hence $t_a = \frac{2{\cdot}62}{0{\cdot}614} t_r = 4{\cdot}27 t_r = \frac{4{\cdot}27}{5{\cdot}27} t$, where t is the total time taken.

The total distance traversed = area of diagram $= \frac{Vt}{2}$.

Hence: $5280=\frac{f_a}{2}\cdot t_a \,.\, t=\frac{0{\cdot}614}{2}\times\frac{4{\cdot}27}{5{\cdot}27}t^2$

$$\therefore\ t=146 \text{ sec.}$$

$$V=\frac{2\times 5280}{146}=72{\cdot}5 \text{ f.p.s.}=\mathbf{49{\cdot}5\ m.p.h.}$$

21. *If the pressure on the pedals of a bicycle is* 100 *lb. acting* vertically *during the downstroke of each pedal, and the combined weight of rider and cycle is* 150 *lb., find the steepest gradient up which the cycle can climb steadily with gears* 10 : 1 *and* 5 : 1 *alternatively Friction may be taken as* 2 *per cent. of the weight*

How long would it take the cyclist to attain from starting a speed of 15 *miles per hour on the level?*

Take the length of crank as 6 in.

Work done per revolution $=100\times 2$ ft. lb.

In high gear, distance covered per revolution of pedals:

$$=10\pi \text{ ft.}$$

Hence at uniform speed, if gradient is 1 in n:

$$100\times 2=150\times\frac{10\pi}{n}+\frac{2\times 150}{100}\times 10\pi \quad n=44\tfrac{1}{2}.$$

In low gear: $100\times 2=150\times\frac{5\pi}{n}+3\times 5\pi \quad n=15\tfrac{1}{2}.$

Mean propelling force on high gear $=100\times\frac{2}{\pi}\times\frac{1}{10}=\mathbf{6{\cdot}4\ lb.}$

Accelerating force $=6{\cdot}4-3=3{\cdot}4$ lb. Acceleration $=\frac{3{\cdot}4}{150}g$.

Distance to attain 15 m.p.h. (22 f.p.s.) on level:

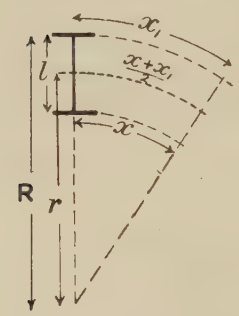
$$22^2=2\times\frac{3{\cdot}4}{150}g\times s \qquad s=\frac{484\times 150}{6{\cdot}8\times g}=\mathbf{330\ ft.}$$

22. *A cage in a mine shaft weighs* 4·3 *tons, and the rope weighs* 5 *lb. per ft. run; find the number of ft.-lb. of work done in lifting the cage* 640 *ft.*

While the cage is lifted 640 ft., the centre of gravity of the rope has only been lifted through a height of half this, viz. 320 ft., hence the total work done:

$$=4{\cdot}3\times 2{,}240\times 640+5\times 640\times 320=\mathbf{7{,}190{,}000\ ft.\text{-}lb.}$$

23. *Two similar cars, one with and one without a differential gear, run round a curve of* 1 *ft. radius. Show that the road resistance of the latter is greater than the former by an amount* $\frac{Pl}{r}$, *where* P *is the tangential force at which a driving wheel slips and* l *is the distance in feet between the driving wheels.*



In the car without a differential, the outer wheels will carry the greater part of the weight when rounding a curve, so that the *inner* driving wheel will slip through a distance $=x_1-x$.

$$\text{Work done due to slipping} = P(x_1-x).$$

$$\text{But } \frac{x_1}{R}=\frac{x}{R-l}=\frac{\frac{1}{2}(x+x_1)}{r}$$

$$\text{Work done}=Px_1\left(1-\frac{R-l}{R}\right)=\frac{Px_1l}{R}$$

$$\text{Increase in road resistance}=\frac{\text{Work done}}{\text{Distance covered}}=\frac{Px_1l}{R\left(\frac{x_1+x}{2}\right)}=\frac{Pl}{R}\cdot\frac{R}{r}=\mathbf{\frac{Pl}{r}}.$$

24. *If a cyclist exerts* $\frac{1}{10}$ *horse-power at* 12 *m.p.h. on the level, find the resistance of the road neglecting air resistance. The rider and machine weigh* 168 *lb. Find the speed when mounting a hill of* 1 *in* 50.

12 m.p.h. = 1056 ft.-min.

$$\text{Road resistance} = \frac{33{,}000}{10\times 1056}=3{\cdot}1 \text{ lb.}$$

$$\text{Resistance due to gravity} = \tfrac{168}{50}=3{\cdot}4 \text{ lb.}$$

Total resistance = 6·5 lb.

Hence the speed when mounting the hill is :

$$\frac{33{,}000}{10\times 6{\cdot}5}=510 \text{ f.m.}=\mathbf{5{\cdot}8 \text{ m.p.h.}}$$

25. *A motor cycle develops* 2 *h.p. when running at* 25 *m.p.h. on the level and when running at* 20 *m.p.h up an incline of* 1 *in* 30. *The friction of the machine and the rolling resistance are taken as constant at all speeds. The*

wind resistance varies as the square of the speed S. *If the weight of the machine and rider is* 300 *lb. show that the value of the total resistance at any speed* S (*m.p.h.*) *on the level is* $23{\cdot}05+\frac{S^2}{90}$.

Work done per min.=66,000 ft.-lb. 25 m.p.h.=2,200 f.p.m.

Total resistance on level=$\frac{66000}{2200}$=30 lb.

If the resistance due to friction is C lb. and that due to the wind is nS^2, then :

$$30=C+n.25^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (i)$$

Total resistance on incline=$\frac{66000}{1760}$=37·5 lb.

$$37{\cdot}5=C+n.20^2+\tfrac{300}{30} \quad . \quad . \quad . \quad . \quad (ii)$$

From (i) and (ii) $30-625n=37{\cdot}5-400n-10$

$$\therefore\ n=\tfrac{1}{90} \qquad C=30-\tfrac{625}{90}=23{\cdot}05$$

Total resistance on the level $=\mathbf{23{\cdot}05+\frac{S^2}{90}}$.

26. *A locomotive travelling at* 40 *m.p.h. picks up water at the rate of* 500 *gal. per min. Find the retarding force due to this operation and the additional horse-power demanded.*

40 m.p.h.=58·7 f.p.s.

The momentum of the train is increased by $\frac{5000}{g}\times 58{\cdot}7$ units per min.

Hence the retarding force in pounds, which is equal to the change of momentum per sec.

$$=\tfrac{1}{60}\times\frac{5000}{32{\cdot}2}\times 58{\cdot}7 \text{ lb.}=152 \text{ lb.}$$

$$\text{Additional h.p. absorbed}=\frac{152\times 58{\cdot}7}{550}=\mathbf{16{\cdot}2}.$$

27. *The resistances which have to be overcome by a cyclist are* (i) *rolling resistance of the tyres on the road, which may be taken at* 3 *lb. per ton at all speeds;* (ii) *the friction of the bearings, which may be taken at* $\frac{1}{2000}$*th of the weight of the rider and machine at all speeds;* (iii) *the wind resistance, which varies as the square of the speed and amounts to* 0·008 *lb. at* 1 *ft. per sec. The weight of the rider and cycle is* 200 *lb. The rider for a short time can exert* 0·6 *horse power. What is the maximum speed he can attain in miles per hour?*

If V=mean velocity in f.p.s.

$$\left(\frac{3\times 200}{2240}+\frac{200}{2000}+0{\cdot}008V^2\right)\frac{V}{550}=0{\cdot}6 \quad \therefore 0{\cdot}368V+0{\cdot}008V^3=330.$$

Solve by trial and error: V=34·1 f.p.s.=**23·3 m.p.h.**

28. *A railway truck weighing* 12 *tons, runs into a buffer stop when travelling at* 5 *m.p.h. The initial compression on the buffer springs is* 1·2 *in. which corresponds to a total load of* 400 *lb. Find the amount of compression on the springs and the corresponding load when the truck is brought to rest.*

5 m.p.h.=7·33 f.p.s.

Kinetic energy of truck $\frac{12\times 7{\cdot}33^2}{2\times 32{\cdot}2}=10$ ft.-tons=22,400 ft.-lb.

The compression rate of the springs is $\frac{400}{1{\cdot}2}=333$ lb. per in.

Net work stored in springs when compressed x in. $\Big\}=\frac{333x^2}{2}-\frac{400\times 1{\cdot}2}{2}$ in.-lb.

Hence the compression when the truck is brought to rest is given by

$$22{,}400\times 12=167x^2-240 \quad x=4{\cdot}01 \text{ in.}$$

Load on springs when the truck is brought to rest:

$$4{\cdot}01\times 333=\mathbf{1335\ lb.}$$

29. *A loaded wagon weighing* 15 *tons when moving at* 4 *m.p.h. runs into a similar stationary wagon. Find the maximum compression load on buffers if the springs compress* 6 *in.*

Maximum compression occurs when wagons move instantaneously at same speed, viz. 2 m.p.h.=2·93 f.p.s.

Let p=load on each buffer spring in tons when compression is x ft.

Total energy shared in 4 buffer springs at any time $\Big\}=4\frac{px}{2}$ ft.-tons.

At point of maximum compression:

Energy stored in buffer springs = Energy stored in one wagon before impact − Energy stored in both wagons after impact

$$=\frac{WV_1^2}{2g}-\frac{2WV_2^2}{2g}$$

$$\frac{4p\times\frac{6}{12}}{2}=\frac{15}{2g}(5{\cdot}86)^2-\frac{30}{2g}(2{\cdot}93)^2 \quad p=\mathbf{4\ tons.}$$

30. *The net accelerating force which acts on a lift weighing* $\frac{1}{2}$ *ton varies according to the following table. If the lift is moving initially with a velocity of* 2 *f.p.s., find its velocity after travelling* 10 *ft.*

Distance, feet .	0	1	2	3	4	5	6	7	8	9	10
Force, pounds .	100	90	80	70	60	50	40	30	20	10	0

If a diagram be plotted showing the relation between distance and force, it will be found to be triangular.

The *space* average force (see Vol. I., p. 10) is 50 lb. Hence the net work done during the travel of 10 ft. $=50\times10$ $=500$ ft.-lb.

This net work is converted into *kinetic energy* and from the principle of the conservation of energy we know :

$$500=\tfrac{1}{2}\cdot\frac{W}{g}V^2-\tfrac{1}{2}\cdot\frac{W}{g}2^2$$

$$\therefore\ V^2=4+\frac{32\cdot2\times1000}{1120}=32\cdot7 \qquad V=\mathbf{5\cdot73\ f.p.s.}$$

31. *A boat-swing and its occupants weigh* 350 *lb. It is suspended from a height of* 25 *ft. Find its velocity at the lowest point of swing when it swings through a total angle of* 180 *degrees.*

What is then the tension in the stayrods ?

The potential energy at the instant when the boat commences its downward swing will be greater than the potential energy at the lowest point by 350×25 ft.-lb.

If we neglect friction, the whole of this loss of potential energy will be converted into kinetic energy. Hence the velocity at the lowest point is given by :

$$350\times25=\tfrac{1}{2}\cdot\frac{350}{g}\cdot v^2 \qquad\qquad \therefore\ v=40\cdot1 \text{ f.p.s.}$$

The tension in the stayrods at the lowest point is due partly to the weight of the boat and partly to centrifugal force.

$$\text{Tension due to centrifugal force}=\frac{W}{g}\cdot\frac{v^2}{R}=\frac{350}{32\cdot2}\times\frac{(40\cdot1)^2}{25}=700 \text{ lb.}$$

Hence total tension $=700+350=\mathbf{1050\ lb.}$

32. *A motor exerts a turning moment on a shaft of* 1,400 *lb.-ft., but the resistance overcome is only* 1,100 *lb.-ft. The weight of the rotating parts is one ton and their radius of gyration* 4·6 *ft. Find the angular acceleration.*

$$A=\frac{Tg}{Wr^2}=\frac{(1400-1100)32\cdot2}{2240\times4\cdot6^2}=0\cdot204 \text{ rads.p.s.p.s.}$$

The speed at the end of the first minute is :

$$0\cdot204\times60=12\cdot24 \text{ rads.p.s.}=\frac{12\cdot24}{2\times3\cdot14}=1\cdot95 \text{r.p.s.}=\mathbf{117\ r.p.m.}$$

The speed will continue to increase at this rate unless the resistance increases, which is usually the case. When the resistance is equal to the turning moment the speed remains constant, and if the resistance becomes greater than the turning moment the speed decreases.

33. *Find the angular acceleration* A *in rads. p.s.p.s. of a series of rotating weights when they are acted upon by a force of* 63 *lb. at a radius of* 8 *in.*

Find the moment of inertia of the body in gravitational units, also the radius of gyration.

$W_1=4$ lb. $r_1=20$ in. $W_3=5$ lb. $r_3=14$ in.
$W_2=7$ lb. $r_2=16$ in. $W_4=8$ lb. $r_4=10$ in.

One is very liable to go wrong with the units in such a case as this—it is safer to convert all radii into *feet* before starting on the solution. Thus the 63 lb. acts at a radius of $\frac{8}{3}$ ft., $r_1=1\cdot67$ ft., $r_2=1\cdot33$ ft., $r_3=1\cdot17$ ft., $r_4=0\cdot83$ ft.

Twisting moment $=63\times\frac{8}{3}=42$ lb.-ft.

$$A=\frac{42\times32\cdot2}{4\times1\cdot67^2+7\times1\cdot33^2+5\times1\cdot17^2+8\times0\cdot83^2}=\frac{1352}{35\cdot92}$$

$=37\cdot6$ rads.p.s.p.s. $=37\cdot6\times57\cdot3=2160$ degs. p.s.p.s.

$$=\frac{37\cdot6}{2\pi}=6 \text{ r.p.s.p.s.}$$

Moment of inertia $=35\cdot92$ in lb.-ft. (gravitational units).

$$(\text{Radius of gyration})^2=\frac{35\cdot92}{4+7+5+8}=1\cdot5.$$

Radius of gyration $=\sqrt{1\cdot5}=\mathbf{1\cdot23\ ft.}$

As a rough check it should be noted that the radius of gyration is somewhere near the mean of the respective radii. The mean in this case is 1·25 ft.

34. *Hailstones fall vertically at a speed of* 29 *f.p.s. on to the deck of a ship which is travelling at a speed of* 14 *knots* (1 *knot*=6,080 *ft. per hr.*). *What will be the apparent velocity and direction of the hailstones to an observer standing on the deck?*

Speed of ship $= \frac{14 \times 6080}{60 \times 60} = 23{\cdot}6$ f.p.s.

Apparent velocity $= \sqrt{29^2 + 23{\cdot}6^2} = 37{\cdot}4$ f.p.s.

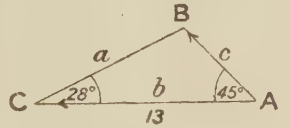
Direction: $\tan \alpha = \frac{23{\cdot}6}{29} = 0{\cdot}814$; $\alpha =$ **39° 10′ to vertical.**

35. *An observer on the deck of a steamer notes that the direction of the smoke is at an angle of* 28° *with the centre line of the ship. The ship is travelling at* 13 *knots* (1 *knot* =6,080 *ft. per hr.*) *due west. The wind is from the S.E. Find the speed of the wind.*

$B = 180 - 45 - 28 = 107°$

$C = b \frac{\sin C}{\sin B} = 13 \times \frac{\sin 28}{\sin 107}$

$C = 13 \times \frac{0{\cdot}469}{0{\cdot}956} = 6{\cdot}38$ knots.



The speed of the wind is 6·38 knots or

$$\frac{6{\cdot}38 \times 6080}{5280} = \textbf{7·35 m.p.h.}$$

NOTE.—A very common mistake is to speak of the speed of a ship as so many knots per hour. The "per hour" should be omitted or the speed given as so many nautical miles per hour.

36. *A shell from a gun is fired at a target* 2,000 *yd. away. The shell falls* 150 *yd. short and is in the air for* $5\frac{1}{2}$ *secs. Neglecting air resistance, find the additional elevation which must be given to the gun in order to hit the target with the next round.*

The trajectory will be so flat that the *horizontal* velocity of the shell may be taken as the same in each case, although strictly it is the actual inclined muzzle velocity which will be the same.

This horizontal velocity will be $\frac{(2000-150)3}{5{\cdot}5} = 1010$ f.p.s.

The vertical component of the muzzle velocity may be regarded as just sufficient to keep the shell in the air for the time necessary to cover the range required.

The vertical velocity of the first shell was therefore such that it was destroyed by gravity in $\frac{5{\cdot}5}{2}$ sec. and is therefore given by:

$$0 = u_1 - g \cdot \frac{5{\cdot}5}{2} \qquad \therefore\ u_1 = 88{\cdot}5 \text{ f.p.s.}$$

The vertical component for the second shell must be sufficient to keep it in the air for $\frac{2000 \times 3}{1010}$ sec. and is therefore given by:

$$0 = u_2 - g \cdot \frac{2000 \times 3}{1010 \times 2} \qquad \therefore\ u_2 = 95{\cdot}6 \text{ f.p.s.}$$

Hence additional elevation must give an increase in the vertical component of 7·1 f.p.s. and will therefore be:

$$\theta = \frac{7{\cdot}1}{1010} \text{ radians} = \frac{7{\cdot}1}{1010} \times \frac{180 \times 60}{\pi} \text{ mins.} = \textbf{24 mins. of angle.}$$

37. *A bombing aeroplane is travelling horizontally at 120 m.p.h. at a height of 3,000 ft. At what angle should a ground target appear when a bomb is released?*

Initially the bomb has no *vertical* component of velocity.

Hence the time taken for the bomb to fall 3,000 ft. is given by:

$$3000 = \tfrac{1}{2} \cdot g \cdot t^2 \qquad \therefore\ t = 13{\cdot}7 \text{ sec.}$$

The horizontal component of the velocity of the bomb will be constant during flight and will be 120 m.p.h. = 176 f.p.s.

Hence the horizontal distance covered during its flight of 13·7 sec. is $176 \times 13{\cdot}7 = 2{,}410$ ft.

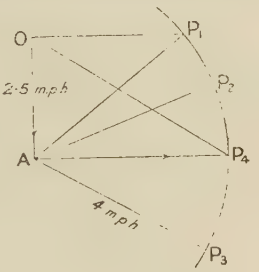
It follows that the angle at which the ground target should appear when the bomb is released will be A with the horizontal, where

$$\tan A = \tfrac{3000}{2410} = 1{\cdot}24 \qquad A = \textbf{51° 7′}.$$

38. *A man can row at the rate of 4 m.p.h. in still water. How long will it take him to row* straight *across a river 400 yd. wide, having a mean surface speed of $2\frac{1}{2}$ m.p.h.?*

What is the least time in which he could cross at all, and where would he land?

His true velocity will be the resultant of the velocity of the stream and his own velocity relative to the water. It will therefore be represented by the vectorial sum of OA, AP in the figure, that is by OP_1, OP_2, etc., according to the direction in which he rows.



(*a*) If he rows *straight* across, his true velocity is represented by OP_1 and $v_1=\sqrt{4^2-2{\cdot}5^2}=3{\cdot}12$ m.p.h.

The time taken in minutes is therefore given by:

$$\frac{400}{1760}=3{\cdot}12\,\frac{t}{60} \qquad \therefore\ t=\mathbf{4{\cdot}37\ min.}$$

(*b*) He will get across in the least time if his velocity vector is OP_4.

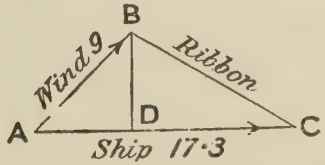
In this case his time for crossing is given by

$$\frac{400}{1760}=4\,\frac{t}{60} \qquad t=\mathbf{3{\cdot}41\ min.}$$

The direction in which he will row in this case is shown by OP_4, and it follows that the distance he will have moved downstream at the point of landing is:

$$\frac{2{\cdot}5}{4}\times 400=\mathbf{250\ yd.}$$

39. *A ship is travelling due east at a speed of* 15 *knots, the wind is blowing from the south-west at* 9 (*land*) *m.p.h. Find the direction taken by a ribbon tied to a mast.*



Speed of ship in land m.p.h. $=\dfrac{15\times 6080}{5280}=17{\cdot}3$

$$\tan C = \frac{BD}{DC} = \frac{9 \sin 45^\circ}{17{\cdot}3 - 9 \cos 45^\circ} = \frac{6{\cdot}36}{10{\cdot}94} = 0{\cdot}581$$

$\therefore$ C = 30° 10′ north of west, which is the direction in which the ribbon flies.

40. *The cars of a roundabout are suspended by two rods in such a manner that the cars can swing outwards when running. The diameter of the circle described by the centre of gravity of the cars is* 60 *ft., and they make* 6 *revs. per min. Find the angle the suspension rods make with the vertical.*

Let l = Distance from the point of suspension to the c. of g. of car.

h = Height of c. of g. of car below point of suspension.

x = Horizontal distance of c. of g. of car from a vertical through the point of suspension.

θ = Angle of suspension rods to the vertical.

$h = l \cos \theta$. $\qquad x = l \sin \theta$.

$$\frac{WV^2}{gR} = 0{\cdot}00034WRN^2 \quad \text{where N} = \text{rev. p. min.}$$

Taking moments about the point of suspension (see Chap. III., Vol. I.)

$$0{\cdot}00034W \times 30 \times 6^2 \times l \cos \theta = Wl \sin \theta$$

$$0{\cdot}3672 = \tan \theta \qquad \theta = \mathbf{20^\circ\ 10'}.$$

41. *A train runs round a curve of* 2,700 *ft. radius at* 40 *m.p.h. Gauge* (G) 4 *ft.* 8½ *ins. Find the necessary super-elevation* (S).

$$\frac{\text{Centrifugal force}}{\text{Weight of train}} = \frac{S}{G} \qquad \frac{WV^2G}{WgR} = S$$

40 m.p.h. = 58·7 f.p.s.

$$S = \frac{58{\cdot}7^2 \times 4{\cdot}7}{32{\cdot}2 \times 2700} = 0{\cdot}186 \text{ ft.} = \mathbf{2{\cdot}24 \text{ ins.}}$$

42. *The speed of a car is* 40 *m.p.h. on a circular level track; it turns through an angle of* 10° *in* 30 *ft. The weight of the car is* 1½ *tons.*

Find the necessary friction between the car wheels and the track to prevent side slip.

40 m.p.h. = 58·7 f.p.s. $\qquad \frac{10}{360} \times \text{circumference} = 30 \text{ ft.}$

Circumference = 1,080 ft. $\qquad R = 171{\cdot}9$ ft.

$$\text{Necessary friction} = \frac{WV^2}{gR} = \frac{1{\cdot}5 \times 58{\cdot}7^2}{32{\cdot}2 \times 171{\cdot}9} = \mathbf{0{\cdot}934 \text{ ton.}}$$

43. *Show that a centrifuge* 24 *ins. diam. when rotating at* 1000 *r.p.m. is* 340 *times as efficient a separator as gravity.*

The force acting on a particle at radius R when moving with an angular velocity ω is the resultant of gravity=W acting vertically and $\frac{W}{g}\omega^2 R$ acting radially.

$$\text{Hence force acting} = W\sqrt{1+\frac{\omega^4 R^2}{g^2}}$$

$$\text{In this case } \omega = 1000 \times \frac{2\pi}{60} \qquad \text{Since } R = 1$$

The centrifugal force is big compared with gravity, hence

$$\text{Force acting} = \frac{W\omega^2 R}{g} \text{ nearly } = \mathbf{340W}$$

Alternative solution : Centrifugal force $=0{\cdot}00034\ WRN^2$
$=0{\cdot}00034W \times 1 \times 1000^2 = \mathbf{340W.}$

44. *Find the greatest speed at which a motor cycle can cross the summit of a hill without leaving the road if the slope changes uniformly from* 1 *in* 8 *up to* 1 *in* 5 *down in* 11 *yds.*

If there were a bend of 50 *ft. radius at the top of the hill, find the greatest speed at which it could be taken.* $\mu = 0{\cdot}4$

Acceleration of bicycle on curved path $= \frac{v^2}{r}$ towards centre

$$\text{Hence, if P = reaction of road, } W - P = \frac{W}{g} \cdot \frac{v^2}{r}$$

Radius of summit of hill : $r = \frac{S}{\theta} = \frac{33}{\frac{1}{8}+\frac{1}{5}} = 102$ ft.

Hence reaction P vanishes and cycle will leave the road if

$\frac{v^2}{gr} > 1$ *i.e.* if $v > 57$ f.p.s. or **39 m.p.h.**

Accelerating force towards centre of bend

$$= \mu P = \mu W\left(1 - \frac{v^2}{gr}\right).$$

Hence, if R is radius of bend : $\frac{W}{g} \cdot \frac{v^2}{R} = \mu W\left(1 - \frac{v^2}{gr}\right)$.

Greatest speed at which it can be taken is given by :

$\frac{v^2}{50} = 0{\cdot}4\left(g - \frac{v^2}{102}\right)$ $v = 23{\cdot}2$ f.p.s. $=$ **15·8 m.p.h.**

CHAPTER II

MENSURATION

1. *Find the length of arc of a circle 22 ft. diam., height from chord to circum. 8 ft.*

$$\text{Length} = \frac{8C_1 - C_a}{3} \text{ approx. (see p. 22, Vol. I.)}$$

$$C_a = 2\sqrt{11^2 - 3^2} = 21{\cdot}17 \text{ ft.}$$

$$C_1 = \sqrt{\left(\frac{21{\cdot}17}{2}\right)^2 + 8^2} = 13{\cdot}27 \text{ ft.}$$

$$\text{Length of arc} = \frac{8 \times 13{\cdot}27 - 21{\cdot}17}{3} = \mathbf{28{\cdot}3 \text{ ft.}}$$

Exact value (see Kempe's "Engineer's Handbook," tables of segments, etc.) is **28·5 ft.**

Find the area of the above-mentioned segment.

$$\text{Area} = \frac{h}{15}(6C_a + 8C_1) \text{ approx.}$$

$$= \frac{8}{15}(6 \times 21{\cdot}17 + 8 \times 13{\cdot}27) = \mathbf{124{\cdot}4 \text{ sq. ins.}}$$

Kempe's exact method gives **125·1 sq. ins.**

2. *Find the circumference of an ellipse diams. 24 and 20 ins.*

$$\text{Length} = \pi d + 2(D - d) - \frac{d(D-d)}{\sqrt{(D+d)(D+2d)}} \text{ approx.}$$

$$= 3{\cdot}14 \times 20 + 8 - \frac{20 \times 4}{\sqrt{44 \times 64}} = \mathbf{69{\cdot}3 \text{ ins.}}$$

Kempe's "Engineer's Handbook" gives

$$\text{Length} = \sqrt{\frac{D^2 + d^2}{2}} \times 3{\cdot}1416 = \mathbf{69{\cdot}4 \text{ ins.}}$$

Find the area of the above-mentioned ellipse.

$$\text{Area} = 0{\cdot}785 \times 20 \times 24 = \mathbf{377 \text{ sq. ins.}}$$

3. *A rod supported at ends sags $\frac{1}{20}$ of its length. Find error in assuming length of rod same as span.*

Rod is very nearly parabolic in form.

$$\left.\begin{matrix}\text{Length of arc}\\ \text{(Vol. I., Fig. 14)}\end{matrix}\right\} = 2\sqrt{\frac{b^2}{4}+\frac{4h^2}{3}} = b\left(1+\frac{8h^2}{3b^2}\right) \text{ nearly.}$$

When $\frac{h}{b}=\frac{1}{20}$. Error $=\frac{2}{3}$ **per cent.**

4. *Find length of parabolic arc, length of base* $=8$ *ins., height* $=12$ *ins.*

$$\text{Length of arc} = 2\sqrt{\frac{64}{4}+\frac{4\times 144}{3}} = 28{\cdot}84 \text{ ins.}$$

(approx., see p. 24, Vol. I.).

Exact value (see books on the calculus).

The equation to the curve may be written $\left(\frac{b}{2}\right)^2 = 4ah$

where a is a constant $(\frac{8}{2})^2 = 4a\times 12$ whence $a=\frac{1}{3}$

Length of arc of half the figure

$$= \frac{\frac{b}{2}\sqrt{4a^2+\left(\frac{b}{2}\right)^2}}{4a} + a\log_e \frac{\frac{b}{2}\sqrt{4a^2+\left(\frac{b}{2}\right)^2}}{2a}$$

$$= \frac{4\sqrt{\frac{1}{9}+16}}{\frac{1}{3}} + \frac{1}{3}\log_e \frac{4\sqrt{\frac{1}{9}+16}}{\frac{2}{3}}$$

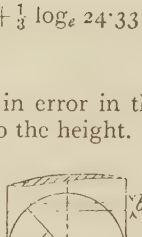
$$= 3\sqrt{16{\cdot}44} + \frac{1}{3}\log_e 6\sqrt{16{\cdot}44} = 12{\cdot}165 + \frac{1}{3}\log_e 24{\cdot}33$$

$$= 12{\cdot}165 + 1{\cdot}064 = 13{\cdot}23$$

Length of whole arc $=$ **26·46 ins.**

Thus the approximation is 9·4 per cent. in error in this case; it depends upon the ratio of the base to the height.

5. *Show that the superficial area of any belt of a sphere cut off by parallel planes equals the surface cut off by the same planes on the enveloping cylinder.*



Consider any small belt made by planes distance b apart.

Sloping breadth of this belt $= \frac{b}{\cos\alpha}$.

Length of belt $= 2\pi R\cos\alpha$.

Area of belt $= 2\pi Rb$.

This is the same as area of belt cut off from enveloping cylinder, for its breadth is b and its length is $2\pi R$.

This is true for any narrow belt, and by adding together a number of them it follows that it is true for a belt of *any* width, up to the whole surface of the sphere :

Surface of sphere $=4\pi R^2$.

Surface of belt on cylinder $=2\pi R \times 2R = 4\pi R^2$ (Vol. I., Fig. 36).

6. *Find the area of a segment of a circle whose chord is 8 ins. and height 3·1 ins. ; also find the length of the arc.*

$C_1 = \sqrt{4^2 + 3{\cdot}1^2} = 5{\cdot}06$ (see Fig. 25, Vol. I.).

$$\text{Area} = \frac{3{\cdot}1}{15}(6 \times 8 + 8 \times 5{\cdot}06) = \mathbf{18{\cdot}28\ sq.\ ins.}\ \text{(nearly)}.$$

Kempe's "Engineer's Handbook" gives **18·36 sq. ins.**

$$\text{Length of arc} = \frac{8 \times 5{\cdot}06 - 8}{3} = \mathbf{10{\cdot}83\ ins.}\ \text{(nearly)}$$

(see Fig. 12, Vol. I.).

Kempe's tables give **10·89 ins.**

In both cases Kempe's values are more accurate than the approximations we have made use of.

7. *Find the area of the wetted surface of the walls of the reservoir described in Example* 15.

If the *sloping* height of the wall is L, and the upper and lower diams. D_1 and D_2, then the inner face would be generated by a line of length L whose centre of gravity is $\frac{D_1+D_2}{4}$ from the centre of the reservoir.

Hence the surface required $= L \times \frac{\pi}{2}(D_1+D_2)$.

In this case $L^2 = 28^2 + \left(\frac{87-34}{2}\right)^2$. $\therefore$ $L = 38{\cdot}6$ ft.

$\therefore$ Area $= 38{\cdot}6 \times \frac{\pi}{2}(87+34) = \mathbf{7330\ sq.\ ft.}$

8. *The hardness of metals is sometimes measured by pressing a hard steel ball into the metal and dividing the load by the area of the curved surface of the indentation. Find the hardness number if the diameter of the indentation is 0·213 in. for a load of 1 ton. Diameter of ball 0·375 in.*

$$\text{Area of surface} = 2\pi R\left(R - \sqrt{R^2 - \frac{d^2}{4}}\right) \text{(see Fig. 34}b\text{, Vol. I.)}$$

$$= 2 \times 3{\cdot}14 \times 0{\cdot}1875\left(0{\cdot}1875 - \sqrt{0{\cdot}1875^2 - \frac{0{\cdot}213^2}{4}}\right)$$

$$= 0{\cdot}0391 \text{ sq. in.}$$

$$\text{Hardness number} = \frac{1}{0{\cdot}0391} = 25{\cdot}6 \text{ tons per sq. in.}$$

The area of the curved surface is sometimes taken from the depth h of the indentation. The area then (see Example 5) is $2\pi Rh$. h in this case $= 0{\cdot}0332$ in., and

$2 \times 3{\cdot}14 \times 0{\cdot}1875 \times 0{\cdot}0332 = 0{\cdot}0391$ sq. in. as before.

In the Brinell system the load is taken in kilograms and the area in square millimetres, 1 ton = 1016 kilos, and 1 sq. mm. $= 0{\cdot}00155$ sq. in.

Therefore the hardness number, if the diameter of the indentation had been taken in mm. and the load 1016 kilos, would be $25{\cdot}6 \times 0{\cdot}00155 \times 1016 = 40{\cdot}3$. The Brinell ball is 10 mm. diam. $= 0{\cdot}394$ in.

9. *Find the area of an indicator diagram 3·2 ins. long, height of admission line above zero pressure 2·12 ins., cut off occurs at $\frac{1}{5}$th stroke. Hyperbolic expansion, no losses. What is the mean height of diagram?*

$$\text{Area} = 2{\cdot}12 \times \frac{3{\cdot}2}{5} + 2{\cdot}12 \times \frac{3{\cdot}2}{5} \log_e 5$$

$$= 1{\cdot}357(1 + 1{\cdot}609) = 3{\cdot}54 \text{ sq. ins.}$$

$$\text{Mean height} = \frac{3{\cdot}54}{3{\cdot}2} = \mathbf{1{\cdot}11 \text{ ins.}}$$

If the back pressure line were 0·18 in. above the zero, the area would be

$3{\cdot}54 - 0{\cdot}18 \times 3{\cdot}2 = 2{\cdot}96$ sq. ins. Mean height $= \mathbf{0{\cdot}93 \text{ in.}}$

10. *Find the area of an indicator diagram 3·2 ins. long, including clearance, admission line 2·12 ins. above zero. Cut off $\frac{1}{5}$ stroke. Clearance 0·25 in. Back pressure line 0·18 in. above zero. No compression. Hyperbolic expansion.*

$$\text{Length of admission line on diagram} = \frac{3{\cdot}2 - 0{\cdot}25}{5} = 0{\cdot}59 \text{ in.}$$

$$\text{Ratio of expansion} = \frac{3{\cdot}2}{0{\cdot}59 + 0{\cdot}25} = 3{\cdot}81.$$

$$\text{Area of diagram} = 2{\cdot}12 \times \frac{2{\cdot}95}{5} + 2{\cdot}12 \times 0{\cdot}84 \log_e 3{\cdot}81 - 2{\cdot}95 \times 0{\cdot}18$$
$$= 1{\cdot}25 + 1{\cdot}78 \times 1{\cdot}34 - 0{\cdot}53 = 2{\cdot}12 \text{ sq. ins.}$$

If compression had started at half the return stroke the pressure reached at the dead centre would be given by

$$p \times 0{\cdot}25 = 0{\cdot}18\left(3{\cdot}2 - \frac{2{\cdot}95}{2}\right) \qquad p = 1{\cdot}24$$

The area under the compression curve is

$$1{\cdot}24 \times 0{\cdot}25 \log_e \frac{1{\cdot}725}{0{\cdot}25} = 0{\cdot}60 \text{ sq. in.}$$

Net area of diagram $= 2{\cdot}12 - 0{\cdot}60 + 1{\cdot}475 \times 0{\cdot}18 =$ **1·79 sq. ins.**

It is of interest to note that under the above-mentioned conditions the engine receives 22·9 per cent. less steam and does 15·5 per cent. less work when compressing, from which one would expect the engine to work more economically with compression than without, but experiments do not always support this theory.

11. *Taking the case mentioned in Example* 9, *find the area when the expansion curve is* $YX^{1{\cdot}4} = \text{constant}$.

We have $YX^{1{\cdot}4} = Y_1X_1^{1{\cdot}4}$ $\qquad 2{\cdot}12\left(\frac{1}{5}\right)^{1{\cdot}4} = Y_1 = 0{\cdot}223''$

$$\text{Area under the curve} = \frac{2{\cdot}12 \times \frac{3{\cdot}2}{5} - 0{\cdot}223 \times 3{\cdot}2}{0{\cdot}4} = 1{\cdot}61 \text{ sq.in.}$$

$$\text{Total area} = 2{\cdot}12 \times \frac{3{\cdot}2}{5} + 1{\cdot}61 = 2{\cdot}97 \text{ sq. ins.}$$

$$\text{Mean height} = \frac{2{\cdot}97}{3{\cdot}2} = 0{\cdot}93 \text{ in.}$$

With back pressure:
Area $= 2{\cdot}97 - 0{\cdot}58 = 2{\cdot}39$ sq. ins. $\qquad$ Mean height $= 0{\cdot}75$ in.

12. *A surface condenser is required with a tubular cooling surface of* 4,500 *sq. ft. Length between plates* 12 *ft.* 6 *ins., diameter of tubes* 1 *in. Find the number of tubes required.*

Total surface of tubes $= 4{,}500 \times 144$ sq. ins.

$$\text{Total length of tubes} = \frac{4500 \times 12}{\pi} \text{ ft.}$$

$$\text{Hence number of tubes required} = \frac{4500 \times 12}{\pi} \times \tfrac{2}{25} = \mathbf{1375}.$$

13. *A three-throw pump with 4-in. rams and 6-in. stroke makes 60 r.p.m. Neglecting slip, calculate the discharge in gals. per min.*

Displacement per stroke of one ram $=\frac{\pi}{4}\times 16\times 6$ cu. ins.

$\therefore$ Displacement per rev. $=3\times\frac{\pi}{4}\times 16\times 6$ cu. ins. $=226$ cu. ins.

$\therefore$ Displacement p.m. $=60\times 226\times\frac{6{\cdot}25}{12^3}$ gals. $=$ **49 gals. p.m.**

14. *A 500-foot coil of stranded copper conductor consisting of seven similar wires weighs 25 lb. Find the diameter of the wire and the current carrying capacity of the conductor, allowing 2,000 amperes per sq. in. Copper weighs 0·32 lb. per cu. in. Allow 2 per cent. in length of wires due to coiling in strand.*

Volume of copper $=\frac{25}{0{\cdot}32}=78{\cdot}2$ cu. ins.

Sectional area of conductor $=\frac{78{\cdot}2}{500\times 12}$ sq. ins.

$\therefore$ Current carrying capacity $=\frac{78{\cdot}2}{6000}\times 2000$ amps. $=$ **26·1 amps.**

If each wire were 500 ft. long, sectional area of each would be $\frac{0{\cdot}0130}{7}=0{\cdot}00186$ sq. in.

Actually each wire is 510 ft. long owing to coiling.

Hence sectional area $=\frac{500}{510}\times 0{\cdot}00186=0{\cdot}00182$ sq. in.

Hence diameter of wire $=\sqrt{\frac{4}{\pi}\times 0{\cdot}00182}=$ **0·0482 in.**

15. *It is required to pump the water out of a circular reservoir by means of a pump which delivers 10,000 gals. of water per min. Find (a) the time taken to empty the reservoir, (b) the number of useful foot-tons of work done. The diameter of the reservoir at the bottom is 34 ft. and 87 ft. at the top; the depth of the water is 28 ft. The centre of gravity of the water is 16·6 ft. above the bottom. The delivery pipe of the pump is 37 ft. above the bottom of the reservoir.*

Area at bottom $=907{\cdot}9$ sq. ft.; midway $2874{\cdot}8$ sq. ft.; top $5944{\cdot}7$ sq. ft.

Volume $=\frac{28}{6}(907{\cdot}9+4\times 2874{\cdot}8+5944{\cdot}7)=85{,}660$ cu. ft.
$=534{,}000$ gals.

(*a*) Time=**53·4 mins.**

(*b*) $\frac{534000 \times 10 \times (37 - 16{\cdot}6)}{2240}$ **=48,700 ft.-tons.**

16. *A reservoir site has been surveyed by taking out contours at every* 20 *ft., the areas of which are :* 3·47, 180·20, 567·82, 889·3, 1,460·2 *sq. ft. Find the cubic contents of the reservoir.*

Volume=$\frac{20}{3}$(3·47+4×180·2+2×567·82+4×889·3 +1460·2)=**45,850 cu. ft.**

17. *Determine the volume in cubic yards of a knoll, and explain what assumption you make in dealing with the top slice.*

Area of contours—

At the datum level		12,430	sq. ft.
10 ft. above datum level		9,840	,,
20 ft. ,, ,, ...		8,520	,,
30 ft. ,, ,, ...		6,130	,,
40 ft. ,, ,, ...		3,170	,,
47 ft. ,, ,,	(*i.e.* top of knoll)	Zero	

Volume=$\frac{10}{3}$(12430+4×9840+2×8520+4×6130+3170) =321,800 cu. ft.

The top may be assumed to be a paraboloid of revolution

$$\text{Volume} = \frac{3170 \times 7}{2} = 11{,}100 \text{ cu. ft.}$$

Total volume=332,800 cu. ft.=**12,300 cu. yds.**

18. *Find the weight of :*

(*a*) *A cast-iron ball,* 8 *ins. diam.*

(*b*) *A pointed cast-iron shell* 12 *ins. diam. consisting of a cylindrical portion* 14 *ins. long, and a pointed portion* 6 *ins. long. The area of the half section of the pointed portion is* 24 *sq. ins. and its c. of g. is at a distance of* 2·25 *ins. from the axis.*

(*c*) *A cast-iron body,* 8 *ins. total length, having sections, taken normal to the length, of* 0, 3·4, 4·8, 5·3, 2·9, 1·7, 0·8 *sq. ins., at equal intervals.*

Cast iron weighs 0·26 *lb. per cu. in.*

(*a*) Weight=$\frac{0{\cdot}26 \times 3{\cdot}14 \times 8^3}{6}$ **=69·7 lb.**

(*b*) Vol. cylindrical portion $=0{\cdot}785\times12^2\times14=1583$ cu. ins.
Vol. pointed portion $=24\times2\times3{\cdot}14\times2{\cdot}25=339$,,
Volume of both $=1922$ cu. ins.
Weight $=1922\times0{\cdot}26=$ **500 lb.**

(*c*) The distance between the sections is $\frac{8}{6}$ in.
Volume $=\frac{8}{18}(0+4\times3{\cdot}4+2\times4{\cdot}8+4\times5{\cdot}3+2\times2{\cdot}9+4\times1{\cdot}7+0{\cdot}8)=25{\cdot}7$ cu. ins.
Weight $=25{\cdot}7\times0{\cdot}26=$ **6·68 lb.**

19. *Find the weight of the rim of a wheel of elliptical section having the major axis radial.*

External diam.	12 *ft.*
Major axis	11 *ins.*
Minor axis	6 *ins.*
Weight of material	0·26 *lb. cu. in.*

Area of section, $11\times6\times0{\cdot}785=51{\cdot}81$ sq. ins.
Diameter of path of c. of g., $144-11=133$ ins.
Wt. of rim, $51{\cdot}81\times3{\cdot}14\times133\times0{\cdot}26=5{,}627$ lb. $=$ **2·51 tons.**

20. *A hemispherical vessel* 8 *ft. diam. is filled with water; find the quantity it contains, also find the depth of water* Y *when* $\frac{2}{3}$ *and* $\frac{1}{3}$ *full.*

Volume of hemisphere $=\frac{\pi d^3}{12}=\frac{3{\cdot}14\times512}{12}=134$ cu. ft.

Volume of slice $=\frac{\pi}{3}(2R^3-3RY^2+Y^3)=\frac{1}{3}\frac{4}{3}$ cu. ft.

Volume $=\frac{3{\cdot}14}{3}(2\times64-3\times4Y^2+Y^3)=$ **44·67 cu. ft.**

Y = 3·1 ft. for $\frac{2}{3}$ full.

For $\frac{1}{3}$ full

Volume of slice $=\frac{\pi}{3}(3RY^2-Y^3)=44{\cdot}67$ cu. ft.

$$=\frac{3{\cdot}14}{3}(3\times4Y^2-Y^3)=44{\cdot}67$$

Y = 2·07 ft. for $\frac{1}{3}$ full.

Equations solved by trial and error.

21. *A* 30 *ft. by* 8 *ft. Lancashire boiler has two furnace tubes each* 2 *ft.* 6 *ins. in diam. Find the weight of water it contains when the surface is* 18 *ins. from the top of the boiler.*

Area of segment above water level (Vol. I., Fig. 25) is nearly $\frac{2}{3}$ Ca. h. Neglecting gusset plates, doors, etc., and also the slight slope of the boiler :

Area of segment $=\frac{2}{3}\sqrt{8^2-5^2}\times 1{\cdot}5=6{\cdot}25$ sq. ft.

Hence sectional area under water

$$=\frac{\pi}{4}\cdot 64-2\cdot\frac{\pi}{4}(2{\cdot}5)^2-6{\cdot}25=34{\cdot}2 \text{ sq. ft.}$$

Weight of water $=34{\cdot}2\times 30\times 62{\cdot}5$ lb. $=$ **64,100 lb.**

22. *A circular concrete storage tank is* 20 *ft. in external diameter and* 12 *ft. in overall height, with vertical outer walls tapering from* 9 *ins. at the top to* 1 *ft.* 6 *ins. at the base, which is* 12 *ins. thick.*

If the concrete is a 6 *to* 1 *mixture by weight and weighs* 130 *lb. per cu. ft., find the weight of cement required.*

Volume of base $=\frac{\pi}{4}\times 20^2\times 1=314$ cu. ft.

Volume of walls 9 ins. thick $=2\pi\times\left(10-\frac{4{\cdot}5}{12}\right)\times 11\times 0{\cdot}75=498$ cu. ft.

Volume of tapered portion $=2\pi\times\left(9{\cdot}25-\frac{0{\cdot}75}{3}\right)\times\frac{11\times 0{\cdot}75}{2}=233$ cu. ft.

Volume of concrete $=1045$ cu. ft.

Weight of cement $=\frac{130\times 1045}{7}$ lb. $=$ **19,400 lb.**

23. *Estimate the number of bricks* 9 *ins. by* $4\frac{1}{2}$ *ins. by* 3 *ins. required in building a circular mill chimney* 150 *ft. high,* 5 *ft. internal diam., and tapering uniformly on the exterior surface from* 6 *ft.* 6 *ins. diam. at the top to* 15 *ft. at the base.*

The volume of the cylindrical interior $=\frac{\pi}{4}(6{\cdot}5^2-5^2)150=2025$ cu. ft.

Area of the triangle which sweeps out the remainder

$$=\tfrac{1}{2}\times 150\times 4{\cdot}25=319 \text{ sq. ft.}$$

Centre of gravity of this triangle lies at a distance from the axis $=2{\cdot}5+0{\cdot}75+\frac{4{\cdot}25}{3}$ ft. $=4{\cdot}67$ ft.

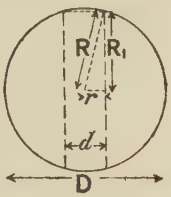
Volume of this portion $=2\pi\times 319\times 4{\cdot}67=9{,}350$ cu. ft.

Total volume of brickwork $=11{,}375$ cu. ft.

Number of bricks required, neglecting mortar,

$$=\frac{11375\times 12\times 144}{9\times 4{\cdot}5\times 3}=\textbf{162,000 nearly.}$$

24. *A hole* 1 *in. diam. is drilled through a* 4 *in. sphere. What proportion of the material is removed by drilling the hole?*



$R_1=\sqrt{R^2-r^2}=\sqrt{4-0{\cdot}25}=1{\cdot}937$

Volume of cylindrical hole $=\pi r^2 2R_1$ (see Fig. 51, Vol. I.).

Volume of slices at ends

$$=\frac{2\pi}{3}\{3R(R-R_1)^2-(R-R_1)^3\}.$$

$$\text{Volume of hole}=2\pi\left\{r^2R_1+R(R-R_1)^2-\frac{(R-R_1)^3}{3}\right\}.$$

$$\left.\begin{matrix}\text{Volume}\\ \text{of hole}\end{matrix}\right\}=6{\cdot}28\left\{\frac{1{\cdot}937}{4}+2(0{\cdot}0635)^2-\frac{0{\cdot}0635^3}{3}\right\}=3{\cdot}09 \text{ cu. ins.}$$

The end slices may, without appreciable error, be assumed to be of parabolic shape.

$$\text{Volume of paraboloid}=\frac{\pi r^2 h}{2}=\frac{3{\cdot}14\times 0{\cdot}0635}{2\times 4}=0{\cdot}025 \text{ cu. in.}$$

Total vol. of hole $=2(\pi r^2R_1+0{\cdot}025)=3{\cdot}09$ cu. ins. as before.

$$\text{Volume of sphere}=\frac{\pi D^3}{6}=33{\cdot}51 \text{ cu. ins.}$$

$$\text{Proportion removed by hole}=\frac{3{\cdot}09}{33{\cdot}51}=\mathbf{9{\cdot}2 \text{ per cent.}}$$

The percentage removed is given approximately by the following expression:

$$\text{Let } n=\frac{d}{D}$$

Percentage removed by drilling $=50n^2(3-n^3)=\mathbf{9{\cdot}3 \text{ p.c.}}$

CHAPTER III

MOMENTS

1. *Find the pressure on the valve when a lever safety valve blows under the following conditions :—*

Length of lever from fulcrum to ball weight	33″
Distance of valve to fulcrum	4″
Diameter of valve	3″
Breadth of lever	$\frac{5}{8}$″
Depth of lever at fulcrum	1·5″
,, *at ball end*	1·0″
Diameter of ball	6″
Wt. of ball hook	1·5 *lb.*
Wt. of material in lever	0·27 *lb. per c. in.*
,, ,, *ball*	0·26 ,, ,,

Wt. of lever $= 33 \times \frac{5}{8} \times 1{\cdot}25 \times 0{\cdot}27 = 7{\cdot}0$ lb.

Distance of c. g. of lever from fulcrum $= \frac{33}{3}\left(\frac{2+1{\cdot}5}{2{\cdot}5}\right) = 15{\cdot}4''$.

Wt. of ball $= \frac{\pi \times 6^3 \times 0{\cdot}26}{6} = 29{\cdot}4$ lb.

Taking moments about fulcrum

$(29{\cdot}4+1{\cdot}5)33+7\times 15{\cdot}4 = 4p$. $\qquad p = 282$ lb.

Area of valve $= 7{\cdot}07$ sq. ins.

Press. at which valve blows $\frac{282}{7{\cdot}07} =$ **40·0 lb. sq. in.**

2. *In a single lever testing machine the lever weighs* 4 *tons, its centre of gravity is* 11 *ins. to the left of the supporting knife edge,* i.e. *on the long side of the lever. Find the load on the specimen when the poise weight* (1 *ton*) *is*

(a) 18 *ins. to the right of the supporting knife edge ;*

(b) 5′ 4″ *to the left of the supporting knife edge ;*

(c) 3′ 7″ *to the left, and, in addition, a weight of* 24 *cwts. is suspended on the end of the long arm at a distance of* 13′ 4″ *to the left of the supporting knife edge. The knife edge from*

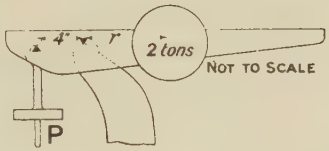
which the specimen shackle is supported is 4″ *from the supporting knife edge.*

(*a*) $P\times4+18=4\times11$ $\quad P=\frac{44-18}{4}=\frac{26}{4}=$ **6·5 tons.**

(*b*) $P\times4=4\times11+64=108$ $\quad P=$ **27 tons.**

(*c*) $P\times4=4\times11+43+\frac{24}{20}\times160$ $\quad P=$ **69·75 tons.**

3. *The lever shown in the sketch weighs* 3·4 *tons, its c. of g. is* 3′ 2″ *from the fulcrum.* $r=84''$. *Find the load* P *on the specimen when the lever is balanced. Also find the position of the poise weight when the load on the specimen is* 4 *tons.*



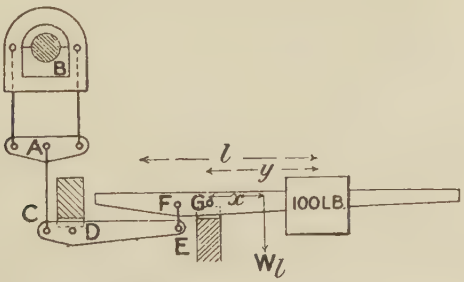
$P=\frac{2\times84+3{\cdot}4\times38}{4}=$ **74·3 tons.** $\quad 4\times4+2\times x=3{\cdot}4\times38$

$x=$ **56·6″** to the left of the fulcrum when the load is 4 tons.

4. *In a friction-testing machine loads are applied to the bearing* B *by the lever mechanism shown. From the data given determine position of jockey weight when the load on bearing is* 15,000 *lb.*

CD=5″ $\quad$ DE=30″ $\quad$ FG=2″

The end lever is balanced when jockey weight is 12″ *to left of knife edge* G.



Let W_l=weight of lever and attached links, the c. of g. of which is distant x from the fulcrum G.

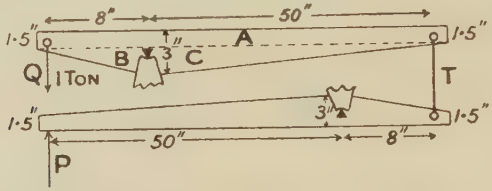
Then $100 \times 12 = W_l x$, when $T_{ef} = 0$, and $100y + W_l x = T_{ef} \times 2$; $100(y+12) = T_{ef} \times 2$.

We also have $T_{ef} \times 30 = 15{,}000 \times 5$ $\quad T_{ef} = 2500$

Hence $100(y+12) = 5000$ $\quad y + 12 = l = 50$ ins.

$y = \mathbf{38}$ **ins.**

5. *In the levers shown, find the force* P *required to balance* 1 *ton at* Q. *Weight of connecting link* 6 *lb. Levers* 1″ *thick. Weight* 0·28 *lb. per cu. in.*



To find c. of g. of levers with regard to fulcrum

Section.	Area.	Arm about fulcrum.	Moment.
A	$58 \times 1{\cdot}5 = 87$	$29 - 8 = 21$	1827
B	$4 \times 1{\cdot}5 = 6$	$-\frac{8}{3}$	−16
C	$25 \times 1{\cdot}5 = 37{\cdot}5$	$\frac{50}{3}$	625
Total . .	130·5		2436

$$x = \frac{2436}{130{\cdot}5} = 18{\cdot}67''$$ to right of fulcrum (top lever).

Weight of beam $= 130{\cdot}5 \times 0{\cdot}28 = 36{\cdot}6$ lb.

Taking moments for top lever :

$2240 \times 8 = 36{\cdot}6 \times 18{\cdot}67 + T \times 50 + 6 \times 50$ $\quad T = 339$ lb.

Also for bottom lever :

$T \times 8 + 18{\cdot}67 \times 36{\cdot}6 = P \times 50$ $\quad P = \mathbf{67{\cdot}9}$ **lb.**

6. *A balanced lever having unequal arms of length* a *and* b *is used to find the weight of a body* B. *When the body is placed in the scale pan at the* a *end a weight of* 14·2 *lb. is required at the* b *end to balance it, and when the body is placed in the scale pan at the* b *end a weight of* 22·3 *lb. is required at the* a *end. Find the weight of* B.

Taking moments about the fulcrum—

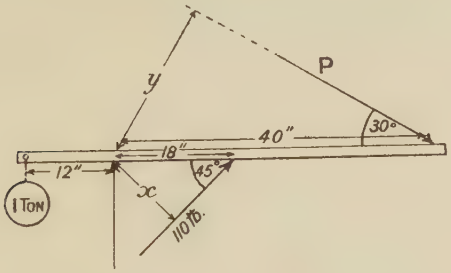
$$B \times a = 14{\cdot}2 \times b \qquad \frac{a}{b} = \frac{14{\cdot}2}{B}$$

and

$$B \times b = 22{\cdot}3 \times a \qquad \frac{a}{b} = \frac{B}{22{\cdot}3}$$

$$B^2 = 22{\cdot}3 \times 14{\cdot}2 \qquad B = \mathbf{17{\cdot}8\ lb.}$$

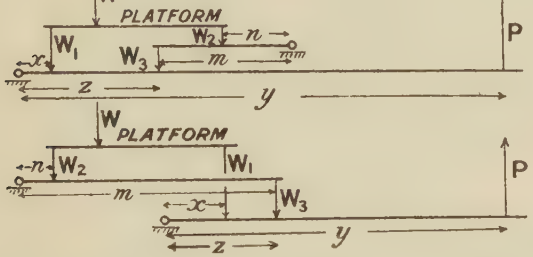
7. *Find the force P required to keep the lever in equilibrium. Neglect weight of lever.*



$x = 18 \sin 45 = 12{\cdot}73''$ $\qquad y = 40 \sin 30 = 20{\cdot}0''$

$$\frac{2240 \times 12 + 110 \times 12{\cdot}73}{20} = P \qquad P = \mathbf{1414\ lb.}$$

8. *The figures show in skeleton the arrangement of the levers of a platform weighing machine, also of a testing*



machine. Show that if the levers be so proportioned that $\frac{n}{m} = \frac{x}{z}$ *the machine will give the same reading for any position of* W *on the platform.*

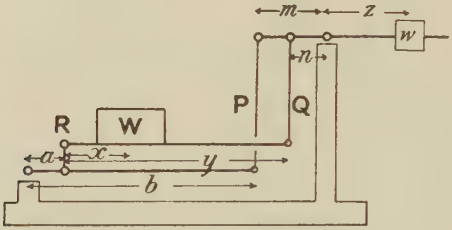
Then $Py = W_1x + W_3z$ $\qquad W_3 = \frac{W_2 n}{m}$

$Py = W_1x + \frac{W_2 n}{m} z$ $\qquad Py = W_1x + \frac{W_2 xz}{z}$

$Py = (W_1 + W_2)x = Wx$

This holds whatever the proportions between W_1 and W_2, consequently P is the same wherever the weight W is placed.

9. *The figure represents the arrangement of the levers in a platform weigh-bridge. Find the necessary proportions that must be adhered to in order that the machine may weigh accurately for any position of* W *on the platform.*



Taking moments about the fulcrum of the top lever,

$$wz = Pm + Qn \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

Taking moments about the platform fulcrum,

$$Wx = Qy \qquad Q = \frac{Wx}{y} \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

Let R be the reaction of the platform fulcrum. Taking moments about the fulcrum of the bottom lever

$Ra = Pb$ also $Rx = Q(y - x)$ $\qquad \frac{Pbx}{a} = Q(y - x)$

$$P = \frac{aQ(y-x)}{bx} = \frac{aW(y-x)}{by} \quad . \quad . \quad . \quad \text{(iii)}$$

Substituting the value of P and Q in (i)

$$w = \frac{aWm(y-x)}{byz} + \frac{Wxn}{yz} \qquad \frac{w}{W} = \frac{1}{yz}\left(\frac{amy - amx + bxn}{b}\right)$$

If we make $\frac{a}{b} = \frac{n}{m}$ the last two terms cancel, and the

velocity ratio $\frac{w}{W}$ is independent of x, *i.e.* the position of W on the platform.

Then $$\frac{w}{W} = \frac{amy}{byz} = \frac{am}{bz} = \frac{n}{z}$$

The arrangement is not very convenient from a practical point of view, since the distance n has to be made very small to secure a large velocity ratio.

10. *Find the position of the centre of gravity of a plate* 4 *ft.* × 2 *ft.* 6 *ins., with a hole* 9 *ins. diameter, whose centre is* 8 *ins. from the long edge of the plate and* 13 *ins. from the short edge.*

Taking moments about a long edge near hole

$$4 \times 2{\cdot}5 \times 1{\cdot}25 - 0{\cdot}785 \times \frac{9^2}{144} \times \frac{8}{12} = 12{\cdot}21$$

Area $= 10 - 0{\cdot}442 = 9{\cdot}56$ sq. ft.

Distance of c. of g. from long edge near hole $= \frac{12{\cdot}21}{9{\cdot}56} =$ **1·28 ft.**

Taking moments about short edge near hole

$$4 \times 2{\cdot}5 \times 2 - 0{\cdot}785 \times \frac{9^2}{144} \times \frac{13}{12} = 19{\cdot}52 \qquad \text{Area } 9{\cdot}56 \text{ sq. ft.}$$

Distance of c. of g. from short edge near hole $\frac{19{\cdot}52}{9{\cdot}56} =$ **2·04 ft.**

11. *A plate rests on three supports* A, B, C. *The distance* AB *is* 4 *feet,* AC 5 *feet, and* BC 2·5 *feet; the centre of gravity is at* D, *which is* 2 *feet* 3 *inches from both* A *and* B; *the plate weighs* 3 *tons. A concentrated weight of* 4 *tons is placed at* E, *which is* 1·5 *feet from* B *and* 1·7 *feet from* C. *Find the load on each support.*

Join AB and taking moments about the line—

$3 \times 1{\cdot}03 + 4 \times 1{\cdot}26 = C \times 2{\cdot}48$ $\quad C = 3{\cdot}3$ tons.

Join BC and taking moments about the line—

$3 \times 2{\cdot}12 + 4 \times 0{\cdot}98 = A \times 3{\cdot}97$ $\quad A = 2{\cdot}6$ tons.

Join AC. $\quad 3 \times 0{\cdot}1 + 4 \times 0{\cdot}5 = 2B$ $\quad B = 1{\cdot}15$ tons.

Total 7·05 tons instead of 7 tons, the error is due to the distances not being taken to three places of decimals. More exact values are—

C = 3·29 tons, $\quad$ A = 2·59 tons, $\quad$ B = 1·12 tons.

12. *The loads on the three axles* A, B, C *of a locomotive are* 8, 12, 10 *tons respectively. The distance between* A *and* B *is* 5′ 6″, *and between* B *and* C 3′ 6″. *If the engine runs over a gap so that there is no load on* B, *what will be the load on* A *and* C?

If the axle B be jacked up with a force of 12 tons when over the gap the axles A and C will support the same load as before, viz. 8 and 10 tons respectively. Hence when the jack is removed the loads on A and C will be due to a downwards force of 12 tons at B.

Take moments about A—

$$W_c \times (5{\cdot}5+3{\cdot}5) = 12 \times 5{\cdot}5 \qquad W_c = 7\tfrac{1}{3} \text{ tons.}$$

$$W_A = 12 - 7\tfrac{1}{3} = 4\tfrac{2}{3} \text{ tons.}$$

Total load on axle A $= 8 + 4\tfrac{2}{3} = \mathbf{12\tfrac{2}{3}}$ **tons.**

„ „ C $= 10 + 7\tfrac{1}{3} = \mathbf{17\tfrac{1}{3}}$ **tons.**

13. *Find the height of the c. of g. of a wire triangle* abc *above the base* ab. ab = 2·1″, ac = 1·2″, bc = 1·5″.

See Fig. 19, Vol. I.

Area enclosed $= \sqrt{2{\cdot}4(0{\cdot}3 \times 1{\cdot}2 \times 0{\cdot}9)} = 0{\cdot}882$ sq. in.

Height of the triangle $h = \dfrac{2 \times 0{\cdot}882}{2{\cdot}1} = 0{\cdot}84''$

Take moments about *ab* $4{\cdot}8h = (1{\cdot}2+1{\cdot}5)\dfrac{0{\cdot}84}{2}$

Height of c. of g. above the base is **0·236″.**

14. *Find the position of the c. of g. of Fig.* 87, *Vol. I., above the line* AB.

$w = 0{\cdot}15''$

$$Y = \frac{0{\cdot}15}{2}\left(\frac{0{\cdot}5+3\times0{\cdot}92+5\times1{\cdot}1+7\times1{\cdot}06+9\times0{\cdot}93+11}{0{\cdot}5+0{\cdot}92+1{\cdot}1+1{\cdot}06+0{\cdot}93+0{\cdot}85}\right.$$

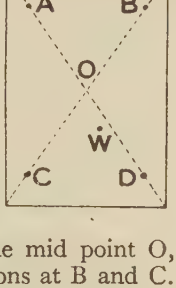
$$\left.\frac{\times0{\cdot}85+13\times0{\cdot}78+15\times0{\cdot}55}{0{\cdot}78+0{\cdot}55}\right)$$

$$Y = 0{\cdot}075 \times \frac{52{\cdot}29}{6{\cdot}69} = 0{\cdot}586''$$ from the line AB.

15. *The body of a motor lorry is supported symmetrically on four similar springs at the corners of a rectangle* 9 *ft. by* 5 *ft. Find the load taken by each spring when a casting weighing* 3 *tons is being carried with its centre of gravity*

6 ins. right and 18 ins. in rear of the centre of the body. The body itself weighs 6 cwt.

Let P_A, P_B, P_C, P_D be the loads on the four springs, due to the casting alone.



Take moments about the horizontal axis AC :

$$(P_B+P_D)5=W\times 3.$$

So for moments about AB :—

$$(P_C+P_D)9=W\times 6.$$

The total load $W=P_A+P_B+P_C+P_D$.

Also, assuming the body to be rigid, it will sink under its load in such a way that the mean of the deflections at A and D, which is the same as the deflection at the mid point O, will be the same as the mean of the deflections at B and C. But since the deflections are proportional to the loads carried, the springs being similar, it follows that :

$$\frac{P_A+P_D}{2}=\frac{P_B+P_C}{2} \qquad P_B=\frac{3W}{5}-P_D \qquad P_C=\frac{6W}{9}-P_D$$

$$W=2P_B+2P_C=\frac{6W}{5}+\frac{12W}{9}-4P_D$$

$$P_D=\frac{23W}{60}=23 \text{ cwt.} \quad \text{Similarly } P_C=\frac{17W}{60}=17 \text{ cwt.}$$

$$P_B=\frac{13W}{60}=13 \text{ cwt.} \qquad P_A=\frac{7W}{60}=7 \text{ cwt.}$$

The load due to the body itself is equally distributed, hence the total load on each spring will be :

D=24·5 cwt. C=18·5 cwt. B=14·5 cwt. A=8·5 cwt.

16. *A pole 40 ft. long, 4″ diam. at one end and 9″ at the other, is carried on cross bars by four men in pairs. The one pair carries the 9″ end, at what distance from this end should the other pair carry in order that the load shall be equally distributed ?*

See Fig. 90, Vol. I. Taper 1″ in 8 ft. or 4″ in 32 ft.

Dist. of c. of g. from apex $\left.\begin{matrix}\\ \end{matrix}\right\}=\frac{3}{4}(40+32)\left(\frac{1-(\frac{4}{9})^4}{1-(\frac{4}{9})^3}\right)=54(\frac{6305}{5985})=56\cdot9$ ft.

or $72-56\cdot9=15\cdot1$ ft. from butt end.

Hence the other point of support should be at $2\times 15\cdot1$ **=30·2 ft.** from butt end.

17. *A trapezoidal wall* 14 *feet thick at the bottom and* 5 *feet thick at the top, height* 40 *feet, is subjected to water pressure on its vertical face; find the height to which the water may rise before the wall will topple over. For the purposes of this question you may neglect the tendency to slide and assume that no water gets under the footing. The material in the wall weighs* 130 *lb. per cubic foot.*

Height of c. of g. above ground $=\frac{40}{3}\left(\frac{14+10}{14+5}\right)=16{\cdot}8$ ft.

Centre line of wall slopes $\frac{9}{2}$ in 40 or 1 in 8·89.

Horizontal dist. of c. of g. from heel $=7+\frac{16{\cdot}8}{8{\cdot}89}=8{\cdot}89$ ft.

Weight of wall $=\frac{14+5}{2}\times 40\times 130=49{,}400$ lb. per ft. run.

The resultant water pressure per foot run on a vertical wall is $\frac{wH^2}{2}$, acting at a distance $\frac{H}{3}$ above the base, w being the weight of a cubic foot of water and H the height of the surface above the base of the wall in feet. (Vol. I. Chap. XVIII.)

Taking moments about the heel of the wall, toppling will take place when H is such that

$$\frac{wH^2}{2}\times\frac{H}{3}=494{,}000\times 8{\cdot}89, \qquad \text{where } w=62{\cdot}4.$$

$$H^3=42{,}200 \qquad H=\mathbf{34{\cdot}8\ ft.}$$

18. *A ball* $\frac{3}{4}''$ *diameter rests in a hole in a flat surface, the surface is tilted until the ball just rolls out of the hole. Find the diam. of the hole. Angle of surface to horizontal* 28° 15′.

The ball rolls when its c. of g. is just vertically over the edge of hole. Let D=diam. of ball. d=diam. of hole.

Then $\frac{d}{2}=\frac{D}{2}\sin 28°\ 15'$. $\qquad d=\frac{3}{4}\times 0{\cdot}4733=\mathbf{0{\cdot}355''}$.

19. *A rod is bent to the arc of a circle of* 6 *feet diameter. Find the position of the c. of g. when it subtends* (*a*) 30°; (*b*) 120°; (*c*) 270°; (*d*) 300°.

In all cases we have the relation:

$$\frac{\text{Arc}}{\text{Radius}} = \frac{\text{Chord of arc}}{\text{Distance of c. of g. from centre}}$$

$$\frac{A}{R} = \frac{C}{Y} \quad \text{and} \quad Y = \frac{CR}{A}$$

(*a*) $\frac{C}{2} = 3 \sin 15$ $\quad C = 1{\cdot}55'$ $\quad A = 1{\cdot}57'$

$Y = \frac{1{\cdot}55 \times 3}{1{\cdot}57} = \mathbf{2{\cdot}96'}$ from the diameter parallel to the chord, and on a line perpendicular to it.

(*b*) $C = 5{\cdot}2'$ $\quad A = 6{\cdot}28'$ $\quad Y = \frac{5{\cdot}2 \times 3}{6{\cdot}28} = \mathbf{2{\cdot}48'}$

(*c*) $C = 4{\cdot}24'$ $\quad A = 14{\cdot}1'$ $\quad Y = \frac{4{\cdot}24 \times 3}{14{\cdot}1} = \mathbf{0{\cdot}90'}$

(*d*) $C = 3{\cdot}0'$ $\quad A = 15{\cdot}7'$ $\quad Y = \frac{3 \times 3}{15{\cdot}7} = \mathbf{0{\cdot}57'}$

Alternative solution $Y = R \frac{\sin \frac{\alpha}{2}}{\frac{\alpha}{2}}$. Where α is the angle subtended.

(*a*) $Y = 3 \frac{\sin 15^\circ}{15 \times \frac{\pi}{180}} = \mathbf{2{\cdot}96'}$ $\quad$ (*b*) $Y = 3 \frac{\sin 60^\circ}{\frac{\pi}{3}} = \mathbf{2{\cdot}48'}$

(*c*) $Y = 3 \frac{\sin 135^\circ}{\frac{3\pi}{4}} = \mathbf{0{\cdot}90'}$ $\quad$ (*d*) $Y = 3 \frac{\sin 150^\circ}{\frac{5\pi}{6}} = \mathbf{0{\cdot}57'}$.

20. *A sphere* 8 *ins. diameter has a flat bottomed hole drilled in it* 2·1 *ins. diameter and* 3 *ins. deep. Find the distance of the centre of gravity from the centre of the sphere. The material weighs* 0·26 *lb. per cu. in. Find the weight of the body.*

Vol. of sphere $= \frac{\pi d^3}{6} = \frac{3{\cdot}14 \times 512}{6} = 268$ cu. ins.

Vol. of hole $= 0{\cdot}785 \times 2{\cdot}1^2 \times 3 = 10{\cdot}4$ cu. ins.

Vol. of spherical cap removed by hole $= 0{\cdot}23$ cu. in.

Height of cap $= 0{\cdot}13$ in.

Taking moments about a point on the surface of the sphere on a line passing through the middle of the hole:

$268\times4-10{\cdot}4\times(1{\cdot}5+0{\cdot}13)=(268-10{\cdot}4)x \qquad x=4{\cdot}096$

The c. of g. is 0·096″ from the centre of the sphere.

Weight $=257{\cdot}6\times0{\cdot}26=$ **67 lb.**

21. *A shaft 20″ long overall has a conical point 6″ long. Find the distance of the c. of g. from the apex of the cone.*

Volume of shaft $=\dfrac{14\pi d^2}{4}$ Volume of cone $=\frac{1}{3}\times\dfrac{6\pi d^2}{4}$

C. of g. of cone is $\frac{3}{4}h$ from apex $=4{\cdot}5''$.

C. of g. of shaft $=7+6=13''$ from apex.

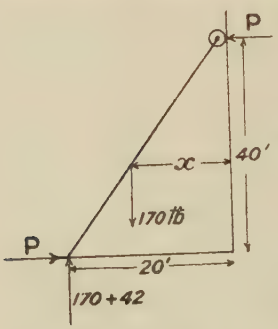
Taking moments about the apex:

$$\frac{13\times14\pi d^2}{4}+\frac{4{\cdot}5\times6\pi d^2}{3\times4}=191\frac{\pi d^2}{4}$$

$$\text{Volume}=\frac{\pi d^2}{4}(14+2)=16\frac{\pi d^2}{4}$$

Distance of c. of g. from apex $=\frac{191}{16}=$ **11·94 ins.**

22. *A ladder fitted with ball-bearing castors at the top which rests against a smooth vertical wall at a height of 40 ft. from the ground is placed with the foot of the ladder 20 ft. horizontally from the base of the wall. The weight of the ladder is 42 lb. Find the vertical height to which a man weighing 170 lb. may ascend before the ladder begins to slip.*



Coefficient of friction between ladder and ground 0·25.

The c. of g. of ladder is $\frac{2}{5}$ height from the ground.

Taking moments about the foot of the ladder:

$$P\times40=42\times\tfrac{2}{5}\times20+170(20-x)$$

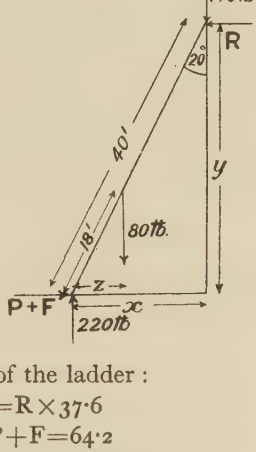
$$P=(170+42)0{\cdot}25=53 \text{ lb.}$$

$$53\times40=336+3400-170x.$$

$x=9{\cdot}5$ ft. or **21·0 ft. above the ground.**

23. *A ladder 40 feet long weighing 80 lb. is placed with its foot on a horizontal floor. The top of the ladder rests against a vertical wall with which it makes an angle of 20°. The c. of g. of the ladder is 18 feet from the foot. A man weighing 140 lb. ascends to the top of the ladder. Find the horizontal force which must be exerted on the foot of the ladder towards the wall to prevent slipping.*

Coefficient of friction at the ground 0·22, the friction on the wall being neglected.



$x = 40 \sin 20°$

$= 40 \times 0{\cdot}342 = 13{\cdot}68'.$

$y = 40 \cos 20°$

$= 40 \times 0{\cdot}94 = 37{\cdot}6$

$z = 18 \times 0{\cdot}342 = 6{\cdot}16'$

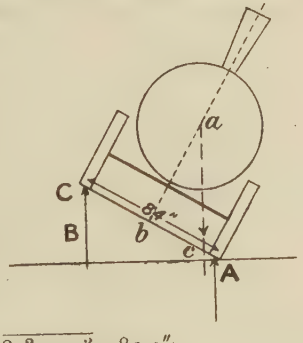
Taking moments about the foot of the ladder:

$$80 \times 6{\cdot}16 + 140 \times 13{\cdot}68 = R \times 37{\cdot}6$$

$$R = 64{\cdot}2 \text{ lb.} \qquad P + F = 64{\cdot}2$$

$$P + 0{\cdot}22 \times 220 = 64{\cdot}2 \qquad P = \mathbf{15{\cdot}8 \text{ lb.}}$$

24. *An engine is run on to a weighbridge and is found to weigh 12 tons. Some balks of timber are then placed beside the weighbridge and two of the side-wheels of the engine are run on to the balks and two on to the platform of the weighbridge, which then records a load of 8 tons. The balks of timber raise the wheels 13 inches. The wheel centres are 7 feet apart. From the above-mentioned data find the height of the centre of gravity of the engine above the ground when standing on a level plane.*



When tilted the base $AB = \sqrt{84^2 - 13^2} = 83{\cdot}0''$.

A perpendicular through the c. of g. cuts AC in the ratio 4 to 8, hence

$$Ac = \tfrac{84}{3} = 28'' \quad \text{and} \quad bc = 14''$$

By similar triangles $\frac{ab}{bc}=\frac{AB}{BC}=\frac{83}{13}$

Height of c. of g.: $ab=\frac{14\times 83}{13}=$ **89·4 ins.**

25. *Find the moment of inertia of a channel section about a line passing through the c. of g. and parallel to the web of the channel.*

Breadth of channel, 8″; *depth,* 4″; *thickness throughout,* 1″.

To find c. of g. take moments about bottom edge.

Section.	Area.	y	Moment.
Two flanges . .	8 sq. in.	2″	16
Web	6	0·5	3
Total . . .	14	—	19

I for flanges above line $=\frac{2\times 2\cdot 643^3}{3}=12\cdot 32$. $y=\frac{19}{14}=1\cdot 357''$

I below axis = rectangle 8″ × 1·357″ − Rectangle 6″ × 0·357

$$=\frac{8\times 1\cdot 357^3}{3}-\frac{6\times 0\cdot 357^3}{3}=6\cdot 55$$

Total I for section = **18·9 inch units.**

26. *Compare the moment of inertia of a hollow concentric circle with one having an eccentric inner circle, both about a line passing through the c. of g. of section and normal to a line joining the two centres. External diam.* 8″, *internal* 4″, *eccentricity* 1·5″.

Concentric: $I=\frac{\pi}{64}(D_e^4-D_i^4)=\frac{3\cdot 14}{64}(4096-256)=188\cdot 5$

Eccentric: $$I=\frac{\pi}{64}\left(D_e^4-D_i^4-\frac{16D_e^2D_i^2x^2}{D_e^2-D_i^2}\right)$$

$$=\frac{3\cdot 14}{64}(4096-256-(\tfrac{3686\cdot 4}{48})=\mathbf{150\cdot 8}$$

27. *Find the moment of inertia of the figure shown in Fig.* 115, *Vol. I.*

$$I_0=\frac{w^3}{4}(b_1+9b_2+25b_3+\text{etc.})$$

$$=\frac{0{\cdot}14^3}{4}(0{\cdot}35+9\times0{\cdot}6+25\times0{\cdot}91+49\times0{\cdot}99+81\times0{\cdot}95$$

$$+121\times0{\cdot}84+169\times0{\cdot}57)$$

$$=0{\cdot}000686\times351{\cdot}92=\mathbf{0{\cdot}241\ inch^4\ units.}$$

As a rough check, assume that the figure is a circle 1″ diam. about an axis touching the circumference.

$$I_0=I+AR^2$$

$$I_0=\frac{\pi d^4}{64}+\frac{\pi d^2}{4}\times\frac{d^2}{4}=\frac{5\pi d^4}{64}=\frac{5\times3{\cdot}14}{64}=\mathbf{0{\cdot}245}$$

28. *Find the polar moment of inertia of a solid disc flywheel* 10″ *diameter,* 3″ *thick. Weight of metal* 0·26 *lb. per cubic inch.* (*See Fig.* 121, *Vol. I.*)

The polar moment of inertia is $I_p=\frac{\pi R^4HW}{2}$

$$=\frac{3{\cdot}14}{2}\left(\tfrac{\pi}{12}\right)^4\times\tfrac{3}{12}\times0{\cdot}26\times12^3=\mathbf{5{\cdot}31}\text{ pounds-feet units.}$$

Let it rotate at 200 r.p.m. Find the energy stored.

200 r.p.m.$=200\times2\pi=1256{\cdot}6$ rads. p.m.$=20{\cdot}94$ rads. p.s.

Energy stored$=\frac{1}{2}\cdot\frac{I_p}{g}\omega^2=\frac{1}{2}\cdot\frac{5{\cdot}31}{32{\cdot}2}\times20{\cdot}94^2=\mathbf{36{\cdot}2\ foot\text{-}lb.}$

or the energy stored may be expressed as $\frac{W_1V^2}{2g}$, where

W_1=wt. of wheel in lb.

V=vely. of a point at the radius of gyration in f.p.s.

$W_1=3{\cdot}14\times5^2\times3\times0{\cdot}26=61{\cdot}25$ lb.

$$V=2\times3{\cdot}14\times\frac{5}{12\sqrt{2}}\times\tfrac{200}{60}=6{\cdot}17\text{ f.p.s.}$$

Energy stored$=\frac{61{\cdot}25\times6{\cdot}17^2}{64{\cdot}4}=\mathbf{36{\cdot}2\ foot\text{-}lb.}$ as before.

29. *Find the moment of inertia of a cylinder* (a) *about its axis ;* (b) *about a diameter on the end plane of the cylinder ;* (c) *about a line parallel to* b *but passing through its centre of gravity.*

Length (L) $18''=1{\cdot}5'$. *Diameter* (D) $4''=0{\cdot}33'$.
Weight per cu. in. (W) $0{\cdot}28$ *lb.*$=484$ *lb. per cu. ft.*

For all energy problems the moment of inertia for convenience should be expressed in feet-pound units.

(*a*) $I'_p=\text{Weight}\left(\frac{R^2}{2}\right)=\frac{484\times3{\cdot}14\times1{\cdot}5}{9\times4}\left(\frac{1}{2\times6^2}\right)$

$=63{\cdot}4\times\frac{1}{72}=\mathbf{0{\cdot}88}$ feet-pound units.

(*b*) $I'_0=\text{Weight}\left(\frac{4L^2+3R^2}{12}\right)=63{\cdot}4\left(\frac{4\times2{\cdot}25+\frac{3}{36}}{12}\right)=\mathbf{48}$

(*c*) $I'=\text{Weight}\left(\frac{L^2+3R^2}{12}\right)=63{\cdot}4\left(\frac{2{\cdot}25+\frac{3}{36}}{12}\right)=\mathbf{12{\cdot}3}$

The energy stored in a rotating body is $\frac{1}{2}\cdot\frac{I_p}{g}\omega^2$.

30. *A solid roller rolls down an inclined plane for a distance (measured on the incline) of* 40 *ft. Find its velocity assuming that there is no slip. Angle of plane to horizontal* $12\frac{1}{2}°$.

The vertical height the roller has fallen$=40\sin 12{\cdot}5°$
$=40\times0{\cdot}216=8{\cdot}64$ ft.

$$\left.\begin{matrix}\text{Work done in falling}\\ \text{vertically}\end{matrix}\right\}=\left\{\begin{matrix}\text{Energy of rotation}\\ \text{of roller}\end{matrix}\right\}+\left\{\begin{matrix}\text{Energy of}\\ \text{translation}\end{matrix}\right.$$

$$W\times8{\cdot}64=\frac{WV_r^2}{2g}+\frac{WV^2}{2g}$$

Where V_r is the velocity of a point at the radius of gyration,

$$\frac{V_r^2}{V^2}=\frac{R^2}{2R^2}\qquad V_r^2=\frac{V^2}{2}$$

$$64{\cdot}4\times8{\cdot}64=\frac{V^2}{2}+V^2=\frac{3V^2}{2}$$

$$V^2=\frac{64{\cdot}4\times8{\cdot}64\times2}{3}\qquad V=\mathbf{19{\cdot}2\ f.p.s.}$$

The average velocity is $\frac{19{\cdot}2}{2}=9{\cdot}6$ f.p.s.

hence it will take $\frac{40}{9{\cdot}6}=4{\cdot}17$ secs. to traverse $40'$.

The acceleration is $\frac{19{\cdot}2}{4{\cdot}17}=4{\cdot}6$ f.p.s.p.s.

The energy of translation of the roller at the end of its

run is 5·76W foot-pounds, and of rotation 2·88W foot-pounds, making 8·64W foot-pounds total.

31. *A pair of wheels weighing* 800 *lb. are attached to an axle* 4 *ins. diameter, they roll down a plane inclined at* 1 *in* 10 *with the axle in contact with the plane at the rate of* 10 *ft. in* 14 *secs. Assuming no slip between the axle and the plane find the radius of gyration, neglecting rolling resistance.*

Mean linear velocity $=\frac{10}{14}$ f.p.s. Final $=\frac{10}{7}$ f.p.s.
Final angular velocity $=6\times\frac{10}{7}$ radians.

$$\text{Total kinetic energy}=\tfrac{1}{2}\times\frac{800}{g}\{(\tfrac{10}{7})^2+K^2(\tfrac{60}{7})^2\}$$

$$800\times10\times\tfrac{1}{10}=\tfrac{1}{2}\times\frac{800}{g}\times\tfrac{100}{49}(1+36K^2)$$

$$1+36K^2=\frac{2\times32\cdot2\times49}{100}=31\cdot5$$

$$K^2=\frac{30\cdot5}{36}\qquad K=0\cdot92\text{ foot}=\textbf{11 ins.}\text{ nearly.}$$

32. *An impact testing-machine consists of a disc provided with a tooth on the periphery. When in use the disc is rotated at a high speed, and the notched test specimen, which is held rigidly at one end, is brought into such a position that the tooth strikes it and breaks it off short. Find the work done in fracturing the test specimen in the following instance :*

Diameter of disc, 30″*; thickness,* 3″*; initial speed of disc,* 1000 *r.p.m.; final speed,* 892 *r.p.m.; density of disc,* 0·27 *lb. per cu. in.*

Weight of disk $=0\cdot785\times30^2\times3\times0\cdot27=572$ lb.

$$\text{Kinetic energy}=\frac{WV^2}{2g},$$

where V is the velocity of a point at the radius of gyration.

$$\left.\begin{matrix}\text{Kinetic energy}\\(\text{initial})\end{matrix}\right\}=\frac{572\times4\times\pi^2\times1\cdot25^2\times1000^2}{64\cdot4\times2\times60^2\times2}=76{,}000$$

$$\left.\begin{matrix}\text{Kinetic energy}\\(\text{final})\end{matrix}\right\}=\frac{76000\times892^2}{1000^2}=60{,}500\text{ ft.-lb.}$$

Difference = Work done = **15,500 ft.-lb.**

Or it may be expressed thus—

$$\left.\begin{matrix}\text{Kinetic energy}\\(\text{initial})\end{matrix}\right\}=\tfrac{1}{2}\cdot\frac{W}{g}K^2\omega^2=\frac{572}{64\cdot4}\times\frac{1\cdot25^2}{2}\times\frac{1000^2\times4\pi^2}{60^2}$$

$=76{,}000$ foot-lb. (final) $=60{,}500$ foot-lb.

CHAPTER IV

RESOLUTION OF FORCES

1. *A rope* ABCD, *whose weight may be neglected, is suspended from the points* A *and* D *and supports weights at* B *and* C. *The inclination of* AB *to the horizontal is* 45°, BC *is level, and* CD *is* 60°. *The weight at* B *is* 10 *tons, find the weight required at* C *and the tension in the three portions of the rope.*

AB$=10\sqrt{2}=$**14·18 tons.** BC$=$**10 tons.**

CD$=$**20 tons.** $W_c=10\sqrt{3}=$**17·3 tons.**

2. *Calculate the maximum safe span for an overhead wire having a sag of* 2 *feet in the direction of the resultant force, diameter of wire* 0·12″, *wind pressure* 20 *lb. per square foot, coefficient of wind resistance* 0·6. *Tensile strength of wire* 100,000 *lb. per square inch. Factor of safety* 5. *Weight of steel* 0·28 *lb. per cubic inch.*

Wt. per ft. run$=0{\cdot}12^2\times0{\cdot}785\times0{\cdot}28\times12=0{\cdot}038$ lb.

Tension due to weight of wire$=\dfrac{wl^2}{8S}=\dfrac{0{\cdot}038l^2}{8\times2}=0{\cdot}0024l^2$.

Wind resistance per ft. run$=\dfrac{0{\cdot}12\times20\times0{\cdot}6}{12}=0{\cdot}12$ lb.

Resultant force$=\sqrt{0{\cdot}12^2+0{\cdot}038^2}=0{\cdot}126$ lb.

Tension due to wt. of wire and wind $\Big\}=\dfrac{0{\cdot}126l^2}{8\times2\times0{\cdot}12^2\times0{\cdot}785}=\dfrac{100000}{5}$

$l=$**169 ft.**

3. *A corner telegraph pole supports two wires at right angles, weight of each is* 0·055 *lb. per foot run. Distance to next pole on each line of wire* 240 *ft. Find the resultant horizontal force acting on the pole in the plane of the wires. Sag of wires* 2·1 *ft.*

$$\text{Horizontal pull} = \frac{wl^2}{8S} = \frac{0{\cdot}055 \times 240^2}{8 \times 2{\cdot}1} = 189 \text{ lb.}$$

$$\text{Resultant} = 189\sqrt{2} = \mathbf{266 \text{ lb.}}$$

4. *A chain weighing* 18 *lb. per foot is attached to the tops of three posts* A, B *and* C *all on the same level and in one line. The span* AB *is* 48 *feet and the dip of the chain is* 5 *feet* 9 *ins., the span* BC *is* 39 *feet and the dip is* 5 *feet. Find the bending moment at the foot of the central post which is* 20 *feet high. You may assume that the chain hangs in a parabolic arc.*

$$\text{Pull in AB} = \frac{18 \times 48^2}{8 \times 5{\cdot}75} = 902 \text{ lb.}$$

$$\text{Pull in BC} = \frac{18 \times 39^2}{8 \times 5} = 684 \text{ lb.}$$

$$\text{Difference} \quad . \quad . \quad . \quad 218 \text{ lb.}$$

$$\text{Bending moment} = 218 \times 20 = \mathbf{4{,}360 \text{ lb.-ft.}}$$

5. The Sag and Tension in Suspended Wire Ropes

The method given below is based on an exact analytical method due to Mr. A. J. V. Umanski, B.Sc., Selected Paper 4,488, Institution of Civil Engineers, but his treatment requires a higher mathematical knowledge to follow than is possessed by most engineers. The following method will be found to give results sufficiently accurate for all practical purposes.

In all cases it is assumed that (i) the sag is small, say not more than one-fifth of the span; (ii) the weight of the rope itself is not great compared with that of the suspended load, say not more than $\frac{1}{20}$th. If the ratio is greater than this a graphical solution is more accurate than the one given below.

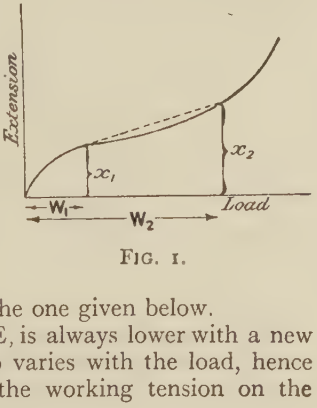


Fig. 1.

The Modulus of Elasticity, E, is always lower with a new than with an old rope. It also varies with the load, hence the value corresponding with the working tension on the rope should be taken.

The load W_1 is the load in pounds on the rope when hanging freely under its own weight, and W_2 is the load on the rope when an external load is placed on the rope. As a rule the load-extension line will not be straight, a chord should then be drawn as shown. The extensions x_1 and x_2 are to be measured in inches from the chord. Then the Modulus of Elasticity, E, in pounds per sq. inch referred to above is

$$E=\frac{(W_2-W_1)l_r}{A(x_2-x_1)} \quad . \quad . \quad . \quad . \quad . \quad (i)$$

Where l_r=the length of rope in ins. on which the extensions were measured in the testing machine.

Where A=sectional area of the wires in the rope in sq. ins.

Freely hanging Wire Rope.—Points of suspension A and B on the same level.

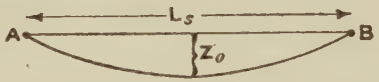


FIG. 2.

T_o=the tension in the rope in pounds.
L_s=length of span in feet.
L_r=length of rope in feet when suspended.
w=weight of one foot of rope in pounds.

The stretch of the rope when freely suspended $=\frac{w^2L_s^3}{24T_0^2}$ (ii)

$$L_r=L_s+\frac{w^2L_s^3}{24T_0^2}=L_s+\tfrac{8}{3}\cdot\frac{Z_0^2}{L_s}$$

$$T_0=\frac{wL_s^2}{8Z_0} \quad (iii) \qquad Z_0=\frac{wL_s^2}{8T_0} \quad (iv)$$

Example.—Let L_s=100 feet, w=1·5 lb. per foot run.

If Z_0=5 feet then $T_0=\frac{1{\cdot}5\times100^2}{8\times5}=375$ lb.

If T_0=1 ton=2,240 lb. then $Z_0=\frac{1{\cdot}5\times100^2}{8\times2240}=0{\cdot}84$ feet.

Length of rope when under a tension of 1 ton $=\frac{1{\cdot}5^2\times100^3}{24\times2240}+100=100{\cdot}019$ ft.

Thus the stretch of the rope is 0·019 foot or 0·228 inch, usually a negligible quantity.

Wire Rope supporting a Single Load.—The points of suspension on the same level.

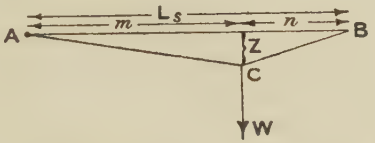


FIG. 3.

Let T=maximum tension in the rope due to its own weight and the suspended weight in pounds.
Z=sag of rope in feet underneath the load.

The excess of the chords AB and AC over the span AB in feet when the rope is supporting the load and its own weight is

$$L_r = L_s + \frac{Z^2}{2}\left(\frac{1}{m} + \frac{1}{n}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(v)}$$

The stretch of the rope due to the additional load is

$$(T - T_0)\frac{L_s}{AE} \quad . \quad . \quad . \quad . \quad . \quad \text{(vi)}$$

The excess of rope over span before applying load $= \frac{w^2 L_s^3}{24 T_0^2}$ (see equation (ii)).

Hence if the load is sufficient to make the curvature of the rope negligible equation (v) represents also the excess of rope over span when loaded, and we can equate the two values of this excess thus :

$$(T - T_0)\frac{L_s}{AE} + \frac{w^2 L_s^3}{24 T_0^2} = \frac{Z^2}{2}\left(\frac{1}{m} + \frac{1}{n}\right) \quad . \quad . \quad \text{(vii)}$$

In any given case T and Z are unknown.

The vertical component of the tension at B due to the weight W is $\frac{Wm}{L_s}$.

The tension T_1 in the rope due to $W = \frac{Wmn}{L_s Z}$

or

$$T_1 Z = \frac{Wmn}{L_s} \quad . \quad . \quad . \quad . \quad . \quad \text{(viii)}$$

or allowing for the weight of the rope (approximately)

$$TZ=\left(W+\frac{wL_s}{2}\right)\frac{mn}{L_s} \quad . \quad . \quad . \quad . \quad (ix)$$

All the quantities in the right of the equation are known, but T and Z are not known, so assume values for T and calculate Z to correspond. We can determine which values satisfy the equation (vii) by tabulating the various quantities.

It will probably be better to take an actual example. The following data were obtained by experiment by Mr. Umanski on a wire rope supplied by Mr. Brunton of Musselburgh :—

$T_0=155$ lb. $Z_0=1{\cdot}76$ ft.
$L_s=55$ ft. $m=18{\cdot}3$ ft.
$n=36{\cdot}7$ ft. $W=896$ lb.
$w=0{\cdot}72$ lb. per foot run. $A=0{\cdot}196$ sq. inch.
$E=6{,}000{,}000$ lb. per sq. inch at the mean load.

TABLE I.

1.	2.	3.	4.
T.	Z.	$(T-T_0)\frac{L_s}{AE}$	$\frac{Z^2}{2}\left(\frac{1}{m}+\frac{1}{n}\right)-\frac{w^2L_s^3}{24T_0^2}$
lb.	Feet.	Feet.	Feet.
1,000	11·2	0·0395	4·96
3,000	3·73	0·133	0·420
4,000	2·80	0·180	0·170
3,900	2·87	0·175	0·188
3,960	2·83	0·178	0·178

There is no necessity to work out the whole table. Start with $T=1{,}000$ lb. Then $Z=11{\cdot}2$; but this is obviously too low, since column 3 must equal column 4.

Try $T=3{,}000$ lb. This is too low since col. 3 is less than col. 4.

$T=4{,}000$ lb. This is too high ; col. 3 is greater than col. 4.

$T=3{,}900$ lb. Too low, but nearly right.

$T=3{,}960$ lb. Right.

Thus the tension in the rope is 3,960 lb. or 20,200 lb. per sq. inch on the wires. The sag beneath the load is 2·83 feet. Experiment gave $T=4{,}340$ lb. and the sag 2·59 feet. The agreement is as good as can be expected

owing to the uncertainty of the exact value of the modulus of elasticity.

For these experimental results—

$$\frac{Z^2}{z}\left(\frac{1}{m}+\frac{1}{n}\right)=0{\cdot}274 \qquad \frac{w^2L_s{}^3}{24T_0{}^2}=0{\cdot}149 \qquad \text{Column } 4=0{\cdot}125$$

$$\therefore E=(T-T_0)\frac{L_s}{0{\cdot}125A}=\frac{4185\times 55}{0{\cdot}125\times 0{\cdot}196}=9{,}400{,}000 \text{ lb. per sq. in.}$$

This is the value of E required to satisfy the experiment.

Rope Loaded with more than One Load.—Both points of support on same level.

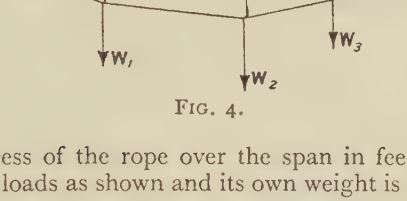


Fig. 4.

The excess of the rope over the span in feet when supporting the loads as shown and its own weight is

$$\tfrac{1}{2}\left\{\frac{Z_1{}^2}{m_1}+\frac{(Z_2-Z_1)^2}{m_2}+\frac{(Z_3-Z_2)^2}{m_3}+\frac{Z_3{}^2}{m_4}\right\} \quad . \quad \text{(x)}$$

The stretch of the rope due to the additional loading is as before

$$(T-T_0)\frac{L_s}{AE} \text{ (also see equation (viii))} \quad . \quad . \quad \text{(xi)}$$

$$(T-T_0)\frac{L_s}{AE}+\frac{w^2L_s{}^3}{24T_0{}^2}=\tfrac{1}{2}\left\{\frac{Z_1{}^2}{m_1}+\frac{(Z_2-Z_1)^2}{m_2}+\frac{(Z_3-Z_2)^2}{m_3}+\frac{Z_3{}^2}{m_4}\right\}$$

To allow approximately for the weight of the rope take as the actual suspended load $W_1+\frac{wL_s}{n+1}$, where n is the number of loads, in this case 3, so $\frac{wL_s}{4}$ is to be added to each load.

The vertical reaction at B due to the weights is

$$R_B=\frac{W_1m_1+W_2(m_1+m_2)+W_3(m_1+m_2+m_3)}{L_s} \quad \text{(xii)}$$

Similarly at A

$$R_A=\frac{W_1(m_2+m_3+m_4)+W_2(m_3+m_4)+W_3m_4}{L_s} \quad \text{(xiii)}$$

Then $TZ_1 = R_A m_1$ . . . . . (xiv)

$TZ_2 = R_A(m_1+m_2) - W_1 m_2$ . . . (xv)

$TZ_3 = R_B m_4$ . . . . . (xvi)

Take the following case and tabulate

$L_s = 100$ feet.
$Z_0 = 5$ feet.
$w = 1{\cdot}6$ lb. foot run.
$T_0 = 375$ lb.
$m_1 = 20$ feet.
$m_2 = 40$,,
$m_3 = 25$,,
$m_4 = 15$,,
$W_1 = 1{,}000+40 = 1{,}040$ lb.
$W_2 = 1{,}400+40 = 1{,}440$ lb.
$W_3 = 1{,}160+40 = 1{,}200$ lb.
$A = 0{\cdot}40$ sq. in.
$E = 8{,}000{,}000$ lb. per sq. in.
$R_a = 1{,}588$ lb. $R_b = 2{,}092$ lb.
$TZ_1 = 1{,}588 \times 20 = 31{,}760$.
$TZ_2 = 1{,}588\ (60) - 1{,}040 \times 40 = 53{,}680$.
$TZ_3 = 2{,}092 \times 15 = 31{,}380$.

From these data compile

TABLE II.

1	2	3	4	5	6	7
T	Z_1	Z_2	Z_3	$(T-T_0)\frac{L_s}{AE}$	$\frac{1}{2}\left\{\frac{Z_1^2}{m_1}+\frac{(Z_2-Z_1)^2}{m_2}+\frac{(Z_3-Z_2)^2}{m_3}+\frac{Z_3^2}{m_4}\right\}$	Ditto $-\frac{w^2 L_s^3}{24 T_0^2}$
lb.						
1,000	31·76	53·68	31·38	0·020	74	73·29
10,000	3·18	5·37	3·14	0·301	$\frac{74}{10^2}=0{\cdot}74$	0·19
8,000	3·97	6·71	3·92	0·238	$\frac{74}{8^2}=1{\cdot}157$	0·397
8,500	3·74	6·32	3·69	0·254	$\frac{74}{8{\cdot}5^2}=1{\cdot}025$	0·265
8,550	3·72	6·28	3·67	0·256	$\frac{74}{8{\cdot}55^2}=1{\cdot}013$	0·255

There is no need to work out the whole table.

Start with T = 1,000 lb. This is obviously too small, since col. 5 ought to be equal to col. 7.

Try 10,000 lb. This is too high.
,, 8,000 lb. ,, ,, low.
,, 8,500 lb. ,, ,, low but nearly right.
,, 8,550 lb. ,, ,, practically right.

Then fill in cols. 2, 3, 4.

Thus the tension in the rope is 8,550 lb. or 21,375 lb. sq. inch on the wires.

The sag under W_1 is 3·72 ft. W_2 is 6·28 ft. W_3 is 3·67 ft

When the Supports are not on the Same Level.

As in equations (iii) and (iv), due to the weight of the rope itself,

$$T_0' = \frac{wL_s^2}{8Z_0}$$

$$Z_0' = \frac{wL_s^2}{8T_0}$$

Equation (v) then becomes

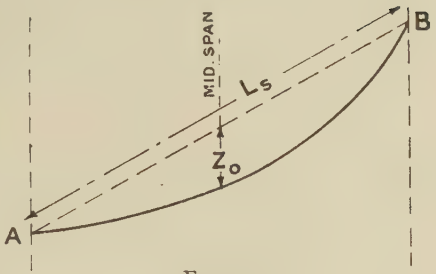


FIG. 5.

$$L_s' = L_s + \frac{Z^2 \cos^2 \alpha}{2}\left(\frac{1}{m} + \frac{1}{n}\right)$$

and (vii) becomes

$$(T - T_0)\frac{L_s}{AE} + \frac{w^2 L_s^3 \cos^2 \alpha}{24 T_0^2} = \frac{Z^2 \cos^2 \alpha}{2}\left(\frac{1}{m} + \frac{1}{n}\right)$$

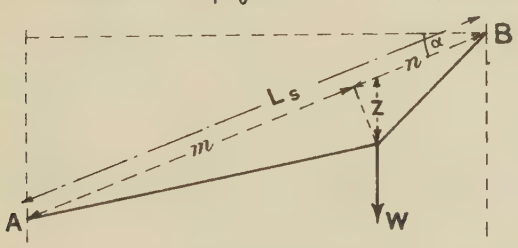


FIG. 6.

If $\alpha = 30°$ and all the other quantities are the same as in Fig. 3, the figures in column 4, Table I, must be multiplied by $(\cos 30)^2 = 0{\cdot}866^2 = 0{\cdot}75$ we then get—

1	2	3	4
T	Z	$(T - T_0)\frac{L_s}{AE}$	$\left\{\frac{Z^2}{2}\left(\frac{1}{m} + \frac{1}{n}\right) - \frac{w^2 L_s^3}{24 T_0^2}\right\} 0{\cdot}75$
1,000	11·2	0·0395	3·720
3,000	3·73	0·133	0·312
4,000	2·80	0·180	0·129
3,800	2·95	0·170	0·155
3,700	3·03	0·165	0·169
3,720	3·01	0·166	0·166

Thus by tilting the rope 30° the tension has been reduced from 3,960 lb. to 3,720 lb. and the sag has been increased from 2·83 feet to 3·01 feet. The difference is small, hence unless the tilt is greater than 30° no great error is introduced by regarding the rope ends as being on the same level.

Readers wishing to draw the exact form taken by unloaded ropes should refer to a paper by Appleyard " Relation between the Bight, Span and Dip of Catenaries," Institution of Civil Engineers Proceedings, Vol. CCVIII, 1918–19.

6. *Find the sectional area of each of the two cables required for a suspension bridge. Span, 75 ft. Width of platform, 8 ft. Dip at mid span, 9 ft. Load on platform, 140 lb. per sq. ft. Stress on cables, 5 tons per sq. in. Weight of cable, 0·28 lb. per cu. in.*

Tension at mid point of span due to applied load

$$= \frac{Wl^2}{8S} = 140 \times 4 \times \frac{75^2}{8 \times 9} = 560 \times 78 \text{ lb.}$$

If the sectional area of cable is a sq. ins. the weight per foot run is $a \times 12 \times 0{\cdot}28 = 3{\cdot}36a$.

Hence tension at mid point of span due to cable weight

$$= 3{\cdot}36a \times 78 \text{ lb.}$$

Assuming the form of the cables to be parabolic, the slope at the ends is given by—

$$\tan \alpha = \frac{4S}{l} = \frac{36}{75}$$

Hence tension at piers due to cables and applied load

$$= (560 + 3{\cdot}36a) \times 78 \times \sqrt{1 + (\tfrac{31}{75})^2} = (560 + 3{\cdot}36a)86{\cdot}6 \text{ lb.}$$

Permissible tension $= 5 \times 2{,}240 \times a$.

Area of cables $= (560 + 3{\cdot}36a)86{\cdot}6 = 11{,}200a$.

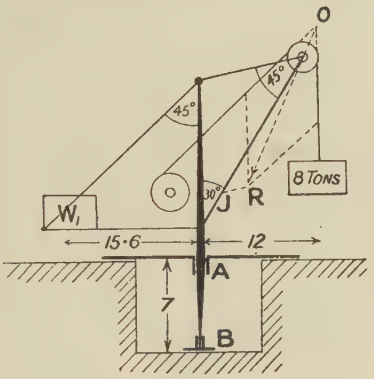
$\therefore a = 4{\cdot}45$ sq. ins.

NOTE.—This solution is not strictly accurate, since the weight of the cable is greater than that shown, owing to the sag. The actual length of cable (see Appleyard, " Relation between Bight, Span and Dip of Catenaries," Proc. Inst. C.E. Vol. CCVIII, 1918–19) is by interpolation 77·8 ft. instead of 75 feet. This only makes a difference of less than 0·1 per cent. in the result.

7. *Find the stresses in the members of the crane shown in the figure, also the forces acting at the bearings* A *and* B. *What counterbalance weight would you use?*

When the load is suspended from the top pulley, the forces are
10·9 tons on jib,
5·6 tons on tie.

In order to reduce the maximum loads on the bearings A and B the counterweight W_1 should be designed to balance one-half of the rated load of 8 tons.



$$W_1 \times 15{\cdot}6 = \frac{8 \times 12}{2}$$

$W_1 = 3{\cdot}1$ tons.

Resultant of chain tensions = OR.

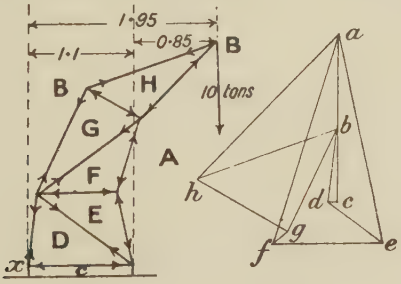
Force on jib, OJ, 16·4 tons } from force diagram.
„ tie, JR, 2·5 „ }

Forces on bearings—

$$\frac{8 \times 12}{2} = P_A \times 7 \qquad P_A = 6{\cdot}85 \text{ tons acting horizontally.}$$

If the weight of the crane is 2 tons, the total downward pressure at B = 8 + 3·1 + 2 = 13·1 tons, and the resultant force on B is $\sqrt{13{\cdot}1^2 + 6{\cdot}85^2} = 14{\cdot}8$ tons, making an angle α with the vertical—$\tan \alpha = \frac{6{\cdot}85}{13{\cdot}1}$. $\alpha = 27\frac{1}{2}°$.

8. *Draw a reciprocal diagram for the structure shown in the figure, and state which bars are in tension.*



Taking moments about x

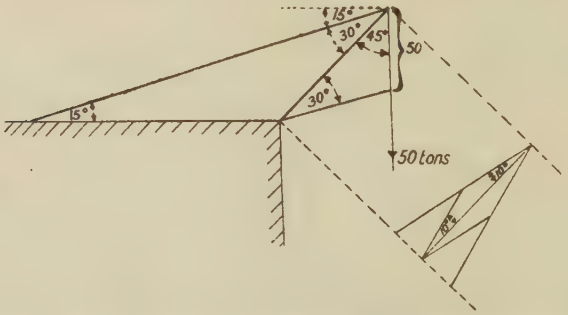
$W \times 1{\cdot}95 = CA \times 1{\cdot}1$ $CA = 17{\cdot}7$ tons $CB = 7{\cdot}7$ tons

Bar.	Force.	Nature.
AH	21·0 tons	Strut
HB	15·4 „	Tie
HG	10·9 „	Strut
GF	2·0 „	Tie
GB	11·9 „	Tie
AF	23·0 „	Strut
FE	11·5 „	Strut
ED	7·2 „	Tie
BD	7·8 „	Tie
EA	22·3 „	Strut
DC	1·0 „	Strut

The force diagram may conveniently be drawn in the following order : *ab*, *bc*, *bh*, *hg*, *ah*, *gf*, *af*, *fe*, *ac*, *de*, *bd*, *dc*.

The closing line *db* on the force diagram should be parallel to the corresponding member of the structure, which gives a check on the accuracy of the drawing.

9. *A load of* 50 *tons is suspended from a shear legs crane. The angle included between the legs is* 20°. *The*



plane of the legs makes an angle of 45° *with the vertical, and the back stay* 15° *with the horizontal.*

Find the force acting on each leg and on the back stay.

Let x be the resultant force in the shear legs, and y the stress in the stay.

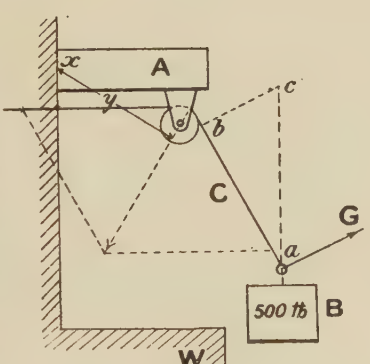
Then $\frac{x}{50}=\frac{\sin 105^\circ}{\sin 30^\circ}=\frac{0{\cdot}966}{0{\cdot}5}$ $x=96{\cdot}6$ tons.

Tension in each leg $=\frac{96{\cdot}6\times\sin 10^\circ}{\sin 160^\circ}=\frac{96{\cdot}6\times 0{\cdot}174}{0{\cdot}342}=$ **49·2 tons.**

$\frac{y}{50}=\frac{\sin 45^\circ}{\sin 30^\circ}=\frac{0{\cdot}707}{0{\cdot}5}$ $y=$**70·7 tons** on back stay.

10. *The diagram illustrates a bale* B *being lifted by means of the chain* C *which passes into the warehouse over a pulley on the beam* A. *A guy rope* G *is being used to keep the bale clear of the wall* W.

Find the tension in C *and* G. *Find also the bending moment in the beam at* x.



On a vertical through *a* mark off *ac*=500 lb. on any convenient scale; draw *cb* parallel to *a*G, then *bc* to the same scale as *ac* is 240 lb., which is the tension in G, and *ab* the tension in the chain is 440 lb., which is also equal to the tension on the horizontal chain. The resultant force on the pulley spindle is 450 lb. The perpendicular distance of x from the resultant, viz. y, is 9·75 ft., hence the bending moment at $x=450\times 9{\cdot}75=$**4,390 lb.-feet.**

11. *A sling chain, diameter of links* $\frac{3}{4}$ *in., is to be used for lifting purposes, the two ends of the chain are attached to eye bolts, and the crane hook passes through a ring in the middle of the chain. The angle between the two portions of the chain is* 60°. *Find the load that may be safely lifted.*

Let P=load on each chain, then $\frac{W}{2P}=\frac{\sqrt{3}}{2}=0{\cdot}86$

$W=1{\cdot}72P$

A convenient practical rule for the safe load for a chain in tons is

$$\frac{(\text{No. of eighths of an inch in link})^2}{10} = \frac{6^2}{10} = 3{\cdot}6 \text{ tons}$$

$$W = \mathbf{6{\cdot}2 \text{ tons.}}$$

12. *A load is suspended by four chains in the same vertical plane. The chains are symmetrical about the line of loading, the inner pair making an angle of* 60° *and the outer* 45° *to the horizontal. The chains are adjusted so that the tension is the same in each. Find the tension* T *for a load of* 5 *tons.*

$$T(\cos 30 + \cos 45) = 2{\cdot}5 \text{ tons}$$

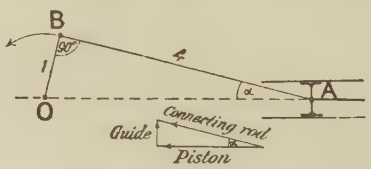
$$T = \frac{2{\cdot}5}{0{\cdot}866 + 0{\cdot}707} = \mathbf{1{\cdot}59 \text{ tons}} \text{ in each chain.}$$

13. *A circular plate, weighing* 2 *tons, is lifted by means of three sling chains attached to the rim of the plate, the points of attachment are equally spaced around the circumference, the upper ends of the chains are threaded over the crane hook. The length of each chain is* 3 *feet* 3 *inches and the diameter of the plate is* 2 *feet* 7 *inches. Find the load on each chain.*

$$\text{Height of hook above plate} = \sqrt{39^2 - 15{\cdot}5^2} = 35{\cdot}8 \text{ ins.}$$

$$\text{Load on each chain} = \tfrac{2}{3} \times \frac{39}{35{\cdot}8} = \mathbf{0{\cdot}725 \text{ ton.}}$$

14. *The stroke of the piston of a steam engine is* 2 *feet and the diameter of the cylinder is* 18 *ins.; length of connecting rod* 4 *feet. The steam pressure is* 80 *pounds per*



square inch. Find graphically or by calculation the pressure on the guide bar, the force acting on the connecting rod, and

the turning effort on the crank pin when the crank and connecting rod are at right angles to each other.

$$OA=\sqrt{16+1}=4{\cdot}12$$

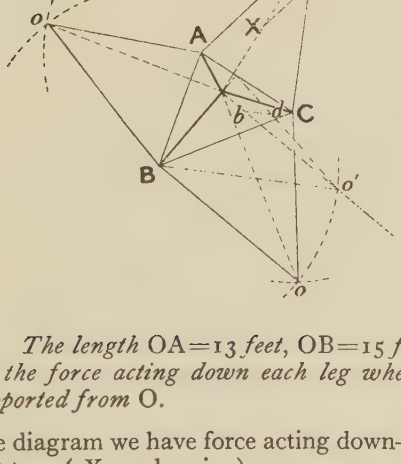
Area of piston$=254{\cdot}5$ sq. ins.

$$\left.\begin{matrix}\text{Force on}\\ \text{con. rod}\end{matrix}\right\}=\frac{\text{piston load}}{\cos\alpha}=\frac{254{\cdot}5\times80\times4{\cdot}12}{4}=\mathbf{21{,}000\ lb.}$$

$$\left.\begin{matrix}\text{Press. on}\\ \text{guides}\end{matrix}\right\}=\text{piston load}\times\tan\alpha=\frac{254{\cdot}5\times80}{4}=\mathbf{5{,}090\ lb.}$$

Turning effort on crank$=$**21,000 lb.-ft.**

15 *In a three-leg lifting tackle, the legs are* OA, OB, OC *respectively,* O *being the apex,* AB$=10$ *feet,* BC$=12$ *feet,*



CA$=9$ *feet. The length* OA$=13$ *feet,* OB$=15$ *feet,* OC$=14$ *feet. Find the force acting down each leg when a load of* 5 *tons is supported from* O.

From the diagram we have force acting down—

OA$=2{\cdot}8$ tons (cX on drawing).

OB$=1{\cdot}0$ ton (bd on drawing).

OC$=1{\cdot}6$ tons (oc on drawing).

OX$=5$ tons which is the resultant of the thrusts in OA, OC.

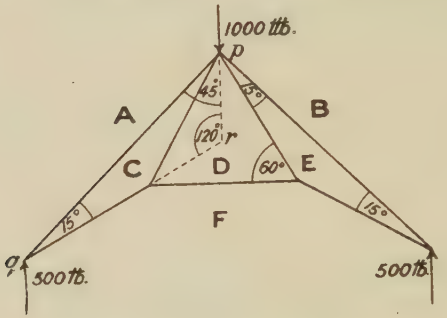
The method of construction is explained in Vol. I. Chap. IV. Fig. 138*a*.

16. *Find trigonometrically the forces acting on the structure shown.*

The triangle of forces for the left-hand abutment is *pqr*, where *pr* represents 500 lb.

Hence force in AC or BE $= \dfrac{500 \sin 120^\circ}{\sin 15^\circ} = \dfrac{500 \times 0{\cdot}866}{0{\cdot}259} =$ **1,672 lb.** (strut)

Force in CF or EF $= \dfrac{500 \sin 45^\circ}{\sin 15^\circ} = \dfrac{500 \times 0{\cdot}707}{0{\cdot}259} =$ **1,364 lb.** (tie)



The vertical components of CF, CD balance.

$$\therefore \text{CD} \sin 60^\circ = \text{CF} \sin 30^\circ$$

$$\text{CD or DE} = \frac{1364 \times 0{\cdot}5}{0{\cdot}866} = \textbf{788 lb.} \text{ (tie)}$$

DF = horizontal component of CF − hor. com. of CD

$= \text{CF} \cos 30^\circ - \text{CD} \cos 60^\circ$

$= 1{,}364 \times 0{\cdot}866 - 788 \times 0{\cdot}5 =$ **790 lb.** (tie).

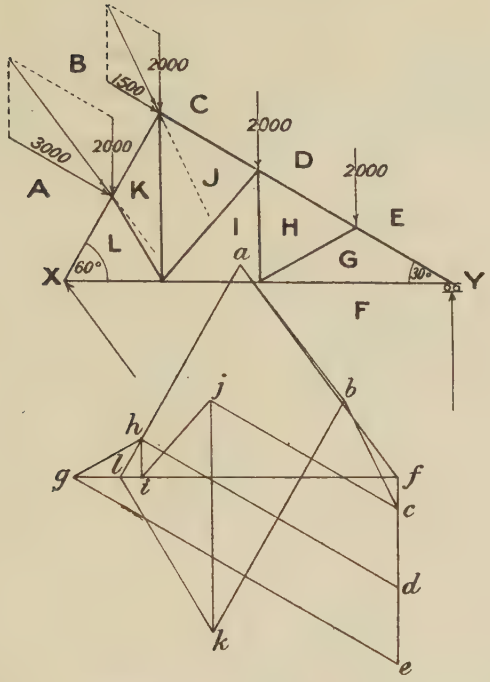
This result might also have been arrived at by the method of sections. Taking moments about the point *p*

$$\text{DF} \times l\frac{\sqrt{3}}{2} = 500l\left(\tfrac{1}{2} + \frac{\sqrt{3}}{2}\right)$$

$$\therefore \text{DF} = \textbf{790 lb.}$$

17. *Draw a reciprocal diagram and find the forces acting on the members of the roof truss shown in the figure.*

By taking the moments of all the external forces about the point X the value of the vertical force, viz. *ef*, is found. Hence the force polygon for external forces can be drawn and the reciprocal diagram built upon it.



Bar.		Load (lb.)		Bar		Load (lb.)
ab		4,400	External forces.	*ge*		9,500 *c*
bc		3,040		*gf*		8,300 *t*
af		6,800		*gh*		2,000 *c*
ef		4,800		*hi*		1,000 *t*
				if		6,500 *t*
al		6,300 *c*		*ij*		2,600 *c*
bk		6,900 *c*		*kj*		6,000 *c*
cj		5,500 *c*		*kl*		4,700 *c*
dh		7,500 *c*		*lf*		7,100 *t*

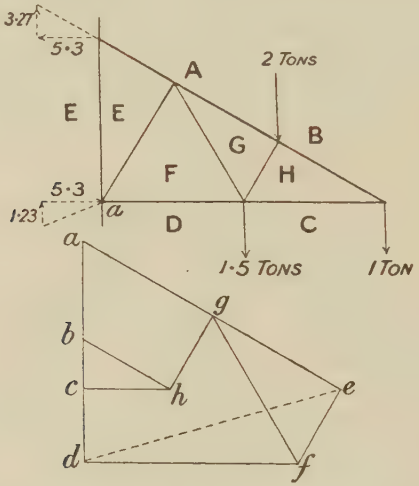
18. *Sketch a reciprocal diagram for the structure shown in the figure and indicate which bars are in tension.*

By measuring the diagram and taking moments about the point *a* we have

$$\begin{aligned} 1 \times 2{\cdot}9 &= 2{\cdot}9 \\ 1{\cdot}5 \times 1{\cdot}45 &= 2{\cdot}175 \\ 2 \times 1{\cdot}8 &= 3{\cdot}6 \\ & \overline{8{\cdot}675} \end{aligned}$$

$\frac{8 \cdot 675}{1 \cdot 63} = 5 \cdot 3$ tons as the horizontal component at the top and bottom joints as shown.

If desired the force diagram can be drawn without calculating the reactions. It is convenient to draw them in



the following order: *bch*, *abhg*, *chgfd*, *agfe*. The lines *ed* and *ea* give the reactions in magnitude and direction at the points of attachment to the wall.

The results are—

Bar.	Force.	Nature.	Bar.	Force.	Nature.
AG	3 tons	T.	HC	1·76 tons	C.
AE	6 tons	T.	FE	1·76 tons	C.
BH	2 tons	T.	FG	3·46 tons	T.
FD	4·3 tons	C.	GH	1·76 tons	C.

19. *In the frame shown find the magnitude and direction of the force at* A *and the magnitude of* B *to keep the frame in equilibrium.*

Resolving the 200 lb. into its vertical and horizontal components and taking moments about B, we have

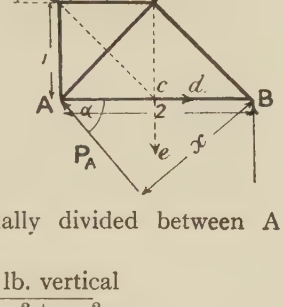
$$100\times 1+\frac{200}{\sqrt{2}}\times 2=\frac{200}{\sqrt{2}}+P_A x \qquad P_A x=241$$

Taking moments about A

$$P_B\times 2=100\times 1+\frac{200}{\sqrt{2}} \qquad P_B=\mathbf{120{\cdot}5\ lb.}$$

$$P_A=\sqrt{141^2+120{\cdot}5^2}=\mathbf{185\ lb.}$$

$$\tan\alpha=\frac{120{\cdot}5}{141} \qquad \alpha=40^\circ\ 30'$$



The problem may also be solved thus—

The two applied forces 200 and 100 lb. may be regarded as acting at their meet point *c*, and may be resolved in vertical and horizontal directions, *ce* and *cd* respectively. The horizontal component=141 lb. must be balanced at A. The vertical component (by moments) is equally divided between A and B.

Hence the reaction at B=120·5 lb. vertical

" " " $A=\sqrt{120{\cdot}5^2+141^2}$

$=185$ lb. at 40° 30′ to AB

20. *A Warren girder of five bays is loaded with* 30 *tons at each of the two bottom joints distant* 15 *and* 30 *feet from the left-hand end. Span* 75 *feet. Find by means other than a reciprocal diagram the forces acting on the booms at the middle of the girder, also on the inclined braces at the right-hand end of the girder.*

Reactions $R_l\times 75=30\times 60+30\times 45$

$R_l=42$ tons $\qquad R_r=18$ tons

Consider the central triangle, apex A, right-hand bottom corner B, left-hand C.

Take a section through A and consider the portion of the girder to the right of this section. By moments about A we have $F_A\times 15\frac{\sqrt{3}}{2}=18\times 37{\cdot}5$, from which $F_A=52$ tons tension, where F_A is the force acting in the boom BC.

By taking moments about B for the portion of the girder

to the right of B we have, $F_B = 41{\cdot}6$ tons compression and $F_C = 26{\cdot}4$ tons compression. F_B and F_C refer to the top booms immediately over B and C respectively.

The problem might have been solved by taking the bending moment on the girder.

Bending moment in middle $= 18 \times 37{\cdot}5$.

Depth of girder $= 15\,\frac{\sqrt{3}}{2}$.

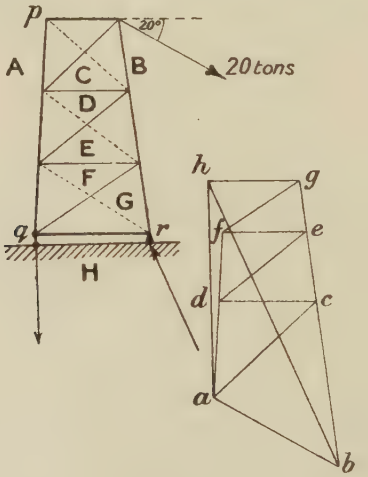
Force in bottom boom $= \frac{18 \times 37{\cdot}5 \times 2}{15 \times \sqrt{3}} = 52$ tons tension.

Bending moment in panel to right of A $= 18 \times 30$.

Force in corresponding top boom $=$ **41·6 tons** compression.

Force in braces $= \frac{18}{\cos 30} = 20{\cdot}8$ tons.

21. *Find the stresses in the members of the trestle shown in the figure. The diagonal bracing is only suitable for tension.*



Since the diagonal bracing is only suitable for tensile stresses, the compression bracing shown in broken lines is neglected. This being so it will be seen that there are only two remaining members which meet at the point *p*. Since there is no applied force at this point and the two members

are not collinear they carry no stress and therefore may be neglected. In the force diagram it will be convenient to start with the triangle *abc* and then determine successively the points *d*, *e*, *f*, *g*, *h*, then the lines *ab*, *hg* will represent the reactions at the base. As a check on the drawing, the line in the structure diagram joining *r* to the point of intersection of the vertical through *q* and the line of the applied force should be parallel to the line *bh* in the force diagram.

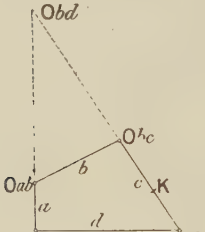
Bar.	Stress in tons.	Bar.	Stress in tons.
BC	+29·3	BE	+37·9
CA	−22·9	FE	+ 7·14
CD	+ 9·64	FG	− 7·86
DA	−19·3	FA	−27·1
DE	−10·4	BG	+42·1

+ indicates compression. − indicates tension.

CHAPTER V

KINEMATICS

1. *In a four-bar mechanism the length of the links are:* a 1, b 2, c $2\frac{1}{4}$, d 3 *fixed. Find the velocity of a point* K *on* c *at a distance of* 1 *from* Ocd, *when the point* Oab *moves at* 3 *f.p.s. and the link* a *is at right angles to* d.

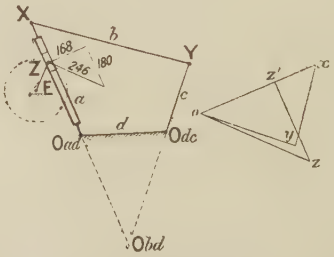


$$\frac{Obd\,Obc}{Obd\,Oab} = \frac{Vbc}{Vab} = \frac{3{\cdot}25}{3{\cdot}6}$$

$$Vbc = \frac{3 \times 3{\cdot}25}{3{\cdot}6} = 2{\cdot}7 \text{ f.p.s.}$$

$$V_K = \frac{2{\cdot}7 \times 1}{2{\cdot}25} = \mathbf{1{\cdot}2\ f.p.s.}$$

2. *Find the velocity of the points* X *and* Y, *also the velocity of sliding of the die* Z *in the slot when the crank* E *rotates at* 70 *r.p.m. Radius of crank* 0·56 *foot.*



The problem is solved below by two methods: (i) by the ordinary virtual centre, (ii) by Smith's velocity diagram, both of which are explained in Vol. I. p. 128, and p. 143.

Velocity of crank pin $=2\pi\times70\times0{\cdot}56=246$ f.p.m.

Velocity component normal to link a $=168$ „

Velocity of X $=168\dfrac{Oad\,X}{Oad\,L}=\dfrac{168\times2{\cdot}55}{1{\cdot}65}=$ **260 f.p.m.**

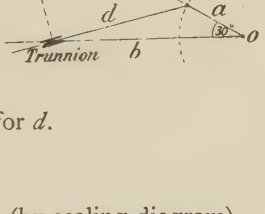
Velocity of Y $=260\dfrac{Obd\,Y}{Obd\,X}=260\times0{\cdot}8=$ **208 f.p.m.**

Velocity of sliding of Z $=$ **180 f.p.m.**

In Smith's diagram, draw $Oz'x$ to represent the velocities of Z (in the slot) and X. Make Oz' represent 168 feet per minute. Then if $Ox : Oz' = O_{ad}X : O_{ad}Z$, Ox will represent in magnitude and direction the velocity of the point X. Hence the velocity of X $=260$ f.p.m. scaled from diagram.

We know that the velocity of Y is perpendicular to O_{dc}, also that its velocity relative to X is perpendicular to XY. Hence if Oy, xy are drawn perpendicular to the links c and b respectively, the vector Oy represents completely the velocity of Y. From the diagram the velocity of Y $=$ **208 f.p.m.**

3. *In an oscillating cylinder engine, find the angular velocity of the cylinder when the crank makes* 18 *radians p.m. The length from the centre of the crank shaft to the centre of the cylinder trunnions is three times the radius of the crank. The angle of the crank from the inner dead centre is* 30°.



Angular velocity of cylinder $=$ ang. vel. of piston rod $d=\omega$.

The point m is the virtual centre for d.

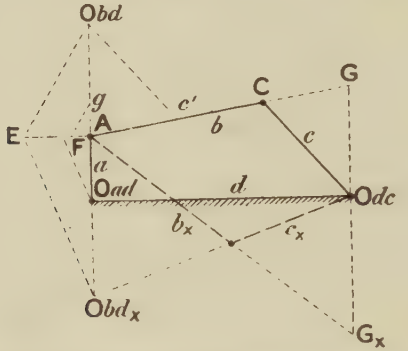
Hence the velocity of $n=\omega\,.\,mn$.

But the velocity of $n=18\,.\,on$

Hence $\omega=\dfrac{on}{mn}\times18=$ **6 rads.p.m.** (by scaling diagram).

4. *In a four-bar mechanism the fixed bar is* 4 *inches long, the crank* 1 *inch, the rocking link* 2 *inches, and the connecting rod* 2·75 *inches. The crank makes* 80 *revolutions per minute. Find the angular velocity of the rocking link and the connecting rod when the crank and fixed links are at right angles to one another both when the mechanism is "open," and when "crossed."*

If drawn in the usual way the virtual centre O*bd* comes off the paper. Take A*c'* $\frac{1}{n}$th AC. Draw a line from *c'* parallel to the bar *c* to meet *a* produced, then O*bd* is an equivalent virtual centre. From A draw a line AE in any



direction (see Vol. I. p. 131), whose length represents the angular velocity of the bar *a*. Set off A*g* = $\frac{1}{n}$ O*ad*. A. From *g* draw a line parallel to O*bd*. E. Then AF is the angular velocity of *b* to the same scale. Produce the bar *b*, from O*dc* draw a line parallel to the bar *a*, then

$$\frac{\omega_c}{\omega_a} = \frac{\text{A} \cdot Oad}{\text{G} \cdot Odc}$$

$$\frac{\omega_c}{\omega_b} = \frac{\text{A} \cdot Oad \times \text{AE}}{\text{G} \cdot Odc \times \text{AF}} = \frac{1 \times 1}{1{\cdot}75 \times 0{\cdot}3} = \frac{1}{0{\cdot}525} \qquad \omega_c = \omega_a \frac{1}{1{\cdot}75}$$

$\omega_c = \frac{80}{1{\cdot}75} = 46$ r.p.m. (Open) $\omega_b = \mathbf{0{\cdot}525\omega_c = 24{\cdot}2}$ **r.p.m.**

When the rods are crossed we have—

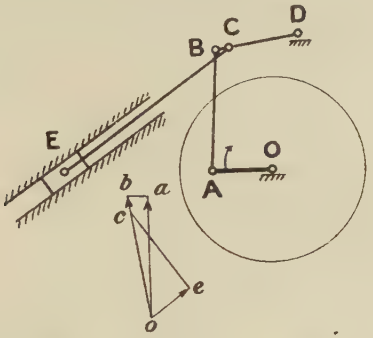
$$\frac{\omega_c}{\omega_a} = \frac{\text{A} \cdot Oad}{\text{G}_x \cdot Odc} = \frac{1}{2{\cdot}16}$$

$$\frac{\omega_c}{\omega_b} = \frac{\text{A} \cdot Oad \times \text{AE}}{\text{G}_x \cdot Ode \times \text{AF}_x} = \frac{1}{2{\cdot}16 \times 0{\cdot}41} = \frac{1}{0{\cdot}886}$$

$\omega_c = \frac{80}{2{\cdot}16} =$ **37 r.p.m.** (Crossed) $\omega_b = 0{\cdot}886 \times 37 =$ **32·8 r.p.m.**

The crossed rods are shown in broken lines and the letters bear the suffix *x*.

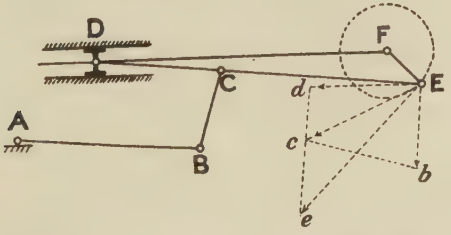
5. *The diagram represents the mechanism of the Atkinson gas engine. Find the speed of the piston* E *when the crank makes* 300 *r.p.m. Throw of crank* 8 *ins.*



Velocity of crank pin $=3{\cdot}14\times1{\cdot}33\times300=1{,}256$ f.p.m.

In the velocity diagram *oa* represents the velocity of the crank pin. *ob* is drawn normal to DB. *ab* normal to AB. $\frac{bc}{ob}=\frac{BC}{BD}$. *ce* is normal to EC. *oe* is parallel to the centre line of the cylinder. *oe* gives the velocity of the piston to the same scale as *oa*. The velocity of the piston is **500 f.p.m.**

6. *In the mechanism shown the crank pin moves with a linear velocity of* 84 *f.p.m. Find the velocity of the points* B, C, D, *also the velocity of* C *relative to* B.



E*e* is drawn at right angles to the crank and is made to represent on any convenient scale the known linear velocity of E, viz. 84 ft. per min. Draw *ed* perpendicular to ED, and E*d* parallel to FD. Then E*d* represents to scale the velocity of the crosshead D.

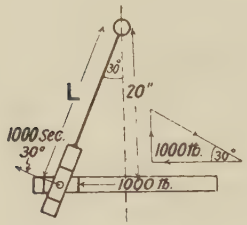
Divide *ed* at *c* so that $\frac{ec}{cd}=\frac{EC}{CD}$. Then E*c* represents the velocity of the point *c*.

Draw E*b* perpendicular to AB and *cb* perpendicular to CB. Then E*b* represents to scale the velocity of the joint B, and *bc* represents the velocity of C relative to B.

From the diagram—

Velocity of B=**40 f.p.m.**
D=**52** „
C=**61** „
Velocity of C relative to B=**54** „

7. *Find the twisting moment on the shaft of the gear shown in the figure. Neglect friction.*



Component of force normal to arm } =1,000 sec. 30°
=1,155 lb.
L=20 sec. 30°=23·1 in.
Twisting moment=1,155×23·1
=**26,700 inch-lb.**
or twisting moment } =1,000×20×sec.² 30°
=**26,700 inch-lb.**

The triangle of forces shown represents the force acting on the pin.

8. *In Fig.* 161, *Vol. I., produce the connecting rod to meet a vertical through* Oad*; let their meet be* e. *Then the triangles* Oad, e, Oab *and* Obd, c, Oab *are similar.*

Hence $\frac{R_a}{R_b}=\frac{e\,.\,Oab}{l}$

and $V_p=r_p\omega_a\left(\frac{R_b+R_a}{R_b}\right)=r_p\omega_a\left(1+\frac{e\,.\,Oab}{l}\right)$

(see p. 135, Vol. I.)

$$V_p=r_p\omega_a\left(\frac{l+e\,.\,Oab}{l}\right)=\frac{ce}{l}$$

If *l* be taken to represent the velocity of rubbing of the crank shaft in its bearings, then *ce* represents to the same scale the velocity of rubbing of the crank pin in the big end

brasses of the connecting rod. As already stipulated $r_c=r_p=r_g$. (See p. 136.)

$$V_g=\omega_a r_g \frac{R_a}{R_b}=\omega_a r_g\left(\frac{e \,.\, Oab}{l}\right)$$

or $e \,.\, Oab$ represents to the same scale the velocity of rubbing of the small end brasses on the gudgeon pin.

In Fig. 161 $l=1{\cdot}7''$	$ce=0{\cdot}4''$	$ec=2{\cdot}1''$
Velocity of rubbing,	gudgeon pin	crank pin
crank shaft 250 ft.-min.	$\frac{250\times 0{\cdot}4}{1{\cdot}7}=59$	$\frac{250\times 2{\cdot}1}{1{\cdot}7}=\mathbf{309}$

Which agrees with the values obtained by the method given on pp. 135–6. Note the values 63 and 314 for the gudgeon pin and crank pin respectively are the maximum values, not for the position shown in the figure.

9. *A rotating cam gives S.H.M. to its follower while turning through an angle of* 120°, *and during the remainder of the revolution the follower is stationary. The follower is kept in contact with the cam by its own weight, and its total travel is* 2·1 *in. Find the maximum speed at which the cam may be rotated and still keep in contact with the follower.*

$$N=\frac{0{\cdot}3\alpha}{\sqrt{R}} \text{ (see p. 157, Vol. I.)} \qquad \alpha=60°$$

$$R=\frac{2{\cdot}1}{2\times 12}=0{\cdot}0875 \text{ ft.} \qquad N=\frac{0{\cdot}3\times 60}{\sqrt{0{\cdot}0875}}=\mathbf{61 \text{ r.p.m.}}$$

Or the question may be more fully dealt with thus—

Assuming that the cam is arranged so that there is no shock at the beginning of the lift, the follower will make a complete (not one-half) cycle of S.H.M. in the time taken by the cam to rotate through 120°, and the radius of this S.H.M. will be

$$R=\frac{2{\cdot}1}{2\times 12}=0{\cdot}0875 \text{ ft.}$$

The equivalent angular velocity of the motion is

$$\omega=\frac{360}{120}\times\frac{2\pi N}{60}$$

Hence the maximum acceleration during the motion of the follower $=\omega^2 R=0{\cdot}0875\left(\frac{\pi N}{10}\right)^2$ f.p.s.p.s.

The follower will lose contact if this is greater than the gravitational acceleration g. Hence the maximum speed is given by $32{\cdot}2 = 0{\cdot}0875\left(\frac{\pi N}{10}\right)^2$ or $N = \mathbf{61\ r.p.m.}$

Some very valuable experimental data on cams will be found in a paper by J. E. Scholes, "Experiments on Cams and Poppet Valves," Institution of Automobile Engineers, April, 1922.

10. *A train of wheels is required to give a gear ratio of* 0·12345. *Find the number of teeth in the wheels to give a close approximation to the ratio.*

First approximation $\frac{12}{100}$.

Error 0·12345 − 0·12000 = 0·00345.

Thus the first approximation must be increased in the ratio $1 + \frac{0{\cdot}00345}{0{\cdot}12000} = 1 + \frac{1}{35}$ very nearly $= \frac{36}{35}$.

Hence the train of wheels gives $\frac{12}{100} \times \frac{36}{35} = \mathbf{0{\cdot}12343}$.

This method is due to Mr. Fisher, see article in *Machinery*, August 15, 1918.

By the use of continued fractions gear ratio problems can often be solved more directly than by Mr. Fisher's method.

In *Machinery*, October 3, 1918, Mr. Pettingill shows by the use of continued fractions how the required gear ratio can be obtained to a closer degree of approximation by two wheels only, viz. $\frac{10}{81} = 0{\cdot}123457$. The use of continued fractions for this purpose is well known, but the article referred to gives an artifice for getting even closer approximations with wheels of practical dimensions.

11. *A train of wheels is required to give a gear ratio of* 0·31416. *Find suitable wheels for a close approximation.*

First approximation $\frac{32}{100}$.

Error 0·31416 − 0·32000 = −0·00584.

Decrease in ratio $1 - \frac{0{\cdot}00584}{0{\cdot}32000} = 1 - \frac{1}{54\frac{3}{4}} = \frac{215}{219}$ nearly.

Required gears $= \frac{32}{100} \times \frac{215}{219} = \frac{43}{60} \times \frac{32}{73} = 0{\cdot}314155$ with a compound train.

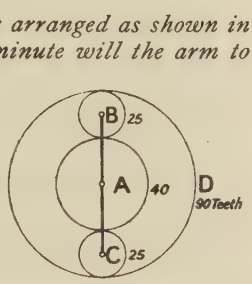
With a triple train a still closer approximation can be obtained.

Decrease in ratio $1 - \frac{0{\cdot}00584}{0{\cdot}32000} = 1 - \frac{1}{54\frac{27}{34}}$ nearly $= \frac{1829}{1863}$.

Required gears $\frac{32}{100}\times\frac{1829}{1863}=\frac{32\times31\times59}{23\times81\times100}=0{\cdot}31415996.$

Method due to Mr. Fisher, *Machinery*, August 15, 1918. By continued fractions we find that two wheels $\frac{71}{226}$ give 0·3141593.

12. *An epicyclic train of wheels is arranged as shown in sketch. How many revolutions per minute will the arm to which the pinions* B *and* C *are attached make—*



(*a*) *when* A *makes one revolution clockwise and* D *makes half a revolution anti-clockwise ;*

(*b*) *when* A *makes one revolution clockwise and* D *makes half a revolution clockwise ;*

(*c*) *when* A *makes a revolution clockwise and* D *is stationary ?*

(*a*) Value of $e=-N_D=1$ $\quad N_F=-0{\cdot}5$

$$-e=\frac{N_F-N_A}{N_D-N_A} \qquad -\tfrac{40}{90}=\frac{-0{\cdot}5-N_A}{1-N_A}$$

$$-40+40N_A=-45-90N_A \qquad 40N_A+90N_A=-45+40$$

$$130N_A=-5 \qquad N_A=-\tfrac{5}{130}=-\tfrac{1}{26}$$

The arm must make $\frac{1}{26}$ of a turn in anticlock direction.

(*b*) Value of $e=-N_D=1$ $\quad N_F=0{\cdot}5$

$$-\tfrac{40}{90}=\frac{0{\cdot}5-N_A}{1-N_A} \qquad -40+40N_A=45-90N_A$$

$$40N_A+90N_A=45+40 \qquad 130N_A=85 \qquad \mathbf{N_A=\tfrac{17}{26}}$$

(*c*) Value of $e=-N_D=1$ $\quad N_F=0$

$$-\tfrac{40}{90}=\frac{0-N_A}{1-N_A} \qquad -40+40N_A=-90N_A$$

$$40N_A+90N_A=40 \qquad 130N_A=40 \qquad N_A=\tfrac{40}{130}=\tfrac{4}{13}$$

Another method of dealing with this problem is as follows :—

(*a*) With the arm fixed, rotate B once clockwise. The resulting displacements are shown in the first line of the table below.

(*b*) Rotate the whole mechanism once clockwise. The result is shown in the lower line of the table.

Arm.	A.	B.	C.	D.	Mult.
o	$-\frac{25}{40}$	$+1$	$+1$	$+\frac{25}{90}$	x
$+1$	$+1$	$+1$	$+1$	$+1$	y

(i) In this case $-\frac{25}{40}x+y=1$ $\quad\frac{25}{90}x+y=-\frac{1}{2}$.

Hence $y=-\frac{1}{26}$ and the arm revolves $\frac{1}{26}$ counter-clockwise.

(ii) In this case $-\frac{25}{40}x+y=1$ and $\frac{25}{90}x+y=+\frac{1}{2}$.

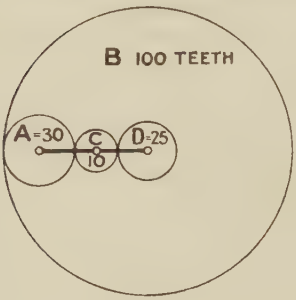
Hence $\mathbf{y=+\frac{17}{26}}$.

(iii) Similarly in this case $y=+\frac{4}{13}$.

13. *Find the number of revolutions made by* B, C, D *when—*

(i) *The arm makes* $+1$ *revol. and* A *is fixed.*

(ii) *The arm makes* $+3$ *r.p.m. and* A *makes* $+8$ *r.p.m.*



(NOTE.—*The centres of* A, C, D *should not have been shown on a straight line.*)

Case (i) A $=0$.

B makes $-\frac{30}{100}+1=\frac{7}{10}$ **r.p.m.**

C „ $+\frac{30}{10}+1=$**4.**

D „ $-\frac{30}{25}+1=-\frac{1}{5}$.

Case (ii) A $=8$ r.p.m.

B makes $+\frac{30}{100}\times 8+\frac{7}{10}\times 3=$**4·5 r.p.m.**

C „ $-\frac{30}{10}\times 8+4\times 3=-$**12.**

D „ $+\frac{30}{25}\times 8-\frac{1}{5}\times 3=$**9.**

This problem can also be solved thus—

Arm.	A.	B.	C.	D.	Mult.
o	$-\frac{10}{30}$	$-\frac{10}{100}$	$+1$	$-\frac{10}{25}$	x
$+1$	$+1$	$+1$	$+1$	$+1$	y

(i) $y=1$ $\quad -\frac{x}{3}+y=0$ $\quad \therefore\ x=3$

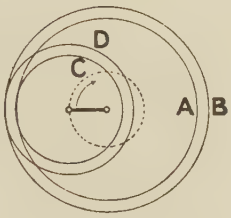
Hence $\quad B=+\frac{7}{10} \quad C=+4 \quad D=-\frac{1}{5}$

(ii) $y=3$ $\quad -\frac{x}{3}+y=8$ $\quad \therefore\ x=-15$

Hence $\quad B=+4\frac{1}{2} \quad C=-12 \quad D=+9$

14. *In the epicyclic train shown in the figure, the annular wheel* A *is fixed, the annular wheel* B *runs free on the same spindle. The wheel* C *gears with* A *and the wheel* D *with* B, *both are fastened to the same spindle.*

Find the r.p.m. made by B *when the arm makes* 40 *r.p.m.*

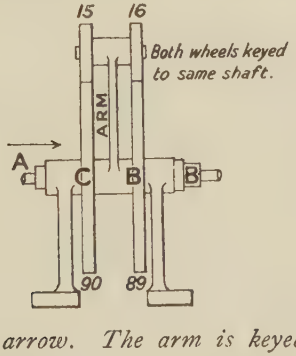


Teeth—A 30, B 31, C 27, D 28.

Revs. of Arm.	Revs. of A.	Revs. of B.
Fixed $=0$	1	$\frac{30\times28}{27\times31}=\frac{280}{279}$
1	0	$1-\frac{280}{279}=-\frac{1}{279}$
40	0	$-\frac{40}{279}$.

Thus B makes $\frac{40}{279}$ r.p.m. contra-clockwise when the arm makes 40 r.p.m. in a clockwise direction.

15. *Find the speed of the wheel* B *when* A *makes* 600 *r.p.m. in a contra-clockwise direction to an observer looking in the*



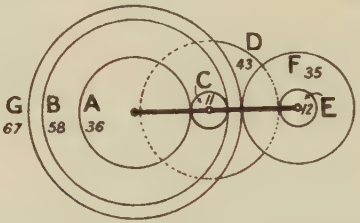
direction of the arrow. The arm is keyed to A *and the wheel* C *fixed to the frame.*

Fixed.	Arm A	C	B
Arm	0	+1	$\frac{90}{15}\times\frac{16}{89}=-\frac{96}{89}$
C	+1	0	$1-\frac{96}{89}=-\frac{7}{89}$

When A makes 600 r.p.m. contra-clockwise B makes $-\frac{600\times 7}{89}=-47{\cdot}2$ r.p.m. in the reverse direction, *i.e.* clockwise.

16. *In the epicyclic gear shown find the number of revolutions per min. made by the wheel* D *when the arm makes* +1 *revolution, the wheel* A *being fixed.*

Fix the arm. Give A one+rev., then the revs. of the other wheels will be



$C=-\frac{36}{11}$ B and G (fastened together) $-\frac{36}{58}$

F and E (fastened together) $+\frac{36}{58}\times\frac{67}{35}$

$D-\frac{36}{58}\times\frac{67}{35}\times\frac{12}{43}=-\frac{28944}{87290}=-0{\cdot}3316=-\frac{1}{3}$ nearly.

The wheel D rotates in the same sense as the arm, hence (see p. 173, Vol. I.) when the arm makes+1 rev. the wheel D makes $+1-\frac{1}{3}=+\frac{2}{3}$ rev.

17. Readers who have to design compound epicyclic gears to specified conditions, such as gear ratio, centres of shafts, numbers of teeth of standard pitch in the various wheels, will do well to consult an article by F. S. Taylor in *Machinery* for July 25, 1918.

CHAPTER VI

DYNAMICS OF ENGINES

1. *A petrol engine is running at* 1,100 *r.p.m. The explosion pressure is* 400 *lb. per sq. in. above atmosphere. The pressure at the end of the impulse stroke is* 50 *lb. per sq. in. above atmosphere. Weight of reciprocating parts per sq. in. of piston* 0·83 *lb. Length of connecting rod* 4·2 *cranks. Throw of crank* 4 *ins. Find the resulting driving pressure at each end of the stroke.*

Acceleration pressure with long rod
$$=0{\cdot}00034\times0{\cdot}83\times\tfrac{4}{12}\times1{,}100^2=113{\cdot}8 \text{ lb. sq. in.}$$

With short rod " in " end, $113{\cdot}8\times\frac{5{\cdot}2}{4{\cdot}2}=140{\cdot}9$ lb. sq. in.

" " " out " end, $113{\cdot}8\times\frac{3{\cdot}2}{4{\cdot}2}=86{\cdot}7$ " "

Resulting driving pressure at beginning of stroke
$$=400-140{\cdot}9=\mathbf{259{\cdot}1 \text{ lb. sq. in.}}$$

Resulting driving pressure at end of stroke
$$=50+86{\cdot}7=\mathbf{136{\cdot}7 \text{ lb. sq. in.}}$$

2. *A flat level table is attached to the tail rod of a vertical engine. Calculate the speed of the engine when a body resting freely on this table begins to jump at the top of the stroke. Radius of crank,* 9 *inches. Length of connecting rod,* 30 *inches.*

Jumping will occur when the inertia of the body at the end of the stroke exceeds the weight.

$$0{\cdot}00034\text{WRN}^2(1+\tfrac{9}{30})=\text{W}$$

$$0{\cdot}00034\times\tfrac{3}{4}\text{N}^2\times1{\cdot}3=1 \qquad \text{N}=\mathbf{54{\cdot}8 \text{ r.p.m.}}$$

This is a good laboratory experiment for demonstrating the inertia of the reciprocating parts of an engine.

3. *Find the tension in the connecting rod bolts of an internal combustion engine. Weight of piston, etc.,* 600 *lb.; diam. of cylinder,* 9 *ins.; pressure below atmosphere on suction stroke,* 2 *lb. sq. in.; friction of piston in cylinder,* 40 *lb.; r.p.m.,* 250; *radius of crank,* 10 *ins.; length of connecting rod,* 38 *ins.*

Total tension = acceln. press. + suction + friction

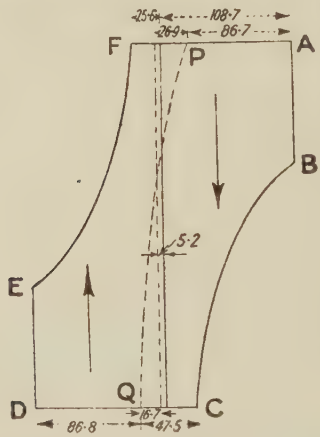
$$= 0{\cdot}00034\,WRN^2\left(1+\frac{1}{n}\right)+pA+40$$

$$= 0{\cdot}00034\times 600\times\tfrac{10}{12}\times 250^2\left(1+\frac{1}{3{\cdot}8}\right)+2\times 0{\cdot}785\times 81+40$$

$$= 13{,}420+127+40 = 13{,}600 \text{ lb.}$$

Tension in bolts $\dfrac{13600}{63{\cdot}6}$ = **214 lb. sq. in. of piston.**

4. *In a vertical inverted steam engine the weight of the reciprocating parts is* 5·2 *lb. per sq. in. of piston area. The initial steam pressure is* 110 *lb. per sq. in. above atmosphere. Cut off at* $\frac{1}{3}$*rd stroke. Clearance negligible. Back pressure* 1·3 *lb. per sq. in. above atmosphere. Stroke,* 42 *ins.; length of connecting rod,* 90 *ins.; r.p.m.,* 84; *assume hyperbolic expansion. Find the effective driving pressures at the ends of the stroke.*



Steam pressure at end of stroke

$$= \frac{110+14{\cdot}7}{3} = 41{\cdot}6 \text{ lb. sq. in.}$$

$$= 26{\cdot}9 \text{ lb. sq. in. above atm.}$$

Effective *steam* pressure on piston at beginning of stroke $= 110-1{\cdot}3 = 108{\cdot}7$ lb. sq. in.

Ditto at end of stroke $= 26{\cdot}9-1{\cdot}3 = 25{\cdot}6$ lb. sq. in.

During the downstroke the weight of the reciprocating parts increases the effective pressure by 5·2 lb. per sq. in. and during the upstroke it decreases it by the same amount.

Pressure at end of stroke due to inertia of reciprocating parts $=0{\cdot}00034\times5{\cdot}2\times\frac{21}{12}\times84^2(1\pm\frac{21}{90})$

$=27$ lb. sq. in. at "in" end; 16·7 lb. sq. in. at "out" end.

The resultant effective pressures during the downstroke are given by the figure ABCQP, and for the upstroke by DEFPQ. The effective pressures at the end of the stroke are—

Downstroke: Top, $110-1{\cdot}3+5{\cdot}2-27=$ **86·7 lb. sq. in.**
„ Bottom, $26{\cdot}9-1{\cdot}3+5{\cdot}2+16{\cdot}7=$ **47·5** „ „
Upstroke: Bottom, $110-1{\cdot}3-5{\cdot}2-16{\cdot}7=$ **86·8** „ „
Top, $26{\cdot}9-1{\cdot}3-5{\cdot}2+27=$ **47·4** „ „

5. *Find the twisting moment in the case of a vertical steam engine when the piston is descending and the crank has turned through* 30° *from the top dead centre. Stroke,* 2 *ft.; diam. of cylinder,* 16 *ins.; r.p.m.,* 200; *total weight of reciprocating parts,* 500 *lb.; effective steam pressure on piston,* 50 *lb. sq. in.; connecting rod-crank ratio,* 4.

Acceleration pressure along centre line of engine

$$=0{\cdot}00034WRN^2\left(\cos 30+\frac{\cos 60}{4}\right)$$

$$=0{\cdot}00034\times500\times200^2(0{\cdot}866+0{\cdot}125)=6{,}740 \text{ lb.}$$

Effective thrust on piston rod

$$=50\times0{\cdot}785\times16^2+500-6{,}740=3{,}810 \text{ lb.}$$

Let α=angle connecting rod makes with centre line of engine.

$$\frac{\sin\alpha}{\sin 30}=\frac{1}{4} \qquad \alpha=7^\circ\ 10'$$

Thrust on connecting rod $=\frac{3810}{\cos 7^\circ\ 10'}=3840$ lb.

Let β=angle connecting rod makes with a line normal to the crank.

$$\beta=180-30-7^\circ\ 10'-90=52^\circ\ 50'.$$

Twisting moment $=3{,}840\times\cos 52^\circ\ 50'=$ **2,320 lb.-ft.**

6. *A toothed wheel is caused to oscillate to and fro by means of a rack which is driven by a crank and very long connecting rod. The wheel rotates each way through an angle of* 120 *degrees* 600 *times p.m. Pitch diameter of wheel,*

18 *ins.; radius of gyration* K, 7 *ins.; weight of wheel* W, 47 *lb. Find the maximum pressure on the teeth of the wheel.*

Stroke of rack $= \frac{18 \times 3{\cdot}14 \times 120}{360} = 18{\cdot}84''$.

Radius of crank $(R_c) = 9{\cdot}42'' = 0{\cdot}785'$.

Pressure on teeth $= 0{\cdot}00034 W R_c N^2 \times \frac{K^2}{R_w^2}$

$= 0{\cdot}00034 \times 47 \times 0{\cdot}785 \times 600^2 \times \frac{49}{81} =$ **2,730 lb.**

7. *Find the necessary clearance at the top of the stroke in the case of a Willans type of engine assuming* (*i*) *hyperbolic,* (*ii*) *adiabatic compression of the air. Area of annulus exposed to air pres.,* 49·5 *sq. ins.; area of underside of piston exposed to atmospheric pres.,* 113·1 *sq. ins.; weight of rec. parts,* 1,000 *lb.; radius of crank,* 3 *ins.; length of con. rod,* 1 *ft.; r.p.m.,* 450.

(See p. 190, Vol. I.)

Net pres. required to retard moving parts without tending to lift crankshaft

$P_1 = 0{\cdot}00034 \times 1000 \times \frac{1}{4} \times 450^2 (1 + \frac{1}{4}) - 1000 + 14{\cdot}7 \times 113{\cdot}1$

$= 22{,}180$ lb.

$C = \frac{29{\cdot}4 \times 49{\cdot}5}{4(22180 - 727)} =$ **0·17″ for hyperbolic compression.**

$C = \frac{0{\cdot}5}{\left(\frac{22180}{14{\cdot}7 \times 49{\cdot}5}\right)^{\frac{1}{1{\cdot}41}} - 1} = \frac{0{\cdot}5}{30{\cdot}5^{\frac{1}{1{\cdot}41}} - 1} = \frac{0{\cdot}5}{10{\cdot}3}$

$=$ **0·49″ for adiabatic compression.**

8. *Find the kinetic energy stored in a disc flywheel when rotating at* 700 *r.p.m. Diam. of wheel,* 20 *ins.; thickness of wheel,* 5 *ins.; diam. of bore,* 3 *ins.; weight of* 1 *cubic inch of metal,* 0·25 *lb.*

Weight of wheel $W = (314{\cdot}16 - 7{\cdot}07)5 \times 0{\cdot}25 = 384$ lb.

Radius of gyration

$K = \sqrt{\frac{R_e^2 - R_i^2}{2}} = \sqrt{\frac{100 - 2{\cdot}25}{2}} = 6{\cdot}99$ ins. $= 0{\cdot}582$ ft.

Kinetic energy stored $\frac{WV^2}{2g} = \frac{W}{2g}\left(\frac{2\pi KN}{60}\right)^2$

$= \frac{384}{2 \times 32{\cdot}2} \times 2^2 \times \pi^2 \times 0{\cdot}582^2 \left(\frac{700}{60}\right)^2 =$ **10,900 ft.-lb.**

9. *Two spheres each weighing* 350 *lb. are attached to the ends of a bar, the distance between the centres of the spheres is* 104″. *The middle of the bar is attached to a shaft at right angles to the bar. The shaft is fitted with a brake drum* 17″ *diam. Find the tangential force which must be exerted on the surface of the drum in order to bring it to rest in two minutes when the shaft is making* 85 *r.p.m.*

Kinetic energy stored in spheres

$$= \frac{2 \times 350(\pi \times \frac{104}{12} \times \frac{85}{60})^2}{2 \times 32{\cdot}2} \text{(nly.)} = 16{,}200 \text{ ft.-lb.}$$

Work done against friction $= \frac{17 \times \pi \times F}{12} \times \frac{85 \times 2}{2} = 16{,}200$

$$F = \frac{16200 \times 12}{17 \times 3{\cdot}14 \times 85} = \mathbf{42{\cdot}8 \text{ lb.}}$$

The energy stored in the bar carrying the spheres; the moment of inertia of each sphere about its own centre; and the energy stored in the shaft and drum are neglected, since all are small compared with the quantity given above.

10. *In the case of a planing machine the table and job together weigh* 15 *tons and travel at* 100 *feet per minute. The driving pulley has a radius of gyration of* 1·5′, *it weighs* 50 *lb. and runs at* 318 *r.p.m. Find the energy stored in each.*

Table—

$$\text{Kinetic energy} = \frac{15 \times (\frac{100}{60})^2}{2 \times 32{\cdot}2} = \mathbf{0{\cdot}65 \text{ ft.-ton.}}$$

Wheel—

$$\text{Kinetic energy} = \frac{50\{2^2 \times \pi^2 \times 1{\cdot}5^2 \times (\frac{318}{60})^2\}}{2 \times 32{\cdot}2 \times 2240} = \mathbf{0{\cdot}86 \text{ ft.-ton.}}$$

It is by no means uncommon to find the energy stored in the driving pulley greater than that stored in the table and job, but the rotational energy should be kept as small as possible to reduce wear and tear on the belt and the machine generally.

11. *The total weight of a* 4*-wheel wagon is* 5 *tons. The wheels are* 40″ *diameter and their radius of gyration* 16″. *Each wheel weighs* 0·13 *ton. Find the kinetic energy stored in the wagon when moving at* 20 *m.p.h.*

Weight of wagon less wheels $= 5 - 0{\cdot}52$ ton $= 4{\cdot}48$ tons

Kinetic energy of wagon only $= \dfrac{4{\cdot}48 \times 29{\cdot}3^2}{2 \times 32{\cdot}2} = 59{\cdot}73$ ft.-tons

Linear k.e. of wheels $= \dfrac{4 \times 0{\cdot}13 \times 29{\cdot}3^2}{2 \times 32{\cdot}2} = 6{\cdot}93$ „

A point on the wheels at the radius of gyration moves with a velocity relative to the axle

$$29{\cdot}3 \times \tfrac{16}{20} = 23{\cdot}44 \text{ f.p.s.}$$

Kinetic energy of rotation of wheels $= \dfrac{4 \times 0{\cdot}13 \times 23{\cdot}44^2}{2 \times 32{\cdot}2} = 4{\cdot}4$ foot-tons.

Total kinetic energy = **71·1 foot-tons.**

This result could also have been arrived at thus—

$$\text{Total kinetic energy} = \frac{(5 + 4 \times 0{\cdot}13 \times (\tfrac{16}{20})^2)}{2 \times 32{\cdot}2} 29{\cdot}3^2$$

$$= \textbf{71·1 foot-tons.}$$

12. *Find the twisting moment required to accelerate a disc flywheel by* 75 *r.p.m. p.m. Weight of wheel,* 12 *lb. Radius of gyration,* 5 *inches.*

$$\text{Moment of momentum of a particle} = \frac{w}{g} vr = \frac{w}{g} \omega r^2$$

Where v is its linear velocity in f.p.s. and ω its angular velocity in rads. p.s.

The moment of momentum of the whole body is the sum of the m. of m. of its particles, *i.e.*

$$\sum \frac{w}{g} \omega r^2 = \omega \sum \frac{w}{g} r^2 = I\omega$$

where I is the moment of inertia of the body about its axis of rotation.

The rate of change of moment of momentum $= I \dfrac{\omega}{t} = I\lambda =$ twisting moment

Where λ is the angular acceleration.

This law is exactly analogous to the principle of linear momentum.

$$\text{The rate of change of linear momentum} = \frac{W}{g} f = \text{force acting.}$$

For this flywheel we have $\omega_0 = \dfrac{2\pi \times 75}{60} = 7{\cdot}85$ rads. p.s.,

but this increase in speed takes place in *one minute*, or the acceleration is 7·85 rads. p.s. *per minute*, or $\omega = \frac{7{\cdot}85}{60} = 0{\cdot}0131$ rads. p.s. *per second*.

(NOTE.—One is very liable to overlook this point.)

In this case $t = 1$ second.

Rate of change of moment of momentum or twisting moment $\left.\right\} I\omega = \frac{W}{g} K^2\omega$

$$= \frac{12}{32{\cdot}2} \times \left(\frac{5}{12}\right)^2 \times 0{\cdot}0131 = \mathbf{0{\cdot}0085\ lb.\text{-}foot.}$$

This could also have been arrived at thus—

Let a force P act on the shaft at a radius R.

Work done by force per minute = increase of kinetic energy per minute.

$$2P\pi RN = \frac{W}{2g}(V_2^2 - V_1^2) = \frac{W(2\pi\rho)^2}{2g}\left(\frac{N_2^2 - N_1^2}{60^2}\right)$$

$$2P\pi RN = \frac{W \times 4\pi^2\rho^2}{2g \times 3600}(N_2 + N_1)(N_2 - N_1)$$

But $\frac{N_2 + N_1}{2} = N$ and $N_2 - N_1 = 75$

Then $PR = \frac{W \times 4\pi^2\rho^2 \times 75}{2g \times 3600 \times \pi} = \frac{12 \times 4 \times 3{\cdot}14 \times 5^2 \times 75}{2 \times 32{\cdot}2 \times 3600 \times 12^2}$

$$= \mathbf{0{\cdot}0085\ lb.\text{-}foot.}$$

13. *An engine running at* 120 *r.p.m. and indicating* 150 *h.p. has some of its load thrown off, which reduces the i.h.p. to* 100. *Find the speed after an interval of* 2 *minutes if the turning moment remains constant. Mechanical efficiency* 80 *per cent. There is no governor. Rotating parts weigh* 2 *tons and have a radius of gyration of* 5·5 *feet.*

Initial mean effective turning moment $= \frac{80}{100} \times \frac{150 \times 550}{4\pi}$ lb.-ft.

If this remains constant while a load absorbing $50 \times 0{\cdot}8 = 40$ b.h.p. is thrown off, the net turning moment tending to increase the speed $= \frac{550}{4\pi} \times 40$ lb.-ft.

Hence the speed will increase according to the relation—

$$\frac{550 \times 40}{4\pi} = \frac{W}{g} K^2\alpha$$

Where α is the angular acceleration in rads. p.s.p.s.

$$\alpha = \frac{550 \times 40 \times 32{\cdot}2}{4 \times 3{\cdot}14 \times 2 \times 2240 \times 5{\cdot}5^2} = 0{\cdot}417$$

At the end of 2 minutes the speed will be

$$\omega = \omega_0 + \alpha \times 2 \times 60 = 4\pi + 0{\cdot}417 \times 120 = 72{\cdot}6$$

$$\therefore \text{ the speed} = \frac{60}{2\pi} \times 72{\cdot}6 = \mathbf{692\ r.p.m.}$$

14. *Find the weight of flywheel required for a three-cylinder gas engine to work under the following conditions :— I.H.P., 250 ; radius of gyration of wheel, 3′ 1″ ; speed, 200 r.p.m. ; coefficient of fluctuation of speed, 3 per cent. ; ratio of excess to mean work done per stroke, 1·4.*

$$W = \frac{48{,}500{,}000 \times m \times \text{I.H.P.}}{KN^3R_w^2} \quad \text{(Vol. I. p. 196)}$$

$$W = \frac{48{,}500{,}000 \times 1{\cdot}4 \times 250}{0{\cdot}03 \times 200^3 \times 3{\cdot}08^2} = \mathbf{7{,}440\ lb.}$$

15. *Find by an approximate method the weight of rim necessary for a flywheel on a single cylinder internal-combustion engine when working under the following conditions : Diameter, 7′ ; r.p.m., 240 ; I.H.P., 90 ; coefficient of speed fluctuation, 0·02.*

The engine misses alternate explosions, $m = 9$.

$$W = \frac{48{,}500{,}000 \times 9 \times 90}{0{\cdot}02 \times 240^3 \times 3{\cdot}5^2} = \mathbf{11{,}600\ lb.}$$

16. *Find the weight of flywheel-rim required for a steam-engine which indicates 90 H.P. when running at an average piston speed of 400 f.p.m. The ratio of the excess work to that done per stroke is 0·33, the fluctuation of velocity is one per cent. on either side of the mean. The radius of gyration of the rim is twice the stroke. R.p.m., 100.*

$$\text{Energy stored in wheel} = \frac{W}{2g}(V_1^2 - V_2^2) = \frac{WKV^2}{g}$$

$$\text{Excess work per stroke} = \frac{90 \times 33000 \times 0{\cdot}33}{200} = 4900 \text{ ft.-lb.}$$

Radius of gyration of rim = 4 feet

$$\left.\begin{matrix}\text{Vely. of a point at} \\ \text{radius of gyration}\end{matrix}\right\} = \frac{3{\cdot}14 \times 2 \times 4 \times 100}{60} = 41{\cdot}9 \text{ f.p.s.}$$

$\frac{WKV^2}{g}$ = 4,900 ft.-lb. $W = \frac{4900 \times 32{\cdot}2}{0{\cdot}02 \times 41{\cdot}9^2}$ = **4,500 lb.**

Check. $W = \frac{48{,}500{,}000 \times 0{\cdot}33 \times 90}{0{\cdot}02 \times 100^3 \times 16}$ = **4,500 lb.**

(See p. 196, Vol. I.)

17. *A flywheel has to be designed from the following data: I.H.P.*, 175; *excess energy to be stored per stroke*, 27 *per cent. of the whole energy developed per stroke; r.p.m.*, 120; *allowable variation of speed on either side of mean speed of flywheel*, 1 *per cent.; radius of gyration of rim of flywheel*, 4′. *What must be the weight of the flywheel rim in pounds?*

(See p. 196, Vol. I.)

$$W = \frac{48{,}500{,}000 \times m \times \text{I.H.P.}}{KN^3R^2} = \frac{48{,}500{,}000 \times 0{\cdot}27 \times 175}{0{\cdot}02 \times 120^3 \times 16}$$

$$W = \mathbf{4{,}140\ lb.}$$

18. *A dynamo provided with an auxiliary disc flywheel is driven by a gas engine. Find the weight of the gas engine flywheel to fulfil the following conditions: Speed of dynamo*, 1,600 *r.p.m.; speed of gas engine*, 170 *r.p.m.: diameter of dynamo disc wheel*, 24″; *thickness of dynamo disc wheel*, 4″; *weight of metal per c. inch*, 0·28 *lb.; I.H.P. of gas engine*, 55; *diameter of gas engine flywheel*, 8 *ft.; excess energy ratio*, 4·3; *coefficient of speed fluctuation*, 0·04.

Weight of dynamo wheel = $452{\cdot}4 \times 4 \times 0{\cdot}28$ = 507 lb.

V^2 of point at radius of gyration = $\frac{2^2 \times \pi^2 \times 1600^2}{2 \times 60^2}$ = 14,000

Energy stored in dynamo disc wheel = $\frac{507 \times 14000}{2 \times 32{\cdot}2}$ = 112,000 ft.-lb.

Average work done per stroke = $\frac{55 \times 33000}{340}$ = 5,350 ft.-lb.

$\frac{\text{Excess energy}}{2 \times \text{speed fluctuation}} = \frac{4{\cdot}3}{0{\cdot}08}$ = 54 strokes to be stored in flywheels

(See Vol. I. p. 198.)

5350×54 = 290,000 foot-lb.

Energy to be stored in engine wheels = 290,000 − 112,000 = 178,000 ft.-lb.

$$\frac{W_1V_1^2}{2g} = \frac{W_1 \times 4 \times 9{\cdot}98 \times 170^2 \times 4^2}{2 \times 32{\cdot}2 \times 60^2} = 17{,}800$$

$$W_1 = \mathbf{2{,}240\ lb.}$$

19. *A shearing machine for cutting red-hot ingots* 15 *inches square was driven by its own two-cylinder steam engine suitably geared down. The machine was originally designed without a flywheel on the crank shaft, but on running it the engine pulled up while shearing the ingot. Two flywheels were therefore mounted on the crankshaft, diameter* 4′ 6″. *R.p.m.* 200. *If the ingots had been sheared cold it would have been necessary to store* $15^2 \times 2 \times 15 = 6{,}750$ *inch-tons of energy in the wheels* (*see p.* 199, *Vol. I.*), *but with red-hot ingots about one-fourth of this, or say* 1,700 *inch-tons, would be sufficient. Find the section of the rim of the wheels.*

$$\text{Then } \frac{1700 \times 2240}{12} = \frac{WV^2}{2g} = \frac{W}{64{\cdot}4}(4{\cdot}5 \times 3{\cdot}14 \times \tfrac{200}{60})^2$$

$$W = 9{,}100 \text{ lb. or } 4{,}550 \text{ lb. each rim.}$$

The circumference at the c. of g. of each rim will be about 150″. Therefore the sectional area of each rim will be $\frac{4550}{150 \times 0{\cdot}26} = 117$ sq. inches or, say, **12″ wide and 10″ deep.**

In order to improve the appearance of the machine the maker put on what he termed a "snug flywheel," viz. a solid sleeve 24″ diameter and 8 feet long. The energy stored was—

$$\text{Weight} = 0{\cdot}785 \times 24^2 \times 0{\cdot}26 \times 96 = 11{,}290 \text{ lb.} \qquad K = \frac{D}{\sqrt{8}}$$

$$\text{Energy stored} = \frac{WV^2}{2g} = \frac{11290}{64{\cdot}4}\left(\frac{2\pi D \times \frac{200}{60}}{\sqrt{8}}\right)^2 = 3{,}850 \text{ ft.-lb.}$$

$$= 20{\cdot}4 \text{ inch-tons.}$$

Or about one per cent. of that stored in the larger flywheels mentioned above. The maker argued that since the weight of his "snug" flywheel was greater than that of the larger wheels, the energy stored must be greater; but the fallacy is obvious.

20. *Find the peripheral speed of a rotating ring required to produce stresses as follows :—*

(i) *Duralumin : Stress,* 2 *tons sq. in. ; wt.,* 0·09 *lb. c. in.*

(ii) *Cast iron : Stress,* 0·5 *ton sq. in. ; wt.,* 0·26 *lb. c. in.*

(iii) *Channel section wound with steel wire : Stress,* 7 *tons sq. in. ; wt.,* 0·28 *lb. c. in.*

In all cases we have $f=\frac{W_r V_w^2}{g}$ and $V_w=\sqrt{\frac{fg}{W_r}}$

(See p. 202, Vol. I.)

(i) $V_w=\sqrt{\frac{2\times 2240\times 32\cdot 2}{12\times 0\cdot 09}}=$ **365 f.p.s.**

(ii) $V_w=\sqrt{\frac{0\cdot 5\times 2240\times 32\cdot 2}{12\times 0\cdot 26}}=$ **107 f.p.s.**

(iii) $V_w=\sqrt{\frac{7\times 2240\times 32\cdot 2}{12\times 0\cdot 28}}=$ **388 f.p.s.**

Taking the energy stored per lb. of metal as 1 for cast iron that for duralumin is 11·5 and for a wire-wound rim 13. Thus the energy which may be stored in a flywheel depends upon the ratio

$$\frac{\text{Safe stress in metal}}{\text{Weight per cubic unit}}$$

21. *Calculate the approximate stress in the rims of flywheels having eight arms. Mean diam. of rim,* 10 *ft.; section of rim* (i) 9·5″ *wide* 12″ *deep*, (ii) 9·5″×2″; *speed,* 240 *r.p.m.*

$V_w=10\times 3\cdot 14\times\frac{240}{60}=125\cdot 6$ f.p.s.

(i) $A=9\cdot 5\times 12=114$ sq. ins., (ii) 19 sq. ins.

(i) $Z=\frac{9\cdot 5\times 144}{6}=228$, (ii) $\frac{9\cdot 5\times 4}{6}=6\cdot 3$.

Bending stress in rim (see p. 204, Vol. I.)

(i) $f=\frac{125\cdot 6^2}{10}\left(0\cdot 56+\frac{114\times 47^2}{680\times 5\times 228}\right)$

$=$ **1,400 lb. sq. in. nrly.**

(ii) $f=\frac{125\cdot 6^2}{10}\left(0\cdot 56+\frac{19\times 47^2}{680\times 5\times 6\cdot 3}\right)=$ **4,000 lb. sq. in. nrly.**

The stress in a simple rotating ring is $\frac{V_w^2}{10}=1{,}570$ lb. sq. in.

22. *A flywheel rotating at* 200 *r.p.m. is brought to rest in four seconds by means of a brake acting on a separate drum on the same shaft, which produces uniform retardation. Find the bending stress on the arms which may be regarded as "built in" at both ends—six. Section of rim,* 12 *ins.*×6 *ins.; mean diam.,* 8 *ft.; diameter of boss,* 12 *ins.; section of arms (rectangular),* 6 *ins.*×3 *ins.*

$$\frac{\text{Force exerted by retarded rim}}{\text{Weight of rim}} = \frac{\text{Retardation of rim}}{\text{Acceleration of gravity}}$$

$$\frac{P}{W} = \frac{\frac{V}{t}}{g}. \qquad \text{Weight of rim} = 5{\cdot}300 \text{ lb.}$$

$$P = \frac{WV}{gt} = \frac{5300 \times 8 \times 3{\cdot}14 \times 200}{32{\cdot}2 \times 4 \times 60} = 3{,}450 \text{ lb.}$$

or $\frac{3450}{6} = 575$ lb. per arm.

Length of arm between rim and boss $= 39''$.

$$\text{Bending moment} = \frac{575 \times 39}{2}. \qquad \text{Modulus of section} = \frac{3 \times 6^2}{6}.$$

$$\text{Bending stress} = \frac{575 \times 39 \times 6}{3 \times 6^2 \times 2} = 620 \text{ lb. per sq. in.}$$

23. *Find the maximum stress in a locomotive coupling rod due to centrifugal force, thrust, and secondary bending in a vertical plane. Length between pin centres,* $84''$*; section of rod,* $5'' \times 2''$*; r.p.m., of driving wheels,* 290*; radius of coupling cranks,* $13''$*; radius of driving wheels,* $36''$*; load on each driving wheel,* 9 *tons; coefficient of friction between wheel and rail,* 0·4.

Under the worst conditions, *i.e.* when the driving wheels are just slipping, the thrust may amount to

$$T = \frac{2 \times 9 \times 0{\cdot}4 \times 36}{13} = 19{\cdot}9 \text{ tons.} \quad \text{(See p. 209, Vol. I.)}$$

$$f = \frac{R_c N^2 l^2 y}{84000 K^2}\left(1 + \frac{Tl^2}{9{\cdot}61 EI}\right) \pm \frac{T}{A}$$

$$f = \frac{13 \times 290^2 \times 84^2 \times 2{\cdot}5 \times 12}{12 \times 84000 \times 5^2}\left(1 + \frac{19{\cdot}9 \times 84^2 \times 12 \times 2240}{9{\cdot}61 \times 30{,}000{,}000 \times 2 \times 125}\right) \pm \frac{19{\cdot}9}{10} \times 2240$$

$f = 9{,}180(1 + 0{\cdot}053) \pm 4{,}460$

$f = 14{,}120$ lb. sq. in. compression $=$ **6·3 tons sq. in.**

24. *Find the bending stress in a connecting rod* $80''$ *long, radius of crank* $15''$*. The rod is of rectangular section,* $5''$ *deep* $1\frac{1}{2}''$ *wide. R.p.m.,* 300.

$$f = \frac{RN^2 l^2 y}{164000 K^2} = \frac{1{\cdot}25 \times 300^2 \times 80^2 \times 2{\cdot}5 \times 12}{164000 \times 5^2}$$

$f =$ **5,280 lb.** sq. inch. (See Vol. I. p. 210.)

If the radius r of the crank be expressed in inches instead of feet, the expression becomes

$$f=\frac{rN^2l^2y}{1{,}968{,}000K^2}.$$

25. *Find the bending stress in a fluted coupling rod when working under the following conditions: Length of rod, 7′ 6″; depth of section, 5″; width of flanges, 2″; depth of flute, 3″; thickness of web, 0·75″; r.p.m., 200; radius of crank, 13″; thrust on rod, 6 tons; Young's modulus, 12,000 tons sq. in.*

$$f=\frac{RN^2l^2y}{84000K^2} \text{ for primary bending}$$

$$K^2=\frac{BH^3-bh^3}{12(BH-bh)}=\frac{2\times5^3-1\cdot25\times3^3}{12(2\times5-1\cdot25\times3)}=2\cdot88$$

$$f=\frac{13\times200^2\times90^2\times2\cdot5}{12\times84000\times2\cdot88}=3{,}630 \text{ lb. sq. in}$$

The deflection due to centrifugal loading (Vol. I. p. 209)

$$\delta=\frac{5cl^4}{384EI}=\frac{cl^2}{8I}\times\frac{5l^2}{48E}=\frac{f}{y}\times\frac{5l^2}{48E}$$

$$\delta=\frac{3630}{2\cdot5}\times\frac{5\times90^2}{48\times12000\times2240}=0\cdot045''$$

Area of section = 6·25 sq. ins.

Compressive stress due to thrust $=\frac{6\times2240}{6\cdot25}=2{,}150$ lb. sq. in.

Bending moment due to thrust $=6\times2{,}240\times0\cdot045$.

Bending stress due to thrust $=\frac{6\times2240\times0\cdot045\times2\cdot5}{6\cdot25\times2\cdot88}=84$ lb. sq. in.

Compressive stress due to all causes $=3630+2150+84=$ **5864 lb. sq. in.**

26. *A disc wheel 12 ins. diam. and weighing 5 lb. is mounted on a shaft, but is found to be $\frac{1}{8}$ inch out of truth sideways at the edge. Find the bending moment to which this will give rise in the shaft when running at 2,000 r.p.m. Is this bending stress repeated or constant?*

For every particle W_1 centrifugal force $=\frac{W_1}{g}\cdot\omega^2r$ and corresponding moment $m=\frac{W_1}{g}\cdot\omega^2r^2\sin\theta$.

$$\text{Total couple} = \frac{\omega^2}{g} \sin\theta \,.\, Wk^2$$

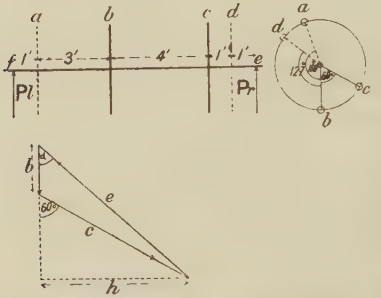
where k=radius of gyration about a diameter of the disc wheel.

$$\omega = 2000 \times \frac{2\pi}{60} \qquad k^2 = \frac{d^2}{16} = \frac{1}{16} \qquad \sin\theta = \frac{\frac{1}{8}}{6} = \frac{1}{48}.$$

$$\text{Hence couple} = \left(\frac{400\pi}{6}\right)^2 \times \frac{1}{32 \cdot 2} \times \frac{1}{48} \times 5 \times \frac{1}{16} = \mathbf{8{\cdot}84\ lb.\text{-}feet.}$$

This couple gives rise to a bending moment of the same value in the shaft, and since the forces always act at the same points relative to the disc and shaft, the stress will be constant and not repeated.

27. *A shaft supported on bearings* 10 *feet apart has two pulleys upon it which are out of balance, the one weighs* 100 *lb. and its c. of g. is* 0·15 *foot from the axis of rotation, the other weighs* 200 *lb. and its c. of g. is* 0·2 *ft. from the axis of rotation. The planes of the pulleys are* 4 *feet and* 8 *feet from the left-hand bearing respectively, and the angle between the radii containing the cs. of g. is* 60°. *Find the forces acting on the two bearings due to centrifugal force when the shaft makes* 100 *r.p.m.*



If the pulleys are balanced by two rotating weights in planes 1 *foot from each bearing the c. of g. of each being at* 1 *foot radius, find the balance weights and their respective positions relative to the* 100 *lb. pulley.*

This problem can be dealt with graphically by the method given in Vol. I. Fig. 223, or analytically thus—

Centrifugal force of 100 lb. weight $= 0{\cdot}00034WRN^2$
$= 0{\cdot}00034 \times 100 \times 0{\cdot}15 \times 100^2 = 51$ lb.

Centrifugal force of 200 lb. weight = 136 lb.

Taking moments about P_l and constructing the moment diagram as shown—

$b = 51 \times 4 = 204$ foot-lb. $\qquad c = 136 \times 8 = 1{,}088$ foot-lb.

$e = P_r \times 10$

Then $b^2+c^2+2bc \cos 60=e^2$

$204^2+1{,}088^2+2\times204\times1{,}088\times0{\cdot}5=e^2$

$e=1{,}203$ foot-lb. $P_r=\frac{1203}{10}=$**120·3 lb.**

Taking moments about P_r, by the same reasoning—

$b=51\times6=306$ foot-lb. $c=136\times2=272$ foot-lb.

$306^2+272^2+2\times306\times272\times0{\cdot}5=f^2=251{,}000$

$f=501$ $P_l=\frac{501}{10}=$**50·1 lb.**

These are the forces acting on the bearing when there are no balance weights.

Taking a plane of reference through a. Since the centrifugal force is proportional to the moment

$b=15\times3=45$ ft.-lb. $c=40\times7=280$ ft.-lb.

Then $45^2+280^2+2\times45\times280\times0{\cdot}5=d^2$

$d^2=93{,}025$ and $d=305$

The balance weight at $d=\frac{305}{8}=$**38·1 lb.**

Taking a plane of reference through d

$b=15\times5=75$ ft.-lb. $c=40\times1=40$ ft.-lb.

Then $75^2+40^2+2\times75\times40\times0{\cdot}5=10{,}225$.

The balance weight at $a=\frac{101}{8}=$**12·6 lb.**

$h=c \sin 60=34{\cdot}6$

$$\text{Sin } a=\frac{h}{e}=\frac{34{\cdot}6}{101}=\mathbf{0{\cdot}343.}$$

$a=$**20°** or a is 180—20=160° ahead of b.

Similarly for the other plane of reference—

$h=280\times0{\cdot}866=242{\cdot}5$

$$\text{Sin } a=\frac{242{\cdot}5}{305}=0{\cdot}795 \qquad a=53°$$

or d is 180—53=**127° ahead.**

The reader would do well to check this graphically. The problem might have been solved thus—

The vector sum $\Sigma WRy=0$.

Taking the plane of reference a—

$8\times W_dR_d \cos a=3\times15+7\times40 \cos 60°=185$

$8\times W_dR_d \sin a=7\times40 \sin 60°$

$\therefore \begin{cases} W_dR_d=38{\cdot}1 & \mathbf{W_d=38{\cdot}1 \text{ lb.}} \\ \tan a=1{\cdot}31 & a=\mathbf{53°} \end{cases}$

Similarly by taking plane of reference d

$\mathbf{W_a=12{\cdot}6 \text{ lb.}}$ $a'=$**20°**

28. *A pulley on a rotating shaft weighs* 1,000 *lb. and has a weight of* 5 *lb. attached to the rim at a distance of* 18″ *from the centre. The pulley is midway between the two bearings. Find the speed at which the pulley must run in order to make the shaft lift out of its bearings.*

The shaft will lift when the upward centrifugal force exceeds the dead weight.

$0{\cdot}00034WRN^2=1{,}005$ $0{\cdot}00034\times5\times1{\cdot}5\times N^2=1{,}005$

$N=$**628 r.p.m.**

When the weight is at the bottom of its path the load on the two bearings will be 2,010 lb.

29. *A shaft* A, B, C, D, *has a rotating weight of* 100 *lb. attached at* A *at a radius of* 3 *feet and another rotating weight of* 200 *lb. at a radius of* 2 *feet at* C. *The arm of the latter is at* 90° *to that of the former. A bearing* B *is one foot from* A *and another bearing* D *is* 4 *feet from* A. *The weight* C *is* 2′ 6″ *from* A. *Find the force acting on each bearing. R.p.m.* 120.

Centrifugal force due to $A=0{\cdot}00034\times100\times3\times120^2=1{,}469$ lb. $(1{,}469+100)12=P_D\times36$.

$P_D=523$ lb. Due to A acting upwards.

$P_B=1{,}569+523=2{,}092$ lb. acting downwards.

Centrif. force due to $B=0{\cdot}00034\times200\times2\times120^2=1958$ lb.

$P_D=P_B=979$ lb. due to B.

When the 2,092 acts downwards at B the 979 acts horizontally, and the resultant is $\sqrt{2092^2+979^2}=2{,}310$ lb.

When the 523 acts upwards at D the 979 acts horizontally, and the resultant is $\sqrt{523^2+979^2}=1{,}110$ lb.

The corresponding bearing pressures can be found for other positions by similar reasoning.

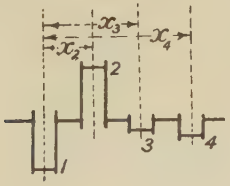
30. *The cylinders of a four-cylinder vertical engine are equally spaced. The weight of each of the reciprocating parts of cylinders* 2 *and* 3 *is* 1 *ton. Cranks* 2 *and* 3 *are* 100 *degrees apart. Find graphically or otherwise the angles of the other cranks relatively to* 2, *also the weights of the reciprocating parts of cylinders* 1 *and* 4, *so that the engine shall be in balance for primary forces and couples.*

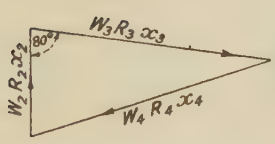
$R=1$ in each case.

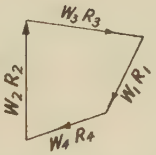
$W_2R_2x_2=1$
$W_3R_3x_3=2$ } Hence the WRx diagram can be drawn.
$W_4R_4x_4=W_4R_4\times 3=2{\cdot}06$ from diagram

$W_4R_4=\dfrac{2{\cdot}06}{3}=0{\cdot}69$ ton-ft. and acts in the direction shown.

Since $R_4=1$ $W_4=0{\cdot}69$ ton.







We can now draw the WR diagram and from it we find $W_4R_4=0{\cdot}69$.

Cranks.	*Angle.*
1 and 2	**208°**
2 and 3	**100°**
2 and 4	**253°**

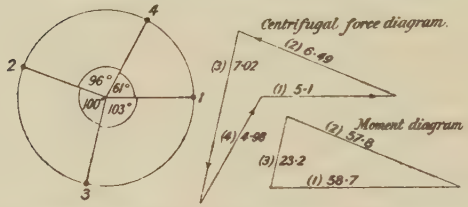
W_1=**0·69 ton,** W_2 and W_3=**1·0 ton each,** W_4=**0·69 ton.**

31. *It is required to balance a four-cranked engine.*

No. of crank.	Distance of centre line of crank from reference plane.	Centrifugal force at crank pin.	Angles between cranks.	
1	11·5	5·10	1 and 3	103°
2	8·9	6·49	3 and 2	100°
3	3·3	7·02	2 and 4	?
4	0·0	?	1 and 4	?

Sequence of cranks 1.3.2.4.
Find the missing quantities.

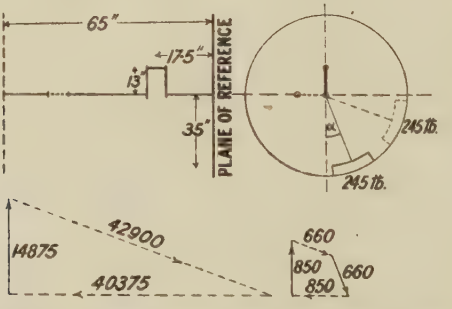
Crank.	Moment.
1	$11{\cdot}5 \times 5{\cdot}1 = 58{\cdot}7$
2	$8{\cdot}9 \times 6{\cdot}49 = 57{\cdot}8$
3	$3{\cdot}3 \times 7{\cdot}02 = 23{\cdot}2$



The moment diagram can therefore be drawn and from it the centrifugal force diagram.

The length of the line 4 on the centrifugal force diagram to the same scale gives the centrifugal force at the crank 4 as 4·98, and the angle between 2 and 4 is 96°, and between 1 and 4 is 61°.

32. *An inside cylinder locomotive (not coupled) is balanced for two-thirds of the reciprocating and the whole of the rotating parts. Find the amount and the angular position of the balance weights on the wheels. Construct balancing*



diagrams to show the couples and the forces acting on the crank-shaft. Weight of reciprocating parts, 600 *lb.; weight of rotating parts,* 450 *lb.; radius of cranks,* 13″; *radius of centre of gravity of balance weights,* 35″; *cylinder centres,* 30″; *wheel centres,* 65″.

$$W_B = \frac{\sqrt{2}W_{BO}}{2y}\sqrt{y^2+c^2} \text{ (see Vol. I. p. 222).}$$

$$W_B = \frac{\sqrt{2}\{\frac{13}{35}(\frac{2}{3}\times 600+450)\sqrt{65^2+30^2}\}}{2\times 65} = 245 \text{ lb. bal. wt.}$$

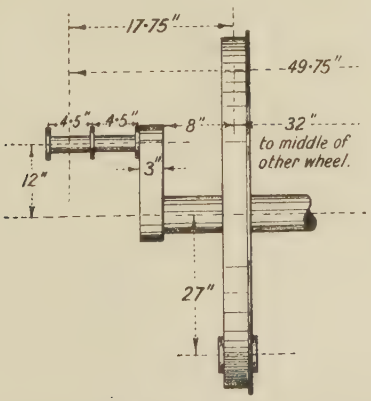
Equivalent balance weight at crank radius $= \frac{245\times 35}{13} = 660$ lb.

$$\tan\alpha = \frac{y-c}{y+c} = \frac{35}{95} = 0{\cdot}368 \qquad \alpha = 20^\circ\ 15'.$$

The following table is deduced from the vector diagrams.

	Distance.	Equivalent weight at crank radius.	Equivalent moment.
R.H. bal. wt.	0	660	0
R.H. crank	17·5	850	14,900
L.H. crank	47·5	850	40,400
L.H. bal. wt.	65	660	42,900

33. *Find the balance weight required and its position for the narrow gauge locomotive as shown in the figure. The weight of the reciprocating parts is* 450 *lb. to be fully balanced. The weight of the rotating parts is* 520 *lb. The weight of each coupling rod is* 250 *lb., i.e.* 125 *lb. on each pin. The coupling rods are on the inner pins.*



The balance weights for the reciprocating and rotating parts on one side only. (For notation, see Vol. I. p. 226.)

$W_p + W_r = 970$ lb. $\qquad W_{B0} = \frac{970 \times 12}{27} = 432$ lb.

On near wheel $W_{B1} = \frac{432 \times 49{\cdot}75}{32} = 672.$

On far wheel $W_{B2} = \frac{432 \times 17{\cdot}75}{32} = 240.$

(Coupling rod) $W_{BC} = \frac{125 \times 12}{27} = 55{\cdot}5$ lb.

On near wheel $W_{BC1} = \frac{55{\cdot}5 \times 45{\cdot}25}{32} = 78$ lb.

On far wheel $W_{BC2} = \frac{55{\cdot}5 \times 13{\cdot}25}{32} = 23.$

Total on near wheel $672 + 78 = 750.$
„ far „ $240 + 23 = 263.$

The total balance weight on each wheel is

$$W_B = \sqrt{750^2 + 263^2} = 795 \text{ lb.}$$

$$\tan \alpha = \frac{263}{750} \qquad \alpha = 19^\circ\ 20'.$$

Check—$W = \frac{(W_r + W_p)R}{R_b} \cdot \frac{\sqrt{2}\sqrt{y^2 + C_g^2}}{2y}$ (see p. 226, Vol. I.)

Where $C_g = \frac{1095 \times 67{\cdot}5 + 125 \times 58{\cdot}5}{1095 + 125} = 66{\cdot}6$

$$W = \frac{(520 + 125 + 450)12}{27} \cdot \frac{\sqrt{32^2 + 66{\cdot}6^2}}{1{\cdot}41 \times 32} = \mathbf{795 \text{ lb.}}$$

34. *Find the amount and position of the balance weights on an outside cylinder locomotive to fulfil the condition that the whole of the rotating parts are balanced and as much of the reciprocating parts as possible consistent with the condition that the wheels shall not lift when the locomotive is running at* 80 *miles per hour. Weight of reciprocating parts per cylinder,* 560 *lb.; weight of rotating parts per cylinder,* 400 *lb.; radius of crank,* 11″; *radius of balance weight,* 32″; *radius of wheel,* 40″; *cylinder centres,* 66″; *wheel centres,* 58″; *weight on each wheel,* 8 *tons. What proportion of the reciprocating parts is balanced?*

For symbols, see p. 222, Vol. I.

Consider first the rotating parts only, then—

$W_{B0}=\frac{400\times 11}{32}=137{\cdot}5$ lb.

$W_{Br}=\frac{137{\cdot}5\times 1{\cdot}41}{2\times 58}\sqrt{58^2+66^2}=147$ lb. for rotating parts.

80 m.p.h.=336 r.p.m.

If the wheels are on the point of lifting at this speed and W is the greatest balance weight permissible for the reciprocating parts, we have—

$0{\cdot}00034\ W\frac{32}{12}\times 336^2=8\times 2{,}240$ $\qquad W=175$ lb.

Total balance weight $147+175=$**322 lb.**

Since 147 lb. balances 400 lb. rotating parts, 175 lb. will balance x lb., where

$$\frac{147}{400}=\frac{175}{x} \qquad x=476 \text{ lb.}$$

Thus $\frac{476}{560}=85$ per cent. of the reciprocating parts are balanced.

Check— $W_{B0}=\frac{(400+476)11}{32}=$**301 lb.**

$W_{Br}=\frac{147\times 301}{137{\cdot}5}=$**322 lb.**

$\tan\theta=\frac{58-66}{58+66}=-0{\cdot}964 \qquad \theta=$**3° 40′.**

35. *Find the speed at which an inside cylinder locomotive driving wheel will lift if all the reciprocating parts are fully balanced when running under the following conditions. Weight of reciprocating parts per cylinder,* 400 *lb.; weight of rotating parts per cylinder,* 380 *lb; radius of cranks,* 13″; *radius of balance weights,* 34″; *radius of driving wheels,* 42″; *cylinder centres,* 30″; *wheel centres,* 74″; *weight on each wheel,* 9 *tons.*

For symbols, see p. 222, Vol. I.

$W_{B0}=(400+380)\frac{13}{34}=300$ lb.

$W_B=\frac{1{\cdot}41\times 300}{2\times 74}(74^2+30^2)^{\frac{1}{2}}=228$ lb.

$W_B'_0=\frac{380\times 13}{34}=145$ lb. for rotating parts only.

$W_B=\frac{1{\cdot}41\times 145}{2\times 74}(74^2+30^2)^{\frac{1}{2}}=110$ lb. for rotating parts only.

Thus the unbalanced portion of the 228 lb. is

$$228-110=118 \text{ lb.}$$

The centrifugal force due to this is 9 tons when—

$$\frac{WV^2}{gR}=0{\cdot}00034\times 118\times\tfrac{34}{12}\times N^2=9\times 2240$$

$$N=421 \text{ r.p.m.} \qquad \frac{421\times 3{\cdot}14\times 7\times 60}{5280}=\textbf{105 m.p.h.}$$

36. *Find the magnitude of the "hammer blow" on the rails, also the maximum "elbowing" or "swaying" moment in the case of an inside cylinder locomotive in which no attempt is made at balancing. Revolving weights at each crank pin, 350 lb.; reciprocating weights at each crank pin, 400 lb.; distance between rail centres, 5 feet; stroke, 25"; cylinder centres, 23"; radius of balance-weights, 2' 6"; radius of driving wheels, 3' 6"; speed, m.p.h., 50.*

50 m.p.h.=200 r.p.m.

Centrifugal force of weights on each crank

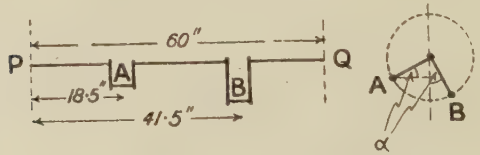
$$=0{\cdot}00034\times 750\times\frac{25}{2\times 12}\times 200^2=10{,}600 \text{ lb.}$$

Horizontal component=10,600 sin θ, where θ is the angle of the crank to the vertical.

This has its maximum value when $\theta=45°$, hence—

$$\text{Maximum swaying moment}=10{,}600\times\frac{1}{\sqrt{2}}\times\tfrac{23}{12}=\textbf{14,400 lb.-ft.}$$

$$\left.\begin{matrix}\text{C.f. of weight at}\\ \text{crank pin}\end{matrix}\right\}=350\times 0{\cdot}00034\times\frac{25}{2\times 12}\times 200^2=4{,}960 \text{ lb.}$$



Hammer blow at Q due to A $=4{,}960 \sin\alpha\,\frac{18{\cdot}5}{60}$

" " " B $=4{,}960\cos\alpha\,\frac{41{\cdot}5}{60}$

" " " both $=\frac{4960}{60}(18{\cdot}5\sin\alpha+41{\cdot}5\cos\alpha)$

Maximum " " " $=\frac{4960}{60}\sqrt{18{\cdot}5^2+41{\cdot}5^2}$

$=$**3,750 lb. at the rail.**

37. *Find the speed at which a governor arm will make an angle of 60° to the axis of rotation when the weight of the ball is 6 lb. and its centre is 11 inches from the suspension pin, and the weight of the arm, which is of uniform section, is 4 lb.*

Diameter of ball, 3·54″.

$$\text{Length of arm to edge of ball} = 11 - \frac{3{\cdot}54}{2} = 9{\cdot}23''$$

$$\left.\begin{matrix}\text{Distance of c. of g. of arm from axis}\\ \text{of rotation}\end{matrix}\right\} = \frac{9{\cdot}23}{2}\sin 60° = 4''$$

$$\text{Height of c. of g. of arm below apex pin} = \frac{9{\cdot}23}{2}\cos 60° = 2{\cdot}31''$$

$$\left.\begin{matrix}\text{Distance of c. of g. of ball from axis}\\ \text{of rotation}\end{matrix}\right\} = 11\sin 60° = 9{\cdot}52''$$

$$\text{Height of c. of g. of ball below apex pin} = 11\cos 60° = 5{\cdot}5''$$

Centrifugal moment of arm and ball

$$= 0{\cdot}00034N^2\left(6\times\frac{9{\cdot}52}{12}\times 5{\cdot}5 + \tfrac{4}{3}\times 4\times\tfrac{4}{12}\times 2{\cdot}31\right)$$

$$= 0{\cdot}00034N^2(26{\cdot}2 + 4{\cdot}1) = 0{\cdot}0103N^2$$

(see p. 242, Vol. I.)

Centripetal moment $= 6\times 9{\cdot}52 + 4\times 4 = 73{\cdot}1$

$$N^2 = \frac{73{\cdot}1}{0{\cdot}0103} = 7{,}100 \qquad N = \mathbf{84{\cdot}3\ r.p.m.}$$

38. *A governor arm consists of a plain bar 12″ long pivoted at a distance of 2·5″ from the axis of rotation (not crossed). Find the speed of rotation when the arm makes an angle of 45° with the vertical. What would be the speed if the commonly used approximation were correct?*

(See p. 240, Vol. I.)

$$\left.\begin{matrix}\text{Centrifugal moment}\\ \text{in inch-pounds}\end{matrix}\right\} = 0{\cdot}00034N^2\Sigma(WRh)$$

$$= \frac{0{\cdot}00034}{12}N^2 w\int_0^L (l\sin\alpha + x)l\cos\alpha\, dh \quad (l \text{ in inches})$$

$$= \frac{0{\cdot}00034}{12}\times wN^2\cos\alpha\left(\frac{\sin\alpha L^3}{3} + \frac{L^2 x}{2}\right)$$

$$\text{Moment of arm about pivot} = \frac{wL^2\sin\alpha}{2}$$

Then $\frac{0{\cdot}00034}{12} w N^2 \cos\alpha \cdot L^2\left(\frac{\sin\alpha L}{3}+\frac{x}{2}\right)=\frac{wL^2\sin\alpha}{2}$

$$\frac{0{\cdot}00034}{12}N^2\times 0{\cdot}707\left(\frac{0{\cdot}707\times 12}{3}+\frac{2{\cdot}5}{2}\right)=\frac{0{\cdot}707}{2}$$

$$N=\mathbf{65{\cdot}7\ r.p.m.}$$

Common approximation—

$$\left.\begin{matrix}\text{Centrifugal moment}\\ \text{in inch-pounds}\end{matrix}\right\}=\frac{0{\cdot}00034 w N^2 L}{12}\left(\frac{L\sin\alpha}{2}+x\right)\frac{L\cos\alpha}{2}$$

$$\frac{0{\cdot}00034}{12} w N^2\left(\frac{L\sin\alpha}{2}+x\right)\frac{L\cos\alpha}{2}=\frac{wL^2\sin\alpha}{2}$$

$$\frac{0{\cdot}00034}{12}N^2\left(\frac{12\times 0{\cdot}707}{2}+2{\cdot}5\right)\frac{0{\cdot}707}{2}=\frac{0{\cdot}707}{2}$$

$$N=\mathbf{72{\cdot}3\ r.p.m.}$$

39. *A bar of uniform section is pivoted at one end and rotates about a vertical axis passing through the pivot like a governor arm. Find the speed when it makes an angle of* 30° *with the axis of rotation. Length of bar,* 20 *ins.*

Find the natural period of oscillation of the bar. Why is it important to arrange for the speed to be materially different from the natural period of oscillation ?

(See p. 241, Vol. I.)

$$\left.\begin{matrix}\text{Centrifugal moment}\\ \text{acting on the arm}\end{matrix}\right\}=\frac{W_a\omega^2R^2}{36g\tan\theta}\left\{\begin{matrix}\text{Where R is measured}\\ \text{in ins.}\end{matrix}\right.$$

Righting moment due to weight of arm $=\frac{W_aR}{2}$

$$\frac{W_aR}{2}=\frac{W_a\omega^2R^2}{36g\tan\theta}$$

$$\omega=\sqrt{\frac{18g}{20\cos\theta}}=\sqrt{\frac{18\times 32{\cdot}2}{20\times 0{\cdot}866}}=5{\cdot}8 \text{ radians per sec.}$$

$$N=\mathbf{55{\cdot}5\ r.p.m.}$$

Time of oscillation (see p. 262, Vol. I.)

$$t=2\pi\sqrt{\frac{K^2+l_g^2}{l_gg}}=6{\cdot}38\sqrt{\frac{\frac{1{\cdot}666^2}{12}+0{\cdot}833^2}{0{\cdot}833\times 32{\cdot}2}}=1{\cdot}17 \text{ secs.}$$

$$N_0=\frac{60}{1{\cdot}17}=\mathbf{51{\cdot}3\ complete\ oscillations\ p.m.}$$

If there is a small cyclical variation in speed in the engine the time of oscillation of the arm may coincide with that of the governor, in which case the arm will oscillate in unison with the engine and thus cause instability in the governor.

40. *Calculate the error involved in taking the weight of the arm as acting at the c. of g. in the case of* (i) *an open armed governor*, (ii) *a crossed arm governor*, (iii) *when the arm is pivoted on the axis of rotation. Length of arm*, L=10″. *Eccentricity*, x=2″. $\alpha=15°, 30°, 45°$.

(i) Let w be the weight of the arm per unit length.
α be the angle between the arm and spindle.
h be the height from centre of ball to apex pin.

$$\left.\begin{matrix}\text{Centrifugal moment on}\\ \text{element of arm}\end{matrix}\right\}=0{\cdot}00034w\,.\,dl\left(\frac{r+x}{12}\right)N^2h$$

$$=K(l\sin\alpha+x)l\cos\alpha\,.\,dl$$

$$\left.\begin{matrix}\text{Centrifugal mo-}\\ \text{ment on arm}\end{matrix}\right\}=K\cos\alpha\left\{\left(\sin\alpha\int_{l=0}^{l=L}l^2\,.\,dl\right)+\int_{l=0}^{l=L}lx\,.\,dl\right\}$$

$$=K\cos\alpha\left(\frac{\sin\alpha L^3}{3}+\frac{L^2x}{2}\right)$$

If the weight of the arm be assumed to act at the c. of g. for the centrifugal force we have—

$$\text{Centrifugal moment}=0{\cdot}00034wL\left(\frac{R}{2}+x\right)N^2\frac{h'}{2}$$

$$=KL\left(\frac{L\sin\alpha}{2}+x\right)\frac{L\cos\alpha}{2}$$

$$\text{Ratio: }\frac{\text{Correct}}{\text{Approximation}}=\tfrac{4}{11}\left(\frac{L\sin\alpha+1{\cdot}5x}{L\sin\alpha+2x}\right)$$

Ratio: $\alpha=15°$, **1·13** ; $\alpha=30°$, **1·18** ; $\alpha=45°$, **1·21.**

Thus the error is not constant as the governor rises.

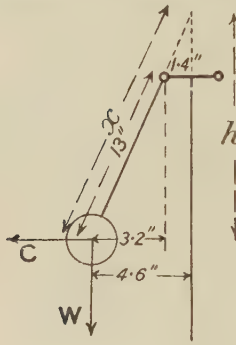
(ii) For a crossed arm governor, the sign of α in the above expression becomes — in both the numerator and denominator. Then we have

Ratio: $\alpha=15°$, **0·28** ; $\alpha=30°$, **2·0** ; $\alpha=45°$, **1·33.**

When $\sin\alpha=\frac{2}{11}$, *i.e.* when the middle of the arm is on the axis of rotation, the approximate method gives no centrifugal moment.

(iii) x is zero in this case and the ratio is $\frac{4}{3}$ for all positions of the governor.

41. *Find the speed at which a Watt governor will lift when each arm is suspended from a pin situated* 1·4 *ins. from the axis of rotation (not crossed). Length of arm* 13 *ins. to centre of ball, radius of ball path* 4·6 *ins.—neglecting the arm.*



$x=\frac{13\times4\cdot6}{3\cdot2}=18\cdot69''$

$h=\sqrt{18\cdot69^2-4\cdot6^2}=18\cdot1''=1\cdot51$ ft.

$Ch=WR$

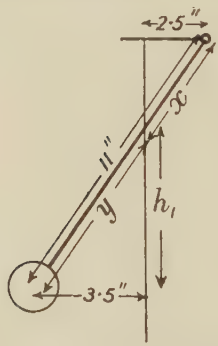
$0\cdot00034\times W\times R\times N^2\times h=WR$

$N^2=1{,}960.$

N=44·2 r.p.m.

Check—$N=\sqrt{\frac{0\cdot816}{h}}=\sqrt{\frac{0\cdot816}{1\cdot51}}=0\cdot738$ r.p.s.=**44·2 r.p.m.**

42. *The arms of a crossed-arm governor are pivoted at points situated* 2·5″ *from the axis of rotation; the length of the arms is* 11″ *to the centre of the balls; at their lowest position the ball centres are* 7″ *apart, and at their highest* 12″ *apart. Find the maximum and minimum speeds for the governor.*



$\frac{x}{y}=\frac{2\cdot5}{3\cdot5}$ $\qquad x+y=11$

$x=4\cdot6''$ $\qquad y=6\cdot4''$

$h_1=\sqrt{6\cdot4^2+3\cdot5^2}=5\cdot36''$

$N_1=\sqrt{\frac{35230}{5\cdot36}}=$**81 r.p.m.**

Similarly $\begin{cases} h_2=4\cdot93'' \\ N_2=\textbf{84 r.p.m.} \end{cases}$

43. *A cut-off gear is actuated by a Hartnell governor, the resistance it has to overcome is* 20 *lb. Find the necessary strength of spring (in lb. per in. of compression) to ensure that the speed shall not vary by more than* 1 *per cent. above or below the mean. Mean speed,* 300 *r.p.m.; weight of*

each ball, 7 lb.; radius of ball path, 8″ to 10″; arms of equal length.

Let S=load on spring in pounds.

When the governor is rising and approaching its highest speed the virtual effect of the friction is to increase the strength of the spring; then—

$$0{\cdot}00034\times7\times\tfrac{10}{12}\times303^2\times h=\left(\frac{S_1}{2}+10\right)h$$

$$(0{\cdot}00034\times7\times\tfrac{10}{12}\times303^2-10)2=S_1=344 \text{ lb.}$$

When the governor is falling and approaching its lowest speed the virtual effect of the friction is to reduce the strength of the spring; then—

$$0{\cdot}00034\times7\times\tfrac{8}{12}\times297^2\times h=\left(\frac{S_2}{2}-10\right)h$$

$$S_2=300 \text{ lb.}$$

The spring should therefore compress 2 inches under a difference of 44 lb. **or 22 lb. per inch.**

44. *Find the distance* x *of the suspension pin from the axis of rotation for a crossed arm governor which is required to run at the same speed when the arms make angles with the spindle of* 30° *and* 45° *respectively. The length of the arm is* 12 *inches.*

Let h=" height " of the governor (feet)

R=radius of the ball path (feet)

$$0{\cdot}00034WRN^2h=W(R+x),\quad N^2=\frac{R+x}{0{\cdot}00034Rh}=\frac{R_1+x}{0{\cdot}00034R_1h_1}$$

$$R+x=1\times\sin 30^\circ=0{\cdot}5' \qquad R_1+x=1\times\sin 45^\circ=0{\cdot}707'$$

$$h=1\times\cos 30^\circ=0{\cdot}866' \qquad h_1=1\times\cos 45^\circ=0{\cdot}707'$$

$$\frac{0{\cdot}5}{(0{\cdot}5-x)0{\cdot}866}=\frac{0{\cdot}707}{(0{\cdot}707-x)0{\cdot}707} \qquad x=0{\cdot}217 \text{ foot}=\mathbf{2{\cdot}6 \text{ inches.}}$$

45. *Find the speed at which the governor shown in the figure will run. The arm weighs* 4 *lb., the ball* 12 *lb., and the lower link* 3 *lb., sleeve* 2 *lb.*

$$\text{Equivalent weight of arm when running}=\frac{4\times4}{3}=5{\cdot}33 \text{ lb}$$

$$\text{Equivalent weight of lower link}=\frac{4\times3}{3}=4 \text{ lb.}$$

Moment of ball, arm and link about pin

$12\times7\cdot5+4\times3\cdot187+3\times2\cdot5+5\times2=120\cdot25$ inch-lb.

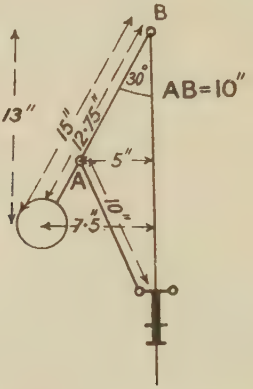
Centrifugal force acting on lower link

$=0\cdot00034\times4\times\frac{3}{12}$ approx. $\times N^2=0\cdot00034N^2$

One half of this acts on the pin A$=0\cdot00017N^2$.

Moment of this force about apex pin

$=8\cdot66\times0\cdot00017N^2=0\cdot0015N^2$



Moment of cen. force on ball about apex pin

$$=0\cdot00034\times12\times\frac{7\cdot5}{12}\times N^2\times13=0\cdot033N^2$$

Moment of cen. force on arm about apex pin

$$=0\cdot00034\times5\cdot33\times\frac{12\cdot75}{12\times4}\times N^2\times\frac{0\cdot866\times12\cdot75}{2}=0\cdot0027N^2$$

Total centrifugal moment$=0\cdot0375N^2=120\cdot25$

$N=3{,}207$ $\quad$ $N=$**56·6 r.p.m.**

If we neglect the weight of the arm and link

$N^2=\frac{35230}{63}=2{,}710$ $\quad$ $N=$**52·1 r.p.m.**

46. *In a Porter governor the links are* 10 *inches long. The resistance to be overcome in moving the automatic expansion gear is equivalent to a force of* 8 *lb. acting in all cases against the movement of the sleeve both when rising and falling. Find* (a) *the travel of the sleeve,* (b) *the speed of the governor when the balls are at their extreme positions*

both when rising and when falling. Weight of each ball, 5 lb.; central weight, 50 lb.; apex angle varies from 30° to 45°.

$h_{30}=8{\cdot}66''$ $h_{45}=7{\cdot}07''$ $\therefore$ travel of sleeve $=2\times1{\cdot}59=3{\cdot}18''$

When beginning to rise

$$N_{30}^2=\frac{35230}{8{\cdot}66}\left(\frac{5+50+8}{5}\right)=51{,}500, \quad N_{30}=227 \text{ r.p.m.}$$

When reaching top position

$$N_{45}^2=\frac{35230}{7{\cdot}07}\left(\frac{5+50+8}{5}\right)=62{,}800, \quad N_{45}=251 \text{ r.p.m.}$$

When beginning to fall

$$N_{45}^2=\frac{35230}{7{\cdot}07}\left(\frac{5+50-8}{5}\right)=46{,}800, \quad N_{45}=216 \text{ r.p.m.}$$

When reaching bottom position

$$N_{30}^2=\frac{35230}{8{\cdot}66}\left(\frac{5+50-8}{5}\right)=38{,}000, \quad N_{30}=195 \text{ r.p.m.}$$

47. *Find the maximum and minimum speeds for a governor in which the maximum and minimum radii of the ball paths are 6″ and 9″.*

Length of ball arms 24″ which are pivoted at points 2″ from the axis of rotation.

(i) *For an open-armed governor,* (ii) *for a crossed-arm governor.*

(i) Height of governor below suspension pin

$$h_1=\sqrt{24^2-4^2}=23{\cdot}66''$$

Equivalent height $=\dfrac{23{\cdot}66\times6}{4}=35{\cdot}5''$

$$n=\sqrt{\frac{35230}{35{\cdot}5}}=\mathbf{31{\cdot}5\ r.p.m.} \text{ for } 6'' \text{ radius}$$

$$h_1=\sqrt{24^2-7^2}=22{\cdot}96''$$

Equivalent height $=\dfrac{22{\cdot}96\times9}{7}=29{\cdot}52''$

$$n=\sqrt{\frac{35230}{29{\cdot}52}}=\mathbf{34{\cdot}5\ r.p.m.} \text{ for } 9'' \text{ radius.}$$

Difference in speed, 3 r.p.m.

(ii) $h_1=\sqrt{24^2-8^2}=22{\cdot}63''$.

Equivalent height $=\dfrac{22{\cdot}63\times6}{8}=16{\cdot}97''$

$$n=\sqrt{\frac{35230}{16{\cdot}97}}=\textbf{45·5 r.p.m.}\text{ for }6''\text{ radius}$$

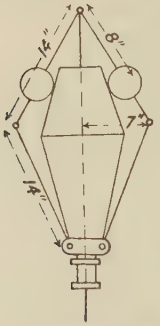
$h_1=\sqrt{24^2-11^2}=21{\cdot}33''$

Equivalent height $=\dfrac{21{\cdot}33\times9}{11}=17{\cdot}4''$

$$n=\sqrt{\frac{35230}{17{\cdot}4}}=\textbf{45 r.p.m.}$$

Difference in speed 0·5 r.p.m.

48. *Find the speed at which the governor shown in the figure will run. Weight of each ball,* 15 *lb. Central weight,* 100 *lb. The weight of the arms may be neglected.*



" Height " of governor $=0{\cdot}866\times8=6{\cdot}93''$

Radius of ball path $=4''$

$0{\cdot}00034\times15\times\frac{4}{12}\times N^2\times6{\cdot}93=15\times4+100\times7$

$N^2=64{,}530$ $\qquad N=\textbf{254 r.p.m.}$

What amount of potential energy is stored in the governor when the central weight rises 2" ?

The link joints will rise 1" and the balls $\frac{8}{14}''$.

Energy stored $=100\times2+2\times15\times\frac{8}{14}=\textbf{217 in.-lb.}$

49. *Find the speed of the governor shown in the figure when at its extreme positions. Weight of ball,* 2·40 *lb.; weight of arm,* 1·01 *lb.; weight of link,* 0·47 *lb.; weight of sleeve,* 5·0 *lb.; uncompressed length of spring,* 5·6" ; *length when balls are in lowest position,* 5·1" ; *length when balls are in highest position,* 3·96" ; *spring compresses* 0·134" *per lb. weight.*

Distance of c. of g. of ball and arm from apex

$$\frac{2{\cdot}4\times8{\cdot}5+\dfrac{1{\cdot}01\times7{\cdot}25}{2}}{2{\cdot}4+1{\cdot}01}=7{\cdot}05''$$

In the lowest position. Radius of path of c. of g. 2·65″.

The link is then nearly vertical and may be regarded as a part of the sleeve.

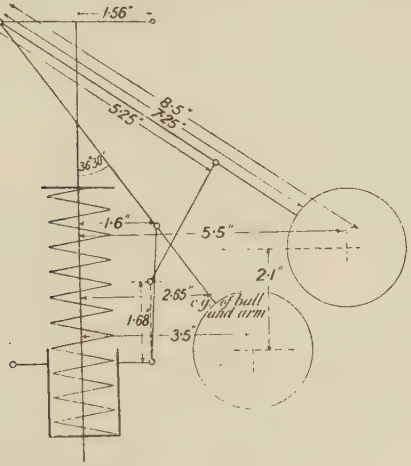
Centripetal moments for one ball and arm about apex pin.

Ball and arm $3{\cdot}41(2{\cdot}65+1{\cdot}56)=14{\cdot}35$ inch-lb.

Sleeve and link (see diagram) $\frac{2{\cdot}1}{1{\cdot}68}(2{\cdot}5+0{\cdot}47)(1{\cdot}6+1{\cdot}56)=11{\cdot}74$,,

Spring $\frac{2{\cdot}1}{1{\cdot}68}\left(\frac{5{\cdot}6-5{\cdot}1}{2\times 0{\cdot}134}\right)(1{\cdot}6+1{\cdot}56)=7{\cdot}37$,,

Total . . . . $33{\cdot}46$,,



Centrifugal moments.

$$h=\sqrt{8{\cdot}5^2-(3{\cdot}5+1{\cdot}56)^2}=6{\cdot}83$$

Ball: $0{\cdot}00034\times 2{\cdot}4\times\frac{3{\cdot}5}{12}\times 6{\cdot}83N^2=0{\cdot}00163N^2$

Arm: $\frac{0{\cdot}00034w}{12}N^2\cos\alpha\left(\frac{\sin\alpha L^3}{3}-\frac{L^2x}{2}\right)$,

$$\frac{0{\cdot}00034\times 1{\cdot}01\,.\,N^2\times 0{\cdot}804}{12\times 7{\cdot}25}\left(\frac{0{\cdot}595\times 381}{3}-\frac{52{\cdot}6\times 1{\cdot}56}{2}\right)$$

$$=0{\cdot}00011N^2$$

Link. One-half the centrifugal force acts on the pin

$$\frac{0{\cdot}00034\times 0{\cdot}47\times 1{\cdot}6}{2\times 12}N^2\times 4{\cdot}2=0{\cdot}00004N^2$$

Total . . . . $0{\cdot}00178N^2$

$$N^2 = \frac{33{\cdot}46}{0{\cdot}00178} = 18{,}800 \qquad N = \mathbf{137\ r.p.m}$$

Experiment gave 140 r.p.m. with spring.
Without the spring we have

$$N^2 = \frac{26{\cdot}09}{0{\cdot}00178} = 14{,}660 \qquad N = 121 \text{ r.p.m.}$$

Experiment gave 112 r.p.m.

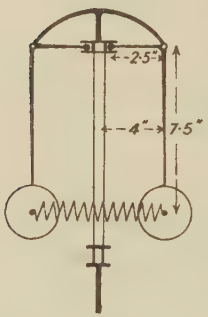
Friction in the joints will probably account for the difference between the calculated and the experimental values.

In the highest position of the balls. The results are arrived at in the same manner.

The calculated speed is **190 r.p.m.** and the experimental value 210 with the spring.

Without the spring the calculated speed is 137 r.p.m. and by experiment 130 r.p.m.

50. *Find the speed for the governor shown in the figure when the radius of the ball path is* 5·5″. *What force acting on the sleeve will be required to keep the balls in this position when the speed of the engine increases* 8 *per cent. Weight of each ball,* 6 *lb.; unstrained length of spring,* 5″; *load required to stretch each spring by* 1 *inch,* 25 *lb.*



Stretch of springs $= 2 \times 5{\cdot}5 - 5 = 6''$
Load on both springs $= 6 \times 25 \times 2 = 300$ lb.

The centripetal force on the balls is negligible.

$$0{\cdot}00034 \times 6 \times \frac{5{\cdot}5}{12} \times N^2 = 300$$

$N = \mathbf{567\ r.p.m.}$ $\qquad N + 8$ per cent. $= 612$ r.p.m.

$$F = 0{\cdot}00034 \times 6 \times \frac{5{\cdot}5}{12} \times 612^2 = 350 \text{ lb.}$$

$350 - 300 = 50$ lb. increase of cen. force on each ball.

Or $\qquad 2 \times 50 \times \frac{7{\cdot}5}{2{\cdot}5} = \mathbf{300\ lb.\ on\ sleeve.}$

51. *Find the speed at which the governor shown in the*

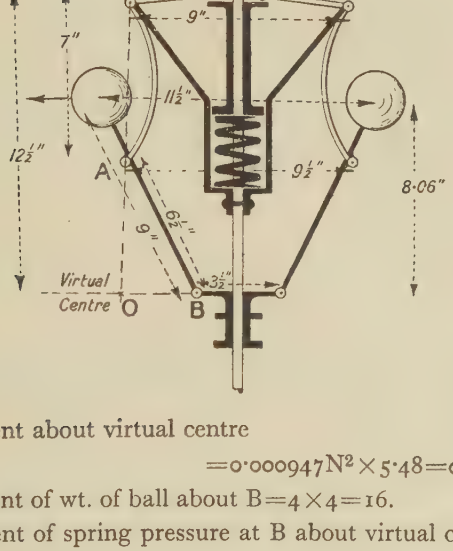
figure will run, neglecting the weight of the arms. Weight of each ball, 4 lb. ; initial load on spring, 70 lb.

Radius of ball path=0·48′.

Cen. force on ball=0·00034×4×0·48×N^2=0·000653N^2

Centrifugal force on ball referred to pin A

$$=0{\cdot}000653N^2\times\frac{9}{6{\cdot}2}=0{\cdot}000947N^2$$



Moment about virtual centre

$$=0{\cdot}000947N^2\times5{\cdot}48=0{\cdot}00519N^2$$

Moment of wt. of ball about B=4×4=16.

Moment of spring pressure at B about virtual centre

$$=35\times3{\cdot}2=112$$

Then $0{\cdot}00519N^2=112-16=96$

$N^2=18{,}380$ $N=$**136 r.p.m.**

52. *In a spring-controlled governor having two balls, the force on the spring is* 900 *lb. when the radius of the ball path is* 8 *ins., and* 1,400 *lb. when it is* 11 *ins. Find the radius when the speed is* 270 *r.p.m. By how much must the initial force be increased in order to make the governor isochronous? At what speed would it then run? The balls weigh* 30 *lb. each.*

Force on spring $F = aR - C$

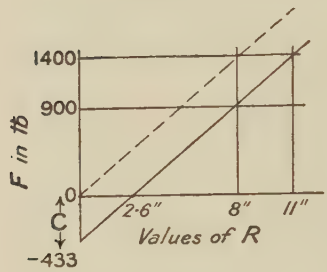
$1{,}400 = 11a - C$

$900 = 8a - C$

By subtraction $500 = 3a$ $\quad a = 166{\cdot}6$ $\quad C = 433$

$$F = 166{\cdot}6R - 433 = 0{\cdot}00034WRN^2 = 0{\cdot}00034 \times 60 \times \frac{R}{12} \times 270^2$$

$R = \mathbf{10{\cdot}1''}$



For isochronous running.

When $F = 0$

$166{\cdot}6R = 433$

$R = 2{\cdot}6''$.

When $R = 0$

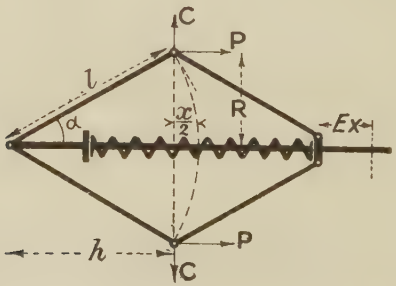
$F = -433$ lb.

Hence the initial force must be increased by 433 lb. (see broken line) to make the force on the spring zero when R is zero.

In that case—

$$166{\cdot}7R = 0{\cdot}00034 \times 60 \times \frac{R}{12} \times N^2 \qquad N = \mathbf{314\ r.p.m.}$$

53. *In the governor shown in the figure find the angle between the arms and the spindle when running at 200 r.p.m. The spring compresses 1″ for 150 lb.* (E). *The weight of each ball is 14 lb.* (W).



The sleeve moves twice as fast horizontally as the balls

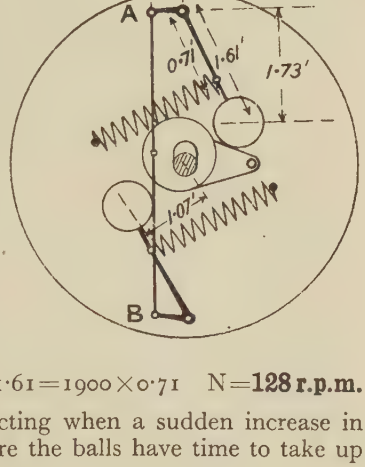
$$l = h + \frac{x}{2} \qquad Ch = PR \qquad h = \frac{PR}{C} = \frac{ExR}{0{\cdot}00034WRN^2}$$

$$\cos\alpha = \frac{h}{l} = \frac{2h}{2h+x} = \frac{2E}{2E+0{\cdot}00034WN^2}$$

$$\cos\alpha = \frac{300}{300+0{\cdot}00034\times14\times40000} = \frac{300}{490{\cdot}4} = 0{\cdot}612 \qquad \alpha = \mathbf{48°}.$$

54. *Find the speed at which a crankshaft governor shown in the figure will run. Show how you would calculate the force acting on the eccentric when there is a sudden increase of speed of 2 r.p.m. Weight of each arm and ball,* 140 *lb.; load on each spring when in position shown,* 1,900 *lb.*

You may assume that the centrifugal force acts at the centre of gravity of the ball.



Moment of centrifugal force=Moment of spring effort.

$$0{\cdot}00034\times140\times1{\cdot}07\times N^2\times1{\cdot}61 = 1900\times0{\cdot}71 \qquad N = \mathbf{128\ r.p.m.}$$

The centrifugal force acting when a sudden increase in speed takes place (*i.e.* before the balls have time to take up their new position)

$$= 0{\cdot}00034\times140\times1{\cdot}07(128+2)^2\times1{\cdot}61 = 1{,}350 \text{ lb.}$$

∴ Increase in centrifugal force $= 1350\left\{\left(\frac{130}{128}\right)^2 - 1\right\} = 42{\cdot}5$ lb.

Moment about pivot $= 42{\cdot}5\times1{\cdot}73 = 73{\cdot}5$.

Force on rod AB $= \frac{73{\cdot}5}{0{\cdot}42} = 175$ lb.

Force on eccentric plate pin $= 2\times175 = \mathbf{350\ lb.}$

55. *Calculate the speed at which a crankshaft governor (Fig.* 251, *Vol. I.) will run. Arm,* $2''\times\frac{3}{4}''\times9''$; *weight* (W), 120 *lb.; tension on spring,* 500 *lb.; distance of spring attachment from pivot,* 8″; *distance of* W *from pivot,* 12″; *radius of c. of g. of arm and weight about centre of shaft,* 4″.

Assume that the centrifugal force acts through the c. of g. of the arm and weight. Taking moments about the pin.

$$120\times12+\frac{2\times3\times9\times0\cdot28}{4}\times4\cdot5=1,457$$

$$\text{Distance of c. of g. from pin}=\frac{1457}{122\cdot5}=11\cdot9''$$

$$0\cdot00034\times123\cdot8\times\tfrac{4}{12}\times N^2\times11\cdot9=500\times8 \qquad N=\mathbf{155\ r.p.m.}$$

56. *In the case of a horizontal governor fitted with two springs attached direct to pins on the centre line of the balls, the weight of the balls is* 20 *lb. each, and when their centres are* 10″ *apart the load on each spring is* 65 *lb. The unstrained length of the springs is* 6″. *Find the speed at which the governor will run when the ball centres are* 8″ *and* 12″ *apart.*

$$\text{Load on both springs per inch extension}=\frac{65\times2}{10-6}=32\cdot5\text{ lb.}$$

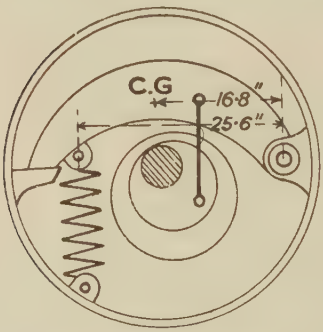
Load at 8″, *i.e.* 2″ extension = 65 lb.

12″ „ 6″ „ = 195 lb.

$$0\cdot00034\times20\times\tfrac{4}{12}\times N_1^2=65 \qquad N_1=\mathbf{170\ r.p.m.}$$

$$0\cdot00034\times20\times\tfrac{6}{12}\times N_2^2=195 \qquad N_2=\mathbf{240\ r.p.m.}$$

Governors of this type are never very sensitive unless some correction device is adopted, since it is impossible to arrange tension springs so that they are unstrained when the radius of the ball path is zero. In one type of governor compression springs are adopted for the purpose of overcoming this difficulty.



57. *Calculate, from the data below, the speed at which the crankshaft governor shown in the figure will begin to lift. The centrifugal moment acting on the eccentric may be neglected. Weight,* 42 *lb.; pull on the governor spring,* 830 *lb.; radius of path of c. of g. of weight,* 8·4″.

$$\left.\begin{matrix}\text{Centrifugal force acting}\\ \text{on weight}\end{matrix}\right\}=0\cdot00034\times42\times\frac{8\cdot4}{12}\times N^2=0\cdot01N^2$$

$$\text{Moment about pivot}=0\cdot01\times16\cdot8N^2=830\times25\cdot6$$

$$\mathbf{N=355\ r.p.m.}$$

CHAPTER VII

VIBRATION

1. *A simple pendulum for a clock* 18 *ins. long is shortened* 0·03 *in. Find the time the clock gains per week.*

Time of one complete swing when normal

$$=t_1=2\pi\sqrt{\frac{l}{g}}=2\times3\cdot1416\sqrt{\frac{18}{12\times32\cdot2}}=1\cdot356103 \text{ secs.}$$

NOTE.—It is necessary to go to many places of decimals because the differences are very small.

$$\left.\begin{matrix}\text{Time when}\\ \text{shortened}\end{matrix}\right\}=t_2=2\times3\cdot1416\sqrt{\frac{18-0\cdot03}{12\times32\cdot2}}=1\cdot354972$$

Difference=0·001131 sec. per swing.

$$\text{Gain per week}=\frac{0\cdot001131\times60\times7\times24}{1\cdot356103}=\mathbf{8\cdot4\ mins.}$$

This problem might also have been solved thus—

$$t=\frac{2\pi}{\sqrt{g}}l^{\frac{1}{2}} \qquad \therefore\ \frac{\delta t}{t}=\tfrac{1}{2}\frac{\delta l}{l}=\tfrac{1}{2}\times\frac{0\cdot03}{18}=\tfrac{1}{1200}.$$

$$\text{Hence gain per swing}=\frac{1\cdot356103}{1200}=0\cdot001131 \text{ sec.}$$

It will be noticed that for a given fractional change $\left(\frac{\delta l}{l}\right)$ in length, the fractional change in period $\left(\frac{\delta t}{t}\right)$ is the same whatever the value of l. Hence the error introduced by a change in temperature on pendulums of similar material is the same whatever their length.

2. *A pendulum makes one complete oscillation to and fro in* 2·42 *secs. The pendulum is pulled sideways so that the bottom point is* 11·2 *ins. horizontally from the vertical*

through the point of suspension, and is then released. Find the speed of the point when it passes the vertical.

This is only another way of stating that the point describes S.H.M. of $\frac{11\cdot2}{12}=0\cdot93$ foot amplitude in a period of 2·42 seconds. The velocity of a point on the circumference of a circle 0·93 foot radius when making $\frac{1}{2\cdot42}$ r.p.s. is

$$V=\frac{2\pi\times0\cdot93}{2\cdot42}=\mathbf{2\cdot41\ f.p.s.}$$

The maximum acceleration is $\frac{V^2}{R}=6\cdot27$ f.p.s.p.s.

If the bottom point of the pendulum be used as a scriber it makes a very convenient arrangement for measuring the peripheral speed of a rotating wheel. The pendulum is so arranged that when it swings the scriber marks the periphery of the wheel (parallel to its axis when the wheel is at rest and at an angle when the wheel is running) by measuring this angle, and knowing the speed of the scriber parallel to the axis the peripheral speed of the wheel is readily obtained. See Question 22.

3. *A* **U** *tube of uniform section* (a *sq. ins.*) *throughout is partially filled with liquid of weight* w *per foot length and* 1 *sq. in. section. If the liquid oscillates by an amount* x *feet above and below its mean level, find the periodic time. The length* l *of the column of liquid in the tube is* 4·6 *feet. Friction neglected.*

Force (F) tending to restore the liquid to its original level $=2wax$.

Weight of liquid column (W)$=wal$

$$F=\frac{W}{g}\times\text{acceleration.}$$

$$\text{Acceleration}=\frac{Fg}{W}=\frac{2waxg}{wal}=\frac{2xg}{l}$$

$$t=2\pi\sqrt{\frac{\text{Displacement}}{\text{Acceleration}}}=2\pi\sqrt{\frac{xl}{2xg}}=2\pi\sqrt{\frac{l}{2g}}$$

$$t=2\times3\cdot14\sqrt{\frac{4\cdot6}{64\cdot4}}=\mathbf{1\cdot68\ seconds.}$$

Thus the period is the same as that of a pendulum whose length is equal to one-half the length of the column of oscillating liquid and is independent of the density of the liquid.

4. *A tube closed at the bottom end with a heavy plug floats vertically in* (i) *water ;* (ii) *mercury. External diam. of tube,* 3·2″ *; weight of tube* (W), 42·4 *lb. ; weight of water* 1 *sq. in. in section and* 1 *ft. long*=0·434 *lb. ; sp. gravity of mercury,* 13·6. *Find the periodic time if the tube bobs up and down vertically without allowing any liquid to enter. Friction neglected.*

Restoring force per foot of displacement (F)
=weight of liquid displaced
$=0{\cdot}785\times3{\cdot}2^2\times0{\cdot}434=3{\cdot}49$ lb. (water)
$=3{\cdot}49\times13{\cdot}6=47{\cdot}5$ (mercury)

$$\text{Acceleration}=\frac{Fg}{W}=\frac{3{\cdot}49\times32{\cdot}2}{42{\cdot}4}=2{\cdot}65 \text{ f.p.s.p.s. (water)}$$
$$=2{\cdot}65\times13{\cdot}6=36{\cdot}05 \text{ (mercury).}$$

$$t=2\pi\sqrt{\frac{\text{Displacement}}{\text{Acceleration}}}.$$ See p. 260, Vol. I.

$$t=2\times3{\cdot}14\sqrt{\frac{1}{2{\cdot}65}}=\mathbf{3{\cdot}86 \text{ secs. (water).}}$$

$$t=2\times3{\cdot}14\sqrt{\frac{1}{36{\cdot}05}}=\mathbf{1{\cdot}05 \text{ secs. (mercury).}}$$

Skin friction on the tube will introduce a small but negligible error.

5. *A thin circular hoop* 3 *ft. diameter hangs on a nail and swings in its own plane : find the time of a complete oscillation.*

Let the radius of the hoop be r, and its weight W.

The polar moment of inertia about a pole passing through its centre and at right angles to its plane

$$=\frac{W}{g}r^2=\frac{W}{g}K^2$$

The length l of a simple pendulum with the same period of oscillation (see p. 262, Vol. I.)

$$=\frac{r^2+r^2}{r}=2r$$

Time of one complete oscillation

$$=2\pi\sqrt{\frac{2r}{g}} \qquad t=6\cdot28\sqrt{\frac{3}{32\cdot2}}=\mathbf{1\cdot91\ secs.}$$

NOTE.—The time of oscillation of a hoop is the same as that of a simple pendulum whose length is equal to the diameter of the hoop.

6. *A flywheel is hung with the inside of its rim resting on a horizontal knife edge parallel to the axis of the wheel and on which it swings in its own plane like a pendulum. Find its moment of inertia about its axis. It makes* 16 *complete oscillations to and fro in* 49·3 *seconds. Diam. of inside of rim,* 7·35 *ft.; weight of wheel* (W), 6·29 *tons.*

This is a case of a compound pendulum. The time t of one complete oscillation is $\frac{49\cdot3}{16}=3\cdot08$ secs. See p. 262, Vol. I.

$$t=2\pi\sqrt{\frac{K^2+l_g^2}{l_g g}} \qquad \left(\frac{t}{2\pi}\right)^2 g l_g=K^2+l_g^2$$

$$\left(\frac{3\cdot08}{6\cdot28}\right)^2 32\cdot2\times\frac{7\cdot35}{2}=K^2+3\cdot675^2$$

$$K^2=14\cdot96 \qquad K=3\cdot87\ \text{ft.}$$

Moment of inertia $=\dfrac{6\cdot29\times2240\times14\cdot96}{32\cdot2}=\mathbf{6{,}546\ pounds\text{-}feet^2}$,

i.e. gravitation or engineer's units.

7. *A pendulum in the form of a straight bar* 5 *ft. long is held in a horizontal position and then released. Find the velocity of a point at the end of the bar when it has turned through an angle of* (i) 40°; (ii) 90°.

The c. of g. is 2·5 ft. from the hinge; when the pendulum has dropped through 40° the c. of g. of the bar has fallen a height of $2\cdot5\sin40^\circ=1\cdot61$ ft. Its kinetic energy is

$$\text{K.E.}=\frac{WV_g^2}{2g}+\frac{I\omega^2}{2}=\frac{W}{2g}\omega^2(l_g^2+K_g^2)=\frac{W}{2g}\omega^2K_0^2=\frac{W}{2g}V_K^2$$

Work done in falling = Kinetic energy of the bar.

$$W\times1\cdot61=\frac{W}{2g}V_K^2$$

$$V_K^2=2\times32\cdot2\times1\cdot61=103\cdot7 \qquad V_K=10\cdot2\ \text{f.p.s}$$

Where V_K is the velocity of a point distant from the hinge equal to the radius of gyration.

$$\text{Radius of gyration } K = \frac{5}{\sqrt{3}}$$

$$\text{Hence the velocity of a point at the end of the bar} = \frac{10{\cdot}2 \times 5}{K} = \mathbf{17{\cdot}65 \text{ f.p.s. at } 40°.}$$

If the bar had fallen through 90° the c. of g. would have fallen 2·5 ft., then

$$W \times 2{\cdot}5 = \frac{WV_K^2}{2g} \qquad V_K = 12{\cdot}7 \text{ f.p.s.}$$

$$\text{Velocity at end of bar} = 12{\cdot}7 \times \sqrt{3} = \mathbf{22 \text{ f.p.s. at } 90°.}$$

8. *A connecting rod hangs vertically from the gudgeon pin which rotates in ball bearings and may be regarded as frictionless. The weight of the rod is* 420 *lb., its c.g. is* 50·2″ *from the centre of the gudgeon pin. Find the moment of inertia, in gravitation units, of the rod* (i) *about the gudgeon pin;* (ii) *about a parallel axis passing through the c.g. of the rod. The period of oscillation of the rod is* 2·61 *secs.*

(i) $t = 2\pi\sqrt{\dfrac{I_o}{Wl_g}}$ (see p. 261, Vol. I.)

$$\left(\frac{2{\cdot}61}{6{\cdot}28}\right)^2 = \frac{I_o \times 12}{420 \times 50{\cdot}2} \qquad I_o = \mathbf{303 \text{ lb.-ft.}^2 \text{ units.}}$$

or $t = 2\pi\sqrt{\dfrac{K_o^2}{l_g g}}$

$$\left(\frac{2{\cdot}61}{6{\cdot}28}\right)^2 = \frac{K_o^2}{4{\cdot}18 \times 32{\cdot}2} \qquad K_o^2 = 23{\cdot}25$$

$$I_o = \frac{420}{32{\cdot}2} \times 23{\cdot}25 = \mathbf{303 \text{ lb.-ft.}^2 \text{ units.}}$$

(ii) $$I_g = I_o - \frac{W}{g} l_g^2 = 303 - \frac{420}{32{\cdot}2}\left(\frac{50{\cdot}2}{12}\right)^2$$

$$= \mathbf{75 \text{ lb.-ft.}^2 \text{ units.}}$$

9. *A connecting rod is* 5 *ft. long between centres and is made for pins* 4 *ins. diameter at big end,* 2 *ins. at small end. The times for* 50 *oscillations when suspended from knife edges through the bushes are* 1 *min.* 55 *secs. and* 1 *min.* 43 *secs. respectively. Find the position of the centre of gravity and the radius of gyration about it.*

$$\frac{115}{50}=2\pi\sqrt{\frac{K^2+l_1^2}{l_1 g}} \qquad \frac{103}{50}=2\pi\sqrt{\frac{K^2+l_2^2}{l_2 g}}$$

$$l_1+l_2=\tfrac{63}{12} \qquad K^2+l_1^2=4{\cdot}32l_1 \qquad K^2+l_2^2=3{\cdot}45l_2$$

$$\mathbf{K=1{\cdot}72'} \qquad \mathbf{l_1=3{\cdot}46'} \qquad \mathbf{l_2=1{\cdot}79'}$$

10. *A solid garden roller 2′ 6″ diam. is mounted on ball bearings. The distance between the axle and the handle is 4·1 ft. The handle is placed on two trestles so that the roller can swing like a pendulum with the handle as a pivot. Neglecting friction, also the weight and inertia of the handle, find the periodic time* (i) *when the roller can rotate on its axle ;* (ii) *when it is clamped to the axle.*

(i) This is a case of a simple pendulum.

$$t=2\pi\sqrt{\frac{l}{g}}=6{\cdot}28\sqrt{\frac{4{\cdot}1}{32{\cdot}2}}=\mathbf{2{\cdot}24\ secs.}$$

(ii) Compound pendulum.

$$t=2\pi\sqrt{\frac{K^2+l_g^2}{l_g g}}=6{\cdot}28\sqrt{\frac{\frac{2{\cdot}5^2}{8}+4{\cdot}1^2}{4{\cdot}1\times 32{\cdot}2}}=\mathbf{2{\cdot}29\ secs.}$$

11. *A helical spring hangs vertically and supports a weight of 33 lb. which stretches it 2·25″. The spring weighs 6 lb. The weight is started oscillating ; find its period.*

The weight in this experiment was suspended by a piece of string and the spring was stretched beyond that due to the dead-weight and then released, when the stretch exceeded a certain amount the string slackened at the top of the travel and allowed the weight to jump : calculate the extension of the spring when this occurred.

See p. 263, Vol. I.

$$t=2\pi\sqrt{\frac{W+\left(\frac{W_s}{3}\right)x}{P_1 g}}=6{\cdot}28\sqrt{\frac{35\times 2{\cdot}25}{33\times 32{\cdot}2\times 12}}=\mathbf{0{\cdot}494\ sec.}$$

Neglecting the weight of the spring $\mathbf{t=0{\cdot}479\ sec.}$

Experiment gave 0·484 sec.

Jumping of the weight takes place when the acceleration due to the spring exceeds that due to gravity, hence

$$g=\omega^2 R=\frac{4\pi^2 R}{t^2} \quad \text{and} \quad \frac{t^2}{4\pi^2}=\frac{\delta}{g} \qquad \therefore \delta=R$$

neglecting the weight of the spring, and where R is the extreme distance of the weight in feet from its position when the extension of the spring is zero. Then

$$R=\frac{32\cdot2t^2}{4\pi^2}=\frac{32\cdot2\times0\cdot479^2}{4\times9\cdot87}=0\cdot188 \text{ foot}=2\cdot25''$$

But if the weight of the spring be taken into account we have

$$R=\frac{W+\frac{Ws}{3}}{W}\delta=2\cdot45 \text{ ins. or } 4\cdot9 \text{ ins. total extension.}$$

An experiment gave 5·1 ins.; but, without special appliances, it is difficult to detect exactly when jumping does take place.

12. *A solid cylindrical weight is attached to the end of a helical spring made of wire of circular section. Show that the natural periods for torsional and longitudinal oscillations will be the same if the ratio*

$$\frac{\textit{Diam. of weight}}{\textit{Diam. of coil}}=\sqrt{\frac{E}{G}}$$

For torsional displacements (see p. 590, Vol. I.)

we have $$\theta_r=\frac{M_tL}{EI}, \quad \text{where}$$

M_t=twisting moment on spring=C in expression for t given below.

L=length of wire in spring.

I=moment of inertia of wire section about the N.A.$=\frac{\pi d^4}{64}$

θ_r=angle of twist in radians.

On p. 266, Vol. I., we show that the time of one complete oscillation t is given by

$$t=2\pi\sqrt{\frac{I_i\theta_r}{C}}$$

where I_i=moment of inertia of the oscillating weight

$$=\frac{WK^2}{g}=\frac{WD_w^2}{8g} \qquad t=2\pi\sqrt{\frac{WK^2M_tL}{gCEI}}$$

$$t=2\pi\sqrt{\frac{64WD_w^2L}{8gE\pi d^4}}=2\pi\sqrt{\frac{8WD_w^2L}{gE\pi d^4}}$$

For longitudinal displacements (see p. 262, Vol. I.)

$$t=2\pi\sqrt{\frac{Wx}{P_1 g}}. \qquad \text{In this case } W=P_1$$

$$x=\frac{2{\cdot}55D^2LW}{Gd^4} \qquad \text{(see p. 587, Vol. I.)}$$

$$t=2\pi\sqrt{\frac{2{\cdot}55D^2LW}{gGd^4}}$$

For equal periods $\dfrac{2{\cdot}55D^2LW}{gGd^4}=\dfrac{8WD_w^2L}{gE\pi d^4}$

$$\frac{E}{G}=\frac{D_w^2}{D^2} \quad \text{and} \quad \frac{D_w}{D}=\sqrt{\frac{E}{G}}$$

13. *It is found that for the comfort of the passengers in a motor car the number of oscillations per minute should not exceed* 100. *Find the corresponding deflection of the springs. Why is a plate spring preferable to a helical spring for this purpose?*

The period of oscillation of a spring is the same as that of a simple pendulum whose length is equal to the static deflection of the spring when loaded.

$$t=2\pi\sqrt{\frac{\Delta}{g}} \qquad \left(\frac{t}{2\pi}\right)^2 g=\Delta$$

$$\left(\frac{60}{100\times 2\times 3{\cdot}14}\right)^2 32{\cdot}2=0{\cdot}294 \text{ ft.}=\mathbf{3{\cdot}5''\ deflection.}$$

The friction between the leaves of a plate spring damps out the oscillations and thereby reduces the number of up-and-down movements of the car.

Readers should refer to a paper "Experiments on Laminated Springs," by H. S. Rowell. Institution of Automobile Engineers. April, 1926.

14. *A weight of* 78 *lb. is suspended from the end of a cantilever at a distance of* 30 *ins. from the built-in end. Breadth of cantilever* 1·9 *ins., thickness* 0·276 *in.* E= 30,000,000 *lb. per sq. in.*

Find the period of oscillation neglecting the weight of the cantilever.

$$t = 0{\cdot}185\sqrt{\frac{WL^3}{EI}} \quad \text{(see p. 264, Vol. I.)}$$

$$= 0{\cdot}185\sqrt{\frac{78 \times 30^3 \times 12}{30{,}000{,}000 \times 1{\cdot}9 \times 0{\cdot}276^3}} = \mathbf{0{\cdot}85\ sec.}$$

An experiment gave 0·88 sec. The neglect of the weight of the cantilever may partly account for the discrepancy. Lack of rigidity in supports may introduce considerable errors.

15. *A wire* 9 *ins. long is clamped horizontally at one end. To the other end a weight is attached, compared with which the weight of the wire itself is negligible. Time of one oscillation* 0·365 *sec. Find the period of oscillation if the wire were held vertically.*

When the wire is held horizontally, the restoring force due to a displacement from its position of equilibrium is purely that due to its elastic properties.

For a certain displacement δ, suppose this force causes an acceleration a_1. Then the period of oscillation is

$$t_1 = 2\pi\sqrt{\frac{\delta}{a_1}}$$

If the wire and weight were suspended as a pendulum with the end unclamped, then the restoring force would be that due to gravity alone, and the period of oscillation would be $t_2 = 2\pi\sqrt{\frac{\delta}{a_2}} = 2\pi\sqrt{\frac{l}{g}}$ in this case.

When the wire and weight are held vertically, but clamped at the end, for a given displacement δ the restoring force is the sum of the elastic restoring force and that due to gravity. Hence the acceleration corresponding to this displacement is $a_3 = a_1 + a_2$, and the time of oscillation is

$$t_3 = 2\pi\sqrt{\frac{\delta}{a_3}}$$

But since $a_3 = a_1 + a_2$, we have $\frac{1}{t_1^2} + \frac{1}{t_2^2} = \frac{1}{t_3^2}$

Hence in the case in point, since

$$t_1 = 0{\cdot}365 \quad \text{and} \quad t_2 = 2\pi\sqrt{\frac{9}{12g}}$$

it follows that $$\frac{1}{t_3^2} = \frac{1}{0{\cdot}365^2} + \frac{12g}{4\pi^2 \times 9} = 8{\cdot}59$$

$$\therefore\ t_3 = \mathbf{0{\cdot}342\ sec.}$$

If the wire were held vertically *upwards*, then the restoring force due to gravity would be negative, and

$$\frac{1}{t_4^2} = \frac{1}{0{\cdot}365^2} - \frac{g}{3\pi^2} = 6{\cdot}41 \qquad t_3 = \mathbf{0{\cdot}395\ sec.}$$

16. *A grooved disc pulley mounted on a horizontal axle has a weight suspended from a cord wrapped round the pulley. Another cord attached to a projecting pin on the side of the disc leads off tangentially to a spring which causes the pulley and weight to oscillate to and fro when the latter is displaced and released. Find the period. Diam. of grooved pulley,* 14″*; weight of grooved pulley,* 64 *lb.; suspended weight,* 40 *lb.; spring stretches* 1″ *for* 50 *lb.; distance of pin from centre of axle,* 5″.

$$t = 2\pi\sqrt{\frac{I\theta}{C}} \quad \text{(see p. 265, Vol. I.)}$$

Effective moment of inertia of pulley and weight

$$= I = \frac{64}{32{\cdot}2}\left(\frac{14}{12}\right)^2 \frac{1}{8} + \frac{40}{32{\cdot}2}\left(\frac{7}{12}\right)^2 = 0{\cdot}761$$

When $\theta = 1$ the force on the spring $= 50 \times 5 = 250$ lb.

$$C = \frac{250 \times 5}{12} = 104{\cdot}2 \qquad t = 6{\cdot}28\sqrt{\frac{0{\cdot}761}{104{\cdot}2}} = \mathbf{0{\cdot}535\ sec.}$$

17. *A valve is kept down by a helical spring which exerts a pressure of* 15 *lb. when the valve is on its seat, and* 22 *lb. when full open. The valve itself weighs* 8 *lb. and the friction of the valve in its guides is* 3 *lb. Lift of valve* 0·6″. *Find the time required to close the valve.*

Mean pressure on valve $= \frac{15+22}{2} = 18{\cdot}5$ lb.

Mean force tending to close valve $= 18{\cdot}5 + 8 - 3 = 23{\cdot}5$ lb.

$$F=\frac{W}{g}a \qquad a=\frac{gF}{W}$$

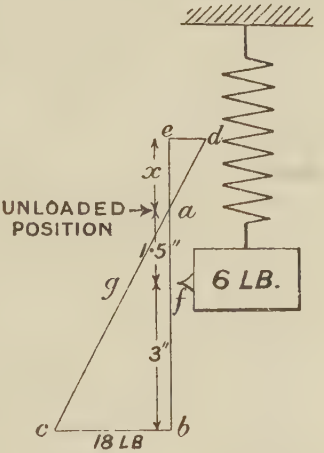
$$a=\frac{32\cdot2\times23\cdot5}{8}=94\cdot6 \text{ f.p.s.p.s.}$$

$$S=\tfrac{1}{2}at^2 \qquad t=\sqrt{\frac{2S}{a}}$$

$$t=\sqrt{\frac{2\times0\cdot6}{12\times94\cdot6}}=\mathbf{0\cdot033 \text{ sec. (nearly).}}$$

The force varies and therefore the acceleration, so that $t=\sqrt{\frac{2S}{a}}$ does not strictly hold. The error due to the assumption is small compared with frictional and other effects.

18. *A helical spring is suspended with a body weighing 6 lb. attached to the lower end which stretches the spring* 1·5″. *The body is pulled down a further* 3″ *and is then suddenly released. Find* (i) *the height* x *to which the body will reach above the unloaded position of the spring;* (ii) *the maximum velocity of the body.*



Total extension of spring = 4·5″.

Force acting on spring = 4·5 × 4 = 18 lb.

Work stored in spring, viz. $abc=\frac{18\times4\cdot5}{2}=40\cdot5$ in.-lb.

When the body is released it oscillates up and down, and when at the top of its traverse it compresses the spring by an amount $ae=x$. The corresponding load $ed=4x$ lb.

Compression energy stored in spring $aed=\frac{4x\times x}{2}=2x^2$ in.-lb.

Work done in lifting body through height $be=(4\cdot5+x)6$ in.-lb.

This is also equal to the energy stored in the spring $=abc-aed=40\cdot5-2x^2$.

Then $(4{\cdot}5+x)6=40{\cdot}5-2x^2$ $\quad \mathbf{x=1{\cdot}5''}$

The total travel of the body up and down$=4{\cdot}5''+1{\cdot}5''=6''$.

When the tension of the spring is greater than the weight of the body the body is accelerated, the acceleration ceases when the two are equal, viz. at half travel f, or when the spring is stretched $1{\cdot}5''$, the velocity is then a maximum, and we have

The work stored in the spring *bcgf* = Kinetic energy of the body at f + Potential energy of the body at f

$$\left(\frac{18+6}{2}\right)3=\frac{12WV^2}{2g}+6\times 3 \qquad \mathbf{V=4\ f.p.s.}$$

If the body were not *attached* to the spring, but simply rested on a collar on the straight lower portion of the spring, and where it was free to slide up and down, it is evident that, neglecting the inertia of the spring, the body would lift off the collar at the position *a*. The weight may leave the collar because: either (i) the tension becomes zero. (If the spring had no inertia this would occur when the natural length l is reached.) Or (ii) if the acceleration during the extension of the spring under load (or the retardation during shortening under load) exceeds g.

The maximum acceleration due to any S.H.M.$=\omega^2$R and the condition that this shall not exceed g is

R not greater than $\frac{g}{\omega^2}$, *i.e.* not greater than $\frac{gt^2}{4\pi^2}$

where R is the semi-amplitude.

If $t=0{\cdot}5$ sec. then R must not exceed $\frac{32{\cdot}2\times 0{\cdot}5^2}{4\pi^2}=0{\cdot}205$ ft.$=2{\cdot}45$ ins

19. *A lever such as that shown in Fig.* 263, *Vol. I.*

$l_1=8''$ $\quad$ *Spring deflects* $3{\cdot}4''$ *for* 50 *lb.*
$l_2=18''$ $\quad$ $W=7$ *lb.* (*suspended by a piece of wire*).
$l_3=19{\cdot}5''$ $\quad$ *Weight of lever*$=2$ *lb.*
$h=1''$
$t=0{\cdot}22''$

Find the time of one complete oscillation.

$$\text{Load on spring}=\frac{7\times 18}{8}+\frac{2\times 9{\cdot}75}{8}=18{\cdot}2 \text{ lb.}$$

$$C=\frac{18{\cdot}2\times 8}{12}=12{\cdot}1 \text{ lb.-feet.}$$

$$I=\frac{7}{32{\cdot}2\times144}(0+18^2)+\frac{2}{32{\cdot}2\times144}\left(\frac{19{\cdot}5^2+1}{12}+\frac{19{\cdot}5^2}{4}\right)=0{\cdot}543$$

$$\theta=\frac{18{\cdot}2\times3{\cdot}4}{8\times50}=0{\cdot}155 \qquad t=6{\cdot}28\sqrt{\frac{0{\cdot}543\times0{\cdot}155}{12{\cdot}1}}=\mathbf{0{\cdot}52\ sec.}$$

Or— when $\theta=1$ deflection $=8''$

$$\text{Load on spring}=\frac{50\times8}{3{\cdot}4}=118 \text{ lb.}$$

$$C=\frac{118\times8}{12}=78{\cdot}7 \text{ lb.-feet} \qquad t=6{\cdot}28\sqrt{\frac{0{\cdot}543\times1}{78{\cdot}7}}=\mathbf{0{\cdot}52\ sec.}$$

An experiment gave **0·56 sec.**

20. *A uniform bar* 12 *feet long and weight* 100 *lb. is hinged at one end so that it can swing in a vertical plane. A vertical spring* 4 *feet from the hinge supports the bar horizontally when at rest. The spring deflects* 0·06″ *per lb. Find the period of oscillation.*

$$t=2\pi\sqrt{\frac{I\theta}{C}} \qquad I=\frac{100\times12^2}{3\times32{\cdot}2}=149$$

When $\theta=1$ the deflection of the spring is 4 ft. and the load on the spring is $\frac{48}{0{\cdot}06}=800$ lb., and the couple is $4\times800=3{,}200$ lb.-feet.

$$t=6{\cdot}28\sqrt{\frac{149}{3200}}=\mathbf{1{\cdot}35\ secs.}$$

21. *A vertical tension rod* 50 *ft. long, free at the bottom end, supports a weight of* 100 *lb. Diameter of rod* $\frac{3}{8}''$. *Find the period of longitudinal vibrations.* $E=30{,}000{,}000$ *lb. sq. in.*

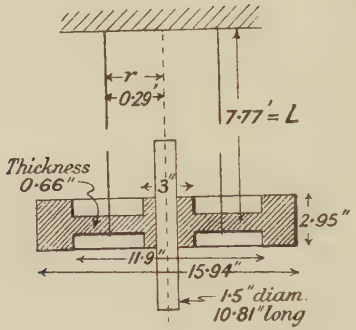
See p. 265, Vol. I.

$$t=2\pi\sqrt{\frac{Wl}{EAg}} \qquad t=2\times3{\cdot}1416\sqrt{\frac{100\times50}{30{,}000{,}000\times0{\cdot}11\times32{\cdot}2}}$$

$$t=\mathbf{0{\cdot}043\ sec.}$$

Unless suitable apparatus is available for measuring the time very accurately this cannot be checked satisfactorily by experiment.

22. *A disc flywheel suspended by two wires as shown in the figure, when twisted and released made* 20 *complete oscillations to and fro in* 1 *min.* $50\frac{3}{5}$ *secs. Find the radius of gyration and moment of inertia in pounds-feet² units. The wheel weighs* 94 *lb.*



Restoring couple $C = W \times \frac{r\theta}{L} \times r = \frac{Wr^2\theta}{L}$

$$\therefore\ t = 2\pi\sqrt{\frac{I\theta}{C}} = 2\pi\sqrt{\frac{IL}{Wr^2}} \qquad I = \frac{Wr^2t^2}{4\pi^2L} = \frac{WK^2}{g}$$

$$K^2 = \frac{0{\cdot}29^2 \times 5{\cdot}53^2 \times 32{\cdot}2}{4 \times 9{\cdot}87 \times 7{\cdot}77} \qquad K^2 = 0{\cdot}27 \qquad K = \mathbf{0{\cdot}52\ ft.}$$

$$I = \frac{94 \times 0{\cdot}27}{32{\cdot}2} = \mathbf{0{\cdot}788.}$$

When making the experiment mentioned above the wheel was given a big amplitude at first when 20 oscillations took place in 2 m. 22·4 secs.; a smaller amplitude gave 2 m. 2 secs., and quite small amplitudes gave 1 m. $50\frac{3}{5}$ secs., thus showing the importance of giving quite small amplitudes when making such experiments.

The moment of inertia of each portion of the wheel may be calculated thus (see p. 98, Vol. I.):

$$I_p = \frac{\pi D^4 H W}{32 \times g} = 1{\cdot}35 D^4 H$$

W = 444 lb. per cubic foot for cast iron.

Rim	$I_p = 1{\cdot}35(1{\cdot}328^4 - 0{\cdot}992^4)0{\cdot}246$	$= 0{\cdot}7120$
Web	$I_p = 1{\cdot}35(0{\cdot}992^4 - 0{\cdot}25^4)0{\cdot}055$	$= 0{\cdot}0716$
Boss	$I_p = 1{\cdot}35(0{\cdot}25^4 - 0{\cdot}125^4)0{\cdot}246$	$= 0{\cdot}0012$
Shaft	$I_p = 1{\cdot}49(0{\cdot}125^4)0{\cdot}9$	$= 0{\cdot}0003$
		$0{\cdot}7851$

Weight—

Rim	$0{\cdot}785 \times 444(1{\cdot}328^2 - 0{\cdot}992^2)0{\cdot}246$	$= 67{\cdot}0$ lb.
Web.	$0{\cdot}785 \times 444(0{\cdot}992^2 - 0{\cdot}25^2)0{\cdot}055$	$= 17{\cdot}7$ lb.
Boss	$0{\cdot}785 \times 444(0{\cdot}25^2 - 0{\cdot}125^2)0{\cdot}246$	$= 4{\cdot}0$ lb.
Shaft	$0{\cdot}785 \times 490(0{\cdot}125^2)0{\cdot}9$	$= 5{\cdot}4$ lb.
	Total	$94{\cdot}1$ lb.

Radius of gyration

$$K^2 = \frac{0{\cdot}785 \times 32{\cdot}2}{94{\cdot}1} = 0{\cdot}269 \qquad \mathbf{K = 0{\cdot}519 \text{ ft.}}$$

The moment of inertia of each portion might have been taken as $I_p = \frac{D^2}{8} \times \text{Vol.} = \frac{\pi D^4 H}{32}$ (inch^5 units)

Then $K^2 = \frac{I_p \text{ for whole wheel (inch}^5\text{)}}{\text{Volume of wheel (inch}^3\text{)}}$ (inch^2 units)

K will then be in inch units.

The wheel in question was mounted in lathe centres with the axle horizontal, a cord 0·18″ diam. was wound round the axle to which a weight of 30 lb. was attached. This weight after descending 11·25 ft. was disengaged, and in its descent it imparted kinetic energy to the wheel. At the moment the weight was disengaged the speed of the periphery of the wheel was measured by a pendulum which scratched a line across the rim of the wheel. The pendulum was pulled to one side against a stop and released at the instant the weight was disengaged. The amplitude of the scriber was 15″, and the period of the pendulum was 2·39 secs. to and fro. Hence the velocity of the scriber was $v = \frac{2\pi \times 1{\cdot}25}{2{\cdot}39}$ $= 3{\cdot}286$ f.s. The angle of the scribed line to a line parallel to the axis was 79° 46′ and tan 70° 46″ = 5·539, hence the velocity of the rim was $5{\cdot}539 \times 3{\cdot}286 = 18{\cdot}2$ ft. per second. And the velocity of a point at the radius of gyration was

$$\frac{18{\cdot}2 \times 0{\cdot}519}{0{\cdot}664} = 14{\cdot}23 \text{ f.s.}$$

Kinetic energy of wheel $= \frac{94 \times 14{\cdot}23^2}{2 \times 32{\cdot}2} = 295{\cdot}5$ ft.-lb.

The velocity of the weight when disengaged was

$$\frac{18{\cdot}2 \times 0{\cdot}07}{0{\cdot}664} = 1{\cdot}92 \text{ f.s.}$$

Kinetic energy of weight when disengaged

$$\frac{30 \times 1{\cdot}92^2}{2 \times 32{\cdot}2} = 1{\cdot}7 \text{ ft.-lb.}$$

A counter on the flywheel registered the total number of revolutions made from the release of the weight till the wheel stopped—in this case it was 198. Hence the mean work done against friction per revolution was $\frac{295{\cdot}5}{198} = 1{\cdot}49$ foot-lb. During the descent of the weight the wheel made 25 revolutions, hence the work done against friction was $1{\cdot}49 \times 25 = 37{\cdot}3$ foot-lb.

The work done by the descending weight was

$$30 \times 11{\cdot}25 = 337{\cdot}5 \text{ foot-lb.}$$

We can account for $295{\cdot}5 + 1{\cdot}7 + 37{\cdot}3 = 334{\cdot}5$ foot-lb., or about 1 per cent. error; the balance is not always so good as in this case, but with care there is no difficulty in getting the energy account to balance within 3 or 4 per cent.

The wheel was suspended on the points of two hooks bearing on the inside of the rim and with the plane of the wheel vertical. The wheel was then set vibrating in its own plane about the points of the hooks as a fulcrum. It made 50 complete oscillations in 56·2 secs. Find the radius of gyration about the axis of the wheel.

$$t = 2\pi\sqrt{\frac{K^2 + l_g^2}{32{\cdot}2 l_g}}$$

See diagram. $l_g = \frac{11{\cdot}9}{2 \times 12} = 0{\cdot}496$ ft. $\qquad l_g^2 = 0{\cdot}246$

$$\left(\frac{56{\cdot}2}{50 \times 2 \times 3{\cdot}1416}\right)^2 = \frac{K^2 + 0{\cdot}246}{32{\cdot}2 \times 0{\cdot}496}$$

$$0{\cdot}32 \times 32{\cdot}2 \times 0{\cdot}496 - 0{\cdot}246 = K^2 = 0{\cdot}265$$

K = 0·515 ft., which agrees closely with the values found by the other methods.

23. *A circular disc is suspended by two long vertical wires of length* l *attached to two opposite points on its circumference. Show that the period of torsional oscillation is* $2\pi\sqrt{\frac{l}{2g}}$.

Suppose the disc is displaced through a small angle α. Then the wires will be inclined to the vertical at an angle

$$\theta=\frac{\alpha r}{l}.$$

The restoring component of the force in each wire is

$$\frac{W}{2}\sin\theta=\frac{W\alpha r}{2l}.$$

The total restoring couple is therefore $\frac{W\alpha r^2}{l}$, and the restoring acceleration $\frac{g}{WK^2}\times\frac{W\alpha r^2}{l}=\frac{g\alpha r^2}{lk^2}$.

$$\left.\begin{matrix}\text{Period of}\\ \text{oscillation}\end{matrix}\right\}=2\pi\sqrt{\frac{\text{angular displacement}}{\text{angular acceleration}}}=2\pi\sqrt{\frac{l}{2g}}$$

The same result can be obtained from the energy equation. Let w be the angular velocity (maximum) at which the disc passes its position of equilibrium.

Then $$W l(1-\cos\theta)=\tfrac{1}{2}\frac{I}{g}w^2$$

Hence $$W l\frac{\alpha^2r^2}{2l^2}=\frac{WK^2w^2}{2g} \qquad \therefore\ wr=\frac{\alpha r^2}{K}\sqrt{\frac{g}{l}}$$

But in a simple Harmonic motion

$$\text{maximum velocity}=\frac{\pi}{2}\times\text{mean velocity}$$

$$\therefore\ \text{period}=\pi\times\frac{\text{amplitude}}{\text{maximum velocity}}=\pi\times\frac{\alpha r}{\frac{\alpha r^2}{K}\sqrt{\frac{g}{l}}}=2\pi\sqrt{\frac{l}{2g}}$$

24. *A flywheel is suspended by two vertical and parallel wires. The wheel is then caused to oscillate freely through a small angle about its axis of rotation, which is vertical. Find the moment of inertia of the wheel. Weight of wheel (plain disc),* 140 *lb.; distance apart of wires* (2r), 8″; *length of wires* (L), 9′; *time taken in making* 15 *complete*

oscillations, 1 *min.* 25 *secs. Weight of metal in disc* 0·26 *lb. per cubic inch*=449 *lb. per cu. ft.*

The wheel and axle are afterwards mounted in ball bearings (friction negligible). A cord is wound round the axle and a weight of 100 *lb. attached to the cord descends through a height of* 18 *feet. What will be the maximum speed of the wheel? The kinetic energy of the* 100 *lb. weight may be neglected.*

$$t=2\pi\sqrt{\frac{\mathrm{LI}}{\mathrm{W}r^2}}=2\pi\sqrt{\frac{\mathrm{LWK}^2}{\mathrm{W}r^2g}}=2\pi\sqrt{\frac{\mathrm{LK}^2}{r^2g}}$$

$$\frac{85}{15}=6\cdot28\sqrt{\frac{9\mathrm{K}^2\cdot3^2}{32\cdot2}} \qquad \mathrm{K}^2=0\cdot324=\frac{\mathrm{D}^2}{8}$$

$$\mathrm{D}=\sqrt{8\times0\cdot324}=1\cdot61 \text{ ft.}=19\cdot3''=\text{Diam. of disc.}$$

$$0\cdot785\times19\cdot3^2\times0\cdot26\times\mathrm{T}=140 \text{ lb.} \qquad \mathrm{T}=1\cdot84''=0\cdot153'$$

$$\mathrm{I}=\frac{\pi\mathrm{D}^4\mathrm{TW}_1}{32g}=\frac{3\cdot14\times1\cdot62^4\times0\cdot153\times449}{32\times32\cdot2}$$

$$=\mathbf{1\cdot41} \text{ in feet and pounds units.}$$

Check— $\dfrac{85}{15\times6\cdot28}=\sqrt{\dfrac{9\mathrm{I}\times3^2}{140}}$ (see above)

$$\mathrm{I}=\frac{85^2\times140}{15^2\times6\cdot28^2\times81}=\mathbf{1\cdot41.}$$

Work done by descending weight=Kinetic energy stored in wheel.

$$18\times100=\frac{\mathrm{WV}_\kappa^2}{2g}=\frac{140\times2^2\times3\cdot14^2\times0\cdot324\times n^2}{2\times32\cdot2}$$

n=8·06 r.p.sec. N=**484 r.p.m.**

25. *Find the natural period of torsional oscillation of a shaft fitted with two flywheels. The weight of one wheel is* 2,000 *lb. and the other* 3,500 *lb., the radii of gyration are* 2·8 *ft. and* 3·2 *ft. respectively. The diameter of the shaft is* 5 *ins. The distance apart of the wheels is* 60 *ins. Coefficient of rigidity* 12,000,000 *lb. per sq. in.*

$$\mathrm{I}_1=2000\times2\cdot8^2=15{,}680 \qquad \mathrm{I}_2=3500\times3\cdot2^2=35{,}840$$

$$15{,}680\,l_1=35{,}840\,l_2$$

$$l_2=2\cdot28\,l_1 \qquad l_1=60-l_2$$

$$l_2=18\cdot3'' \qquad l_1=41\cdot7''$$

$$t=2\pi\sqrt{\frac{12\mathrm{I}l}{\mathrm{GI}_p}}=2\pi\sqrt{\frac{12\mathrm{WK}_i^2l\times32}{144g\mathrm{G}\pi d^4}}=1\cdot03\sqrt{\frac{\mathrm{WK}_i^2l}{\mathrm{G}d^4}}$$

$$= 1{\cdot}03\sqrt{\frac{2000 \times 33{\cdot}6^2 \times 41{\cdot}7}{12000000 \times 625}} = \mathbf{0{\cdot}115\ sec.}$$

or **8·7** vibrations p.s.

26. *A vertical rod fixed at its upper end has a disc pulley attached to the lower end, find the period of torsional oscillation of the disc when given a small displacement. Length of rod,* 12′; *weight of disc,* 12·09 *lb.; diam. of disc,* 9·9″; *diam. of rod,* 0·255″; G=12,000,000 lb. sq. in.

$$t = 1{\cdot}03\sqrt{\frac{WK_i^2 l}{Gd^4}} = 1{\cdot}03\sqrt{\frac{12{\cdot}09 \times 9{\cdot}9^2 \times 144}{12000000 \times 8 \times 0{\cdot}255^4}}$$

$$t = \mathbf{0{\cdot}67\ sec.}$$

An experiment gave **0·68 sec.**

27. *An oil engine crank shaft is* 3·5 *ins. in diameter, it carries two flywheels at equal distances on each side of the crank, the distance between the centres of the flywheels being* 24 *ins.*

If the radius of gyration of each flywheel is 20 *ins., determine the frequency of the natural torsional vibration of the shaft.*

You may neglect the moment of inertia of crank, etc., and take the modulus of transverse elasticity of the steel of the shaft as 11,500,000 *lb. per sq. in.*

Weight of each wheel 0·5 *ton.* (Edinburgh University.)

(See p. 266, Vol. I.)

$$t = 1{\cdot}03\sqrt{\frac{WK_i^2 l}{Gd^4}} = 1{\cdot}03\sqrt{\frac{1120 \times 20^2 \times 12}{11500000 \times 3{\cdot}5^4}}$$

$$= 1{\cdot}03\sqrt{\frac{1}{321}} = \frac{\mathbf{1}}{\mathbf{17{\cdot}4}}\ \mathbf{sec.}$$

or the shaft makes 17·4 oscillations p.s. or 1044 p.m. If one larger wheel, weighing one ton, had been used instead of the two smaller wheels the W in the above expression would be 2240 lb. and $t = \frac{1}{12{\cdot}3}$ sec. or 12·3 oscillations p.s. or 738 p.m. The oscillations p.m. of the crank shaft should not approximate to the r.p.m. of the engine or trouble will ensue.

28. *A vertical reciprocating engine is bolted on to the end of a cantilever consisting of two rolled joists which are*

rigidly built into a wall. Find the speed of the engine which will set up dangerous vibrations. Moment of inertia of the two joist sections, 54 inch⁴ units; weight of the joists, 320 lb.; length of joists, 120 ins.; weight of engine, 1500 lb.; E for joists, 30,000,000 lb. per sq. in.

Neglecting the weight of the joists

$t=2\pi\sqrt{\frac{\Delta}{g}}$ Where Δ is the deflection in ft.

δ „ „ ins.

$$\delta=\frac{WL^3}{3EI}=\frac{1500\times120^3}{3\times30000000\times54}=0\cdot53'' \qquad \Delta=0\cdot044'$$

$$t=6\cdot28\sqrt{\frac{0\cdot044}{32\cdot2}}=0\cdot23 \text{ sec.} \quad \text{or} \quad \mathbf{260\ r.p.m.}$$

Allowing for the weight of the joists—

$\delta=0\cdot57''$ $\Delta=0\cdot048'$

$t=0\cdot25$ sec. or **240** vibrations p.m.

Thus if the engine runs at about 240 r.p.m. dangerous vibrations will be set up. If the engine must run at about this speed trouble may be avoided by placing a heavy weight between it and the wall, thus altering the natural period.

It should be noted that every section has *two* moments of inertia, one about a horizontal and one about a vertical section, and that either of these gives a periodicity which should be avoided.

29. *A mast of a ship is of uniform section 25 ins. external diameter and 0·5 in. thick. Its total height above the level of the water is 70 ft., of which 60 ft. is above the wedging where it is rigidly held. The vessel makes a single roll in 5 secs. through an angle of 30° on each side of the vertical. Find the bending stress at the wedging due to the weight of the mast and to its inertia.*

Sectional area of mast $=38\cdot5$ sq. ins.

$$\text{Weight above wedging}=\frac{38\cdot5\times0\cdot28\times12\times60}{2240}=3\cdot47 \text{ tons}$$

Height of its c. of g. above wedging $=30$ ft.

Bending moment due to weight of mast when at extreme position $=3\cdot47\times30\times\sin30°=52$ tons-ft. $=M_1$

$$\left.\begin{matrix}\text{Moment of inertia of 60 ft. portion}\\ \text{of mast about its own c. of g.}\end{matrix}\right\}=\frac{3\cdot47}{g}\times\frac{60^2}{12}$$

Ditto about base of 70 ft. mast

$$=\frac{3\cdot47}{g}\times\frac{60^2}{12}+\frac{3\cdot47}{g}\times 40^2=205 \text{ tons-ft. units.}$$

Assume the axis of rolling passes through the base of the mast and that the motion is angular S.H.M. Then the angular displacement from the vertical at any instant is—

$$\theta=a\cos\frac{2\pi t}{T}$$

Angular acceleration is $\frac{d^2\theta}{dt^2}=-a\frac{4\pi^2}{T^2}\cdot\cos\frac{2\pi t}{T}$

Bending moment at wedging due to rolling $\Big\}=\Big\{$Couple acting on the upper part of mast$\Big\}=I\frac{d^2\theta}{dt^2}$

where I is moment of inertia of upper part of mast about the axis of rolling=205 tons-ft. units.

Maximum bending moment at wedging due to rolling $\Big\}=M_2=205a\frac{4\pi^2}{T^2}$

and occurs when the mast is at maximum displacement.

Total maximum bending moment at wedging

$$=M_1+M_2=52+205\frac{\pi}{6}\cdot\frac{4\pi^2}{5^2}=221 \text{ tons-ft.}$$

Modulus of section (Z) of mast (see Chap. XI. Vol. I.)
=231 in. units.

$$\text{Maximum bending stress}=\frac{221\times12}{231}=\mathbf{11\cdot5} \text{ tons sq. in.}$$

30. *Find the approximate whirling speed of an unloaded hollow shaft running in swivelled bearings at each end. External diam.* 3″, *internal* 2″. *Length* 10 *ft. between bearing centres.* E=30,000,000 *lb. sq. in.*

Weight per inch run=1·1 lb.

$$I=\frac{3\cdot14(81-16)}{64}=3\cdot19$$

$$\delta=\frac{5wl^4}{384EI}=\frac{5\times1\cdot1\times120^4}{384\times30{,}000{,}000\times3\cdot19}=0\cdot031''$$

$$N=\frac{187\cdot8}{\sqrt{0\cdot031}}=\frac{187\cdot8}{0\cdot176}=\mathbf{1067 \text{ r.p.m.}}$$

(Dunkerley, 1185.)

31. *A shaft rotates in swivelled bearings, find the speeds at which it will whirl in one, two, and three bows when unloaded, also when loaded with a central pulley. Length between bearing centres,* 8′ *; diam. of shaft,* 1·5″ *; weight of pulley,* 50 *lb.;* E, 14,000 *tons sq. in.*

Unloaded— $\delta = \dfrac{L^4}{25920d^2}$

$$N_1 = \frac{187{\cdot}8}{\sqrt{\delta}} = \frac{30200d}{L^2} = \frac{30200 \times 1{\cdot}5}{64} = \mathbf{709\ r.p.m.}$$

$N_2 = 709 \times 4 = 2836$ r.p.m. $N_3 = 709 \times 9 = \mathbf{6381\ r.p.m.}$

Loaded— $\delta = \dfrac{36WL^3}{EI}$ $N = \dfrac{187{\cdot}8}{\sqrt{\delta}} = \dfrac{37900d^2}{\sqrt{WL^3}}$

Neglecting the weight of the shaft $\left.\right\} N = \dfrac{37900 \times 1{\cdot}5^2}{\sqrt{50 \times 512}} = 547$ r.p.m.

Taking the weight of shaft into account $\left.\right\} N = \dfrac{37900 \times 1{\cdot}5^2}{\sqrt{79{\cdot}6 \times 512}} = \mathbf{432\ r.p.m.}$

32. *Find the whirling speed of a shaft* 4″ *diameter,* 4′ 6″ *long, when running in short or swivelled bearings and loaded centrally with a wheel weighing* 1000 *lb.* (i) *Neglecting the weight of the shaft;* (ii) *allowing for the weight of the shaft.*

(i) $N = \dfrac{37700 \times 4^2}{\sqrt{1000 \times 4{\cdot}5^3}} = \mathbf{2000\ r.p.m.}$

(ii) Equivalent weight of shaft $\left.\right\} = 0{\cdot}785 \times 4^2 \times 0{\cdot}28 \times \frac{48}{77} \times 54 = 118$ lb.

$$N = \frac{37700 \times 4^2}{\sqrt{1118 \times 4{\cdot}5^3}} = \mathbf{1{,}900\ r.p.m.\ (nearly).}$$

33. *Find the whirling speed of a tapered shaft when loaded with a central wheel weighing* 15 *tons. The bearings are short or swivelled. Total length between centres of bearings* 60 *ins.*

Diameter 1, 4″	*Diameter* 4, 5·5″	*Diameter* 7, 7·0″
„ 2, 4·5″	„ 5, 6″	„ 8, 7·5″
„ 3, 5″	„ 6, 6·5″	„ 9, 8″

Diameter 10, 8″

$E = 12{,}000$ *tons sq. inch.*

See p. 273, Vol. I.

$$\delta = \frac{15\times5^3}{1\cdot82\times12000}\left(\frac{1}{256}+\frac{9}{410}+\frac{25}{625}+\frac{49}{915}+\frac{81}{1296}+\frac{121}{1785}+\frac{169}{2401}+\frac{225}{3136}\right.$$

$$\left.+\frac{289}{4096}+\frac{361}{4096}\right) = \frac{15\times125\times0\cdot55}{1\cdot82\times12000} = 0\cdot047''$$

Check.—If the beam had tapered in such a manner that the stress was constant at all sections the deflection would have been $\delta = \frac{Wl^3}{32EI} = \frac{15\times60^3\times20\cdot4}{32\times12000\times8^4} = 0\cdot041''$, which shows that the above is not far from the truth.

$$N = \frac{187\cdot8}{\sqrt{0\cdot047}} = \mathbf{866\ r.p.m.}$$

If the shaft had been designed for constant stress conditions, the whirling speed would have been

$$N = \frac{187\cdot8}{\sqrt{0\cdot041}} = \mathbf{927\ r.p.m.}$$

34. *A vertical shaft is* 0·4″ *diameter and* 10″ *long between the inside edges of long bearings which fix its direction at the ends. A pulley midway between the bearings weighs* 12 *lb., its centre of gravity is* 0·02″ *from the axis of the shaft. Find* (i) *the whirling speed;* (ii) *the bending stress when the shaft is running at* 0·85 *of its critical speed.*

See p. 270, Vol. I.

$$\text{Whirling speed} = \frac{75400d^2}{\sqrt{WL^3}} = \frac{75400\times0\cdot16}{\sqrt{12\times(\frac{10}{12})^3}}$$

$$N = \mathbf{4600\ r.p.m.}$$

To get the deflection at any other speed.

Let R = Initial radius of the c. of g. of rotating weight (ft.).

y = Additional radius (ft.) when the shaft is bent, due to the centrifugal force.

e = Force required to produce unit deflection, *i.e.* 1 ft. of the shaft.

$$\delta = \frac{WL^3}{192EI} \qquad W = \frac{192EI\delta}{L^3} \qquad e = \frac{192EI}{L^3}$$

Then equating the centrifugal force to the elastic righting force of the shaft—

$$\frac{W\omega^2}{g}(R+y) = ey \qquad y = R\frac{W\omega^2}{eg - W\omega^2}$$

This is infinitely great when $eg = W\omega^2$ at the critical speed ω_c, putting $\omega_c^2 = \frac{eg}{W}$, we have $y = R\frac{\omega^2}{\omega_c^2 - \omega^2}$

(See Morley's "Strength of Materials," p. 459, 1917 edition.)

In this case $\omega_c = 0{\cdot}85\omega$, hence—

$$y = 0{\cdot}02\frac{\omega^2}{0{\cdot}72\omega^2 - \omega^2} = 0{\cdot}071''$$

Centrifugal force $= 0{\cdot}00034 \times 12 \times \frac{0{\cdot}071 + 0{\cdot}02}{12}(4600 \times 0{\cdot}85)^2$
$= 473$ lb.

Bending moment (see p. 526, Vol. I.) $= \frac{473 \times 10}{8} = 590$ in.-pounds.

Bending stress (see pp. 433 and 452, Vol. I.) $= \frac{590 \times 10{\cdot}2}{0{\cdot}064} =$ **94,000 lb. sq. in.**

Unless the shaft was made of exceptionally good steel it would certainly fracture under this stress if prolonged; but if it were only momentarily subjected to this stress as the speed was being increased, it is possible it might last for a long time.

It is very questionable whether any bearings, however long, hold a shaft perfectly rigid at the ends; if there is only a very small amount of shake it materially affects the results.

Whirling of Shafts

35. The expression given on p. 269, Vol. I., is correct for some arrangements of loading, but not for all. A coefficient K, however, can be introduced to bring the calculated and experimental values into agreement: thus $N = \frac{187{\cdot}8K}{\sqrt{\delta}}$, where

K = 1 for a shaft centrally loaded between the bearings, whether the ends are "fixed" or "free."

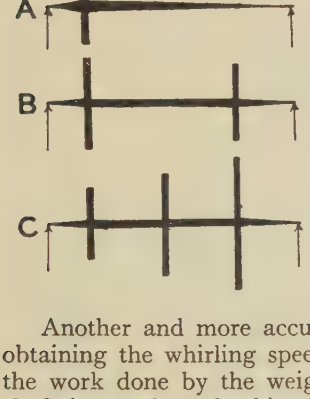
K = 1·125 for an evenly distributed load, "free" ends.

K = 1·138 for an evenly distributed load, "fixed" ends.

For turbine shafts, Baumann (*Journal of Institute of Electrical Engineers*, vol. xlviii., p. 807, 1912) gives K = 1·06 to 1·07.

In all cases δ is the maximum deflection.

For tapered shafts, Naylor (*Selected Engineering Papers, Institution of Civil Engineers*, " Whirling Speeds of Loaded Tapered Shafts ") gives



K=0·97 (mean of 3).

K=1·10 (mean of 2 unequal loads).
K=1·16 (equal loads).

K=1·05 (mean of 2).

Another and more accurate method (due to Morley) of obtaining the whirling speed is based on the principle that the work done by the weights in bending or straining the shaft is equal to the kinetic energy stored in the whirling weights.

Let a shaft be loaded with weights W_1, W_2, W_3, etc., and δ_1, δ_2, δ_3, etc., be the respective deflections in inches immediately under those weights. Let V_1, V_2, V_3, etc., be the corresponding linear velocities in feet per second of the centre points of the shaft which describe small circles of radii δ_1, δ_2, δ_3, etc., when the shaft whirls. Then $V=\frac{2\pi\delta N}{12\times 60}$

$$\left.\begin{matrix}\text{Elastic strain energy}\\ \text{in ft.-pounds}\end{matrix}\right\}=\left\{\begin{matrix}\text{Kinetic energy of whirling}\\ \text{weights in ft.-pounds.}\end{matrix}\right.$$

$$\tfrac{1}{2}\times\tfrac{1}{12}(W_1\delta_1+W_2\delta_2+W_3\delta_3+\text{etc.})=\left(\frac{W_1V_1^2}{2g}+\frac{W_2V_2^2}{2g}+\frac{W_3V_3^2}{2g}+\text{etc.}\right)$$

$$\tfrac{1}{24}(W_1\delta_1+W_2\delta_2+W_3\delta_3+\text{etc.})=\frac{2^2\pi^2N^2}{12^2\times 60^2\times 2g}(W_1\delta_1^2+W_2\delta_2^2+W_3\delta_3^2+\text{etc.})$$

$$N=187{\cdot}8\sqrt{\frac{W_1\delta_1+W_2\delta_2+W_3\delta_3+\text{etc.}}{W_1\delta_1^2+W_2\delta_2^2+W_3\delta_3^2+\text{etc.}}}$$

When there is one load only $N=\frac{187{\cdot}8}{\sqrt{\delta}}$, which is our

original expression. But when there are more than one load the above expression must be used. It holds for parallel or tapered shafts, but, of course, the corresponding values for the deflection must be inserted.

36. *Find the whirling speed of a shaft 7″ diameter, 10 ft. long, free ends.*

Loaded (1) 2 *tons at* 3′ (2) 3 *tons at* 5′ (3) 2 *tons at* 7′

E=12,000 *tons sq. in.* I=117·7.

First find the deflection (see p. 523, Vol. I.), or make use of a Livingstone diagram.

Load.	Position of load.	Position of deflection.	*k*
2 tons	0·3L	0·3L (1)	0·0147
	,,	0·5L (2)	0·0165
	,,	0·7L (3)	0·0124
3 tons	0·5L	0·3L (1)	0·0165
	,,	0·5L (2)	0·0208
	,,	0·7L (3)	0·0165
2 tons	0·7L	0·3L (1)	0·0124
	,,	0·5L (2)	0·0165
	,,	0·7L (3)	0·0147

$$\delta_1=\frac{(0\cdot0147\times2+0\cdot0165\times3+0\cdot0124\times2)120^3}{12{,}000\times117\cdot7}=0\cdot127''$$

$$\delta_2=\frac{(0\cdot0165\times2+0\cdot0208\times3+0\cdot0165\times2)120^3}{12{,}000\times117\cdot7}=0\cdot157''$$

$$\delta_3=\delta_1$$

$$N=187\cdot8\sqrt{\frac{2\times0\cdot127+3\times0\cdot157+2\times0\cdot127}{2\times0\cdot127^2+3\times0\cdot157^2+2\times0\cdot127^2}}$$

$$N=187\cdot8\times2\cdot66=\mathbf{499\ r.p.m.}$$

By our original expression $N=\frac{187\cdot8}{\sqrt{0\cdot157}}=474$

This gives $K=\frac{499}{474}=\mathbf{1\cdot05}$.

The loading is somewhat similar to Naylor's C shaft, which gave the same value of K.

CHAPTER VIII

GYROSCOPIC ACTION

1. *A steam turbine on a boat makes* 1,800 *r.p.m.* (*clockwise to observer looking toward the bow*). *The moment of inertia of the rotor is* 3,800 *lb.-ft. units. The boat is steered in a circle to starboard* (i.e. *to right facing bow*) *and makes one complete turn in* 1·5 *mins. Find the couple acting on the hull and state how it acts.*

Gyro. couple $= 0{\cdot}00034 WK^2Nn$

$$= 0{\cdot}00034 \times 3{,}800 \times 1{,}800 \times \frac{1}{1{\cdot}5} = \mathbf{1{,}550\ pounds\text{-}feet.}$$

The couple tends to lift the stern.

2. *A dynamo on a train has the armature shaft parallel with the axles and rotates in the same sense. The armature is fixed midway between the bearings, which are* 48 *inches apart. The weight of the armature is* 1,700 *lb. and its radius of gyration is* 13 *inches. Revolutions per minute,* 920. *Find the pressure on the bearings and state in which direction it acts on both the right- and left-hand bearing when looking towards the engine and when the train runs round a curve of* 560 *feet radius at a speed of* 50 *miles per hour, the centre of the curve being to the left of the observer.*

$$\text{Gyro. couple} = 0{\cdot}00034 \times 1{,}700 \times \left(\frac{13}{12}\right)^2 \times 920 \times \frac{50 \times 5280}{560 \times 6{\cdot}28 \times 60}$$

$= 781$ foot-pounds $= 9{,}372$ in.-pounds.

Press. on bearings $= \frac{9372}{48} = 195$ lb. downwards on the right-hand bearing and upwards on the left.

Net press. on right-hand bearing $= 195 + \frac{1700}{2} = \mathbf{1{,}045\ lb.}$

Net press. on left-hand bearing $= \frac{1700}{2} - 195 = \mathbf{655\ lb.}$

3. *The propeller on a boat makes* 300 *r.p.m. and has a moment of inertia of* 2,240 *lb.-ft.*^{2} *units* (WK^2). *The boat makes a complete turn in* 90 *seconds. Find the gyro. couple.*

$$\text{Gyro. couple} = 0{\cdot}00034 \times 2{,}240 \times 300 \times \tfrac{60}{90}$$
$$= \mathbf{152{\cdot}3\ foot\text{-}pounds.}$$

4. *In an experiment with a gyroscopic top the weight of the wheel was* 1·123 *lb. The radius of gyration* 0·11 *foot. It precessed* 1·5 *r.p.m. when a weight of* 0·073 *lb. was hung on the gyroscope at a radius of* 0·16 *foot. Find the speed of rotation of the wheel.*

$$\text{Gyro. couple} = 0{\cdot}073 \times 0{\cdot}16 = 0{\cdot}0117 \text{ pound-foot.}$$
$$\text{Then } 0{\cdot}0117 = 0{\cdot}00034 \times 1{\cdot}123 \times 0{\cdot}0121 \times N \times 1{\cdot}5$$
$$N = \mathbf{1{,}690\ r.p.m.}$$

Experiment gave 1,800 r.p.m., but there is great difficulty in determining the exact speed of the wheel without special appliances which were not available.

5. *A dynamo on board ship is arranged with its armature athwart the ship. Find the pressure on the bearings due to the gyroscopic action each time the ship rolls, and state the direction in which the pressure acts. Weight of armature,* 160 *lb.; radius of gyration,* 0·5 *foot; distance apart of bearings,* 2·2 *feet; r.p.m. of armature,* 1,500; *angle of roll on each side of vertical,* 30°; *time of one complete roll,* 1·7 *secs.*

If the roll is S.H. angular motion the maximum angular velocity will occur at the instant when the ship is erect and will be $\frac{\pi}{2} \times$ mean angular velocity. This ratio is assumed to be 1·5.

The ship rolls 60° in 0·85 sec., or at a mean rate of 1 rev. in 5·1 secs., *i.e.* $\frac{1}{5{\cdot}1}$ r.p.s. The maximum speed is greater than the mean, say, 1·5, or the maximum speed of rotation is $\frac{1{\cdot}5}{5{\cdot}1}$ r.p.s.

$$\text{Gyro. couple} = 0{\cdot}00034 \times 160 \times 0{\cdot}5^2 \times 1{,}500 \times \frac{1{\cdot}5 \times 60}{5{\cdot}1}$$
$$= 360 \text{ foot-lb.}$$

Thus the pressure on each bearing is

$$\frac{360}{2{\cdot}2} = \textbf{164 lb. acting horizontally.}$$

6. *A motor car of* 10 *ft. wheel base is travelling at* 34 *miles an hour round a curve of* 250 *ft. radius to the left-hand of the driver. The crank shaft is arranged at right angles to the road wheel axles and rotates in clockwise direction at the starting handle. The weight of the flywheel is* 100 *lb., and the radius of gyration* 9 *ins. The speed of the engine is* 1,200 *r.p.m. Find the change of load on the steering wheels, and state whether it is increased or decreased.*

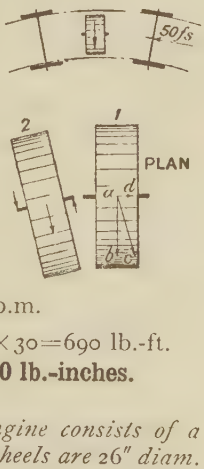
If the above-mentioned car gets a side slip which causes it to turn through a right angle in half a second, find the bending moment on the crank shaft due to the gyroscopic couple.

34 m.p.h. = 50 f.s.

Precession: 1 rev. in $\frac{250 \times 2\pi}{50} = 31{\cdot}4$ secs. $= \frac{60}{31{\cdot}4}$ r.p.m.

Gyro. couple $= 0{\cdot}00034 WK^2Nn$

$$= 0{\cdot}00034 \times 100\left(\frac{9}{12}\right)^2 1{,}200 \times \frac{60}{31{\cdot}4} = 44 \text{ pounds-feet.}$$



The wheel precesses from *ab* to *ac*, hence the external couple acting on the crank shaft tends to lift the front end, so that the reaction of the steering wheels will be increased. If the flywheel were parallel to the road wheels and rotating in the same sense the gyro. couple would increase the load on the outer wheels and decrease it on the inner wheels.

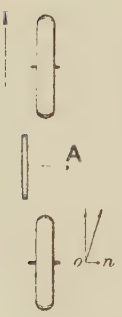
Side slip.—Precession $= \frac{1}{2}$ r.p.s. $= 30$ r.p.m.

Gyro. couple $= 0{\cdot}00034 \times 100(\frac{9}{12})^2 1{,}200 \times 30 = 690$ lb.-ft.

Bending moment on crank shaft $=$ **8,280 lb.-inches.**

7. *The flywheel on a motor bicycle engine consists of a steel disc* 12″ *diam.,* $\frac{1}{2}$″ *thick. The road wheels are* 26″ *diam. and the gear is* 5 *to* 1. *When travelling at* 15 *m.p.h. the*

bicycle turns to the right round a curve of 45 ft. radius. The flywheel rotates in the same sense as the road wheels. Find the gyroscopic couple due to the flywheel. Does it act with or against the centrifugal force?



Speed, 15 m.p.h.=1,320 f.p.m.

Precession of cycle $= \frac{1320}{45 \times 2 \times 3 \cdot 14} = 4 \cdot 67$ r p.m.

Weight of flywheel=14·6 lb.

$K^2 = \frac{D^2}{8} = \frac{1}{8}$ foot^2 units.

R.p.m. of road wheels $= \frac{1320 \times 12}{26 \times 3 \cdot 14} = 194$

R.p.m. of flywheel $= 194 \times 5 = 970$

Gyro. couple $= 0 \cdot 00034 \times 14 \cdot 6 \times \frac{1}{8} \times 970 \times 4 \cdot 67$

$= \mathbf{2 \cdot 8}$ **foot-lb.**

A downward force at A makes it precess in the direction required. The gyro. couple acts in the same direction as the centrifugal couple.

8. *In a ball thrust bearing the balls rotate round the axis of the bearing and at the same time rotate on their own axes, thus there is a gyroscopic couple acting on the balls which is only possible when there is friction between the ball and the races; if the friction is insufficient the balls skid. Find an expression for the pressure on the balls required to prevent skidding. Let* P=*pressure in lb. on each ball;* r=*radius of balls in ins.;* R=*radius of ball track in ins.;* μ=*coefft. of friction between ball and track;* W=*weight of one ball* $= \frac{4 \times 0 \cdot 28 \times \pi r^3}{3} = 1 \cdot 17 r^3$ *lb.;* N_1=*number of r.p.m. of ball round axis of bearing;* n=*number of r.p.m. of balls on own axis;* k^2=*square of radius of gyration of balls* $= \frac{2r^2}{5 \times 144}$ (*feet*^{2} *units*).

Gyro. couple $= 0 \cdot 00034 W K^2 N n$.

Resistance to skidding $= \frac{2P\mu r}{12}$

When skidding occurs the gyro. couple is equal to the friction couple.

$$P=\frac{12}{2\mu r}\times 0{\cdot}00034\times 1{\cdot}17r^3\times\frac{r^2}{360}\times N_1\times\frac{N_1R}{r}$$

$$P=\frac{0{\cdot}00000662r^3N_1^2R}{\mu}$$ Where P is the least pressure per ball which will prevent skidding.

Let N = r.p.m. of shaft $=2N_1$

$$P=\frac{0{\cdot}00000165r^3N^2R}{\mu}$$

Let $R=5''$; $r=1''$; $N=1{,}000$; $\mu=0{\cdot}33$.

Then P = **25 lb.** on each ball.

9. *Taking the safe load on each ball for a flat-raced thrust bearing as* $P_s=\frac{500000d}{N_sD+200d}$ *in which* $D=6''$, $d=1''$, $\mu=0{\cdot}1$. *Find the speed at which skidding of the balls will occur.*

Assume $N_s=2{,}000$ r.p.m.

Then $P=\frac{0{\cdot}00000165\times 0{\cdot}5^3\times 2000^2\times 3}{0{\cdot}1}$ $P=24{\cdot}8$ lb.

$$P_s=\frac{500000\times 1}{2000\times 6+200\times 1}=41 \text{ lb.}$$

The balls will skid if the load on each is less than 24·8 lb.; but the safe load is 41 lb., therefore they are quite safe against skidding at this speed when working near their safe load.

Take $N_s=2{,}200$ r.p.m. Then

$P=30$ lb. $P_s=37$ lb.

The balls are safe against skidding but the margin is small.

Take $N_s=2{,}300$ r.p.m. Then $P=37$ lb. $P_s=36$ lb.

Thus if this speed be exceeded the balls will skid at any safe load.

CHAPTER IX

FRICTION

1. *Find the effort required to push a block weighing* 5 *tons up an inclined plane and to pull it down, also the work done when the line of action of the force is* (a) *horizontal*, (b) *parallel to plane*, (c) *minimum. The plane rises* 8 *feet vertical in* 20 *feet horizontal. Coefficient of friction* $\mu = 0{\cdot}25$.

$\tan \alpha = \frac{8}{20}$, $\alpha = 21^\circ\ 50'$, $\cos \alpha = 0{\cdot}928$, $\mu = 0{\cdot}25 = \tan \phi$, $\phi = 14^\circ$, $\cos \phi = 0{\cdot}97$, $\alpha + \phi = 35^\circ\ 50'$, $\tan (\alpha + \phi) = 0{\cdot}722$, $\sin (\alpha + \phi) = 0{\cdot}585$, $\tan (\alpha - \phi) = 0{\cdot}138$.

(See p. 291, Vol. I.)

(*a*) *Horizontal effort.*

$P_b = W \tan (\alpha + \phi) = 5 \times 0{\cdot}722 = 3{\cdot}61$ tons.

Resolve parallel to plane

$P_{hp} = W \tan (\alpha + \phi) \cos \alpha = 3{\cdot}61 \times 0{\cdot}928 = 3{\cdot}35$ tons.

Length of plane $= \sqrt{8^2 + 20^2} = 21{\cdot}54$ feet.

Work done in pushing block up plane $= 3{\cdot}35 \times 21{\cdot}54$
$= \mathbf{72{\cdot}2\ foot\text{-}tons.}$

Effort parallel to plane when pulling block down
$= W \tan (\alpha - \phi) \cos \alpha = 5 \times 0{\cdot}138 \times 0{\cdot}928 = 0{\cdot}64$ ton.

Work done by the block in sliding down the plane
$= 0{\cdot}64 \times 21{\cdot}54 = \mathbf{13{\cdot}8\ foot\text{-}tons.}$

This might also have been arrived at thus—

Total work done in lifting block $\times \frac{1}{\text{efficiency}}$

$$= 5 \times 8\ \frac{\tan (\alpha + \phi)}{\tan \alpha} = \mathbf{72{\cdot}2\ foot\text{-}tons.}$$

(See p. 294, Vol. I.)

Work done by the block in sliding down the plane

$$= \frac{40(\tan \alpha - \phi)}{\tan \alpha} = \mathbf{13{\cdot}8\ foot\text{-}tons.}$$

The block slides down *against* the pull and does not require pulling down.

(*b*) *Parallel to plane.*

Effort when pulling block up

$P_p = W(\mu \cos \alpha + \sin \alpha) = 5(0{\cdot}25 \times 0{\cdot}928 + 0{\cdot}372) = 3{\cdot}02$ tons.

Work done (up) $= 3{\cdot}02 \times 21{\cdot}54 =$ **65 foot-tons.**

Effort when letting the block slide down

$= W(\sin \alpha - \mu \cos \alpha)$

$= 5(0{\cdot}372 - 0{\cdot}25 \times 0{\cdot}928) = 0{\cdot}7$ ton acting up the plane.

Work done (down) $0{\cdot}7 \times 21{\cdot}54 =$ **15 foot-tons** by block.

This might have been arrived at thus—

Work done in pulling block up plane

= Work done in lifting block + work done in dragging block through the horizontal distance

$= 5 \times 8 + 5 \times 20 \times 0{\cdot}25 =$ **65 foot-tons.**

Work done by block in being lowered down plane

= Work done in lowering block − work done in dragging block through the horizontal distance

$= 40 - 25 =$ **15 foot-tons.**

(*c*) *Minimum effort.*

(Up) $P_{min.} = W \sin (\phi + \alpha)$ (acting at an angle ϕ upwards from the plane) $= 5 \times 0{\cdot}585 = 2{\cdot}93$ tons.

Resolve parallel to plane—

(Up) $P_{min.\ p} = P_{min.} \cos \phi = 2{\cdot}93 \times 0{\cdot}97 =$ **2·84 tons.**

Work done (up) $= 2{\cdot}84 \times 21{\cdot}54 =$ **61·2 foot-tons.**

(Down) $P_{min.} = W \sin (\alpha - \phi)$ (acting at an angle ϕ downwards from the plane) $= 5 \times 0{\cdot}136 =$ **0·68 ton.**

Work done (down) $= 0{\cdot}68 \times 0{\cdot}97 \times 21{\cdot}54 =$ **14·2 foot-tons.**

2. *Find the force required to haul a railway wagon steadily up an incline of* 1 *in* 45. *The coefficient of friction between the axle and the bearing is* 0·07, *and the rolling resistance of the wheels on the rail is* 4·6 *pounds per ton. The weight of the vehicle* = 27 *tons; diameter of the axle* $= 3\frac{1}{2}''$; *diameter of the wheel* $= 42''$. *What additional force would be required to accelerate this vehicle* 4·1 *feet per second per second, neglecting rotational inertia of the wheels?*

Effort required to lift vehicle	$\frac{27 \times 2240}{45} =$	1344 lb.
Rolling resistance	$4{\cdot}6 \times 27 =$	124 ,,
Axle friction	$27 \times 2{,}240 \times 0{\cdot}07 \times \frac{3{\cdot}5}{42} =$	353 ,,
	Total .	**1821 lb.**

$$\frac{\text{Force required to accelerate (F)}}{\text{Weight (W)}} = \frac{\text{Acceleration } (f_a)}{\text{Accel. due to gravity } (g)}$$

$$F = \frac{Wf_a}{g} = \frac{27 \times 2240 \times 4{\cdot}1}{32{\cdot}2} = \mathbf{7{,}700\ lb.}$$

3. *A motor cycle gives off 2 effective horse-power at all speeds. On a level road it runs at 25 miles per hour, but on an incline of 1 in 30 it only runs at 20 miles an hour. The weight of the cycle and rider together is 300 lb. The rolling resistance of the tyres is constant at all speeds, but the air resistance varies as the square of the speed. Show that the total resistance in pounds weight is* $\frac{1}{90}(2075+S^2)$. *Where S is the speed in miles per hour. Find the speed at which the cycle will mount a slope of 1 in 10.*

2 H.P. = 66,000 foot-lb. per minute.

$$25 \text{ m.p.h.} = \frac{25 \times 5280}{60} = 2200 \text{ ft.-min.}$$

$$R_{25} = \tfrac{66000}{2200} = 30 \text{ lb. resistance at 25 m.p.h.}$$

$30 = c + n \times 25^2$, where c is the rolling resistance and ns^2 is the air resistance.

$$R_{20} = \frac{30 \times 25}{20} = 37{\cdot}5 = c + n \times 20^2 + \tfrac{300}{30}$$

$$30 - 625n = 37{\cdot}5 - 400n - 10 \qquad n = \tfrac{1}{90}$$

$$30 - \tfrac{625}{90} = c = \tfrac{2075}{90} = 23{\cdot}05$$

$$R = 23{\cdot}05 + \frac{S^2}{90} = \tfrac{1}{90}(2075 + S^2).$$

The speed at which it will mount a slope of 1 in 10.

$$\frac{66000}{\text{ft.p.min.}} = \tfrac{1}{90}(2075 + S^2) + \tfrac{300}{10}$$

$$\frac{66000 \times 90}{88 \times S} = 2075 + S^2 + 2700$$

$67500 = 4775S + S^3$. Solving by trial and error.

$S = \mathbf{13{\cdot}6\ m.p.h.}$

4. *The resistances overcome by a motor car are as follows:* (a) *The internal friction of the engine and gearing, also the rolling resistance of the wheels, which may be assumed to be constant at all speeds, and a fraction K of the weight of the*

car. (b) *The wind resistance, which varies as the square of the speed and may be written* nv^2. *Find an expression for the resistance of the car in terms of the speed, also find the incline the car can mount at* 14 *m.p.h. Weight of car,* 2,500 *lb.; I.H.P. of engine,* 24; *speed,* 27 *m.p.h. on level and* 19 *m.p.h. on an incline of* 1 *in* 10.

Let V=speed in miles per hour.

Then $24\times33000=\frac{27\times5280}{60}(K\times2500+n27^2)$.

Resistance at 27 m.p.h. on the level in pounds

$$=\frac{24\times33000\times60}{27\times5280}=2500K+27^2n=333.$$

Resistance at 19 m.p.h. on an incline of 1 in 10 in pounds

$$=\frac{24\times33000\times60}{19\times5280}=2500K+19^2n+\tfrac{2500}{10}=474.$$

By subtraction $109=368n$ $\qquad n=0\cdot296$

Substituting the value of n in the above expression

$$\frac{333-729\times0\cdot296}{2500}=\tfrac{117}{2500}=K=0\cdot0468.$$

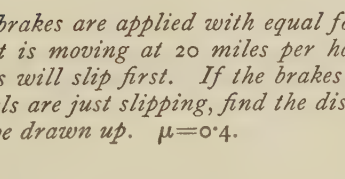
Resistance in pounds $=117+0\cdot296V^2+2{,}500\times$inclination.

Horse-power $=\frac{V}{375}(117+0\cdot296V^2+2500\times\text{incl.})$.

Incline at 14 m.p.h.: $24=\frac{14}{375}(117+0\cdot296\times14^2+2500\times\text{incl.})$.

Inclination=**0·187 or 1 in 5·3.**

5. *A lorry has a wheel base of* 10 *feet, and its centre of gravity is* 4 *feet above ground and midway between the*



axles. If the brakes are applied with equal force to all four wheels when it is moving at 20 *miles per hour, show that the back wheels will slip first. If the brakes are kept on so that these wheels are just slipping, find the distance in which the lorry will be drawn up.* $\mu=0\cdot4$.

Taking moments about the point A—

$$R_2 \times 10 - W \times 5 = -\frac{W}{g} \cdot f \times 4 \quad . \quad . \quad . \quad (i)$$

By moments about B—

$$R_1 \times 10 - W \times 5 = +\frac{W}{g} \cdot f \times 4 \quad . \quad . \quad . \quad (ii)$$

Hence R_2 is always less than R_1.

But if the brakes are applied equally the retarding forces at A and B must be the same. Hence $F_1 = F_2$.

But $F_1 = \mu_1 R_1$, where μ_1 is the effective coefficient of friction at A. Hence $\mu_1 R_1 = \mu_2 R_2$.

And since $R_2 < R_1$, μ_2 will rise to the limiting value before μ_1, and the rear wheels will slip first.

If the brakes are kept at the pressure just necessary to cause this slipping of the rear wheels, the values of F_1 and F_2 will each be μR_2.

Whence $$\frac{W}{g} \cdot f = 2\mu R_2$$

and from the equations (i) and (ii) $R_2 = \dfrac{5W}{10+8\mu}$

and $$f = \frac{2\mu R_2 g}{W} = \frac{2\mu g}{W} \times \frac{5W}{10+8\mu}$$

$$= \frac{5g\mu}{5+4\mu} = \frac{5 \times 32 \times 0{\cdot}4}{6{\cdot}6} = 9{\cdot}7 \text{ f.p.s.p.s.}$$

Hence distance required to draw up is

$$S = \frac{\left(\frac{20 \times 22}{15}\right)^2}{2 \times 9{\cdot}7} = \textbf{44 feet} \text{ nearly.}$$

6. *In the case of a train fitted with air brakes the driver can vary the pressure on the brake blocks at will. Find* (i) *The pressure on the brake blocks for pulling up the train in the shortest distance possible.* (ii) *The distance run after the application of the brakes when* (a) *the wheels skid,* (b) *the wheels are on the point of skidding but do not actually do so. Weight of train,* 350 *tons ; speed,* 60 *m.p.h. ; coefficient of friction—between wheel and rail when not skidding,* 0·13 ; *coefficient of friction—between wheel and rail when skidding,* 0·05 ; *coefficient of friction between wheel and brake shoe,* 0·26.

Maximum resistance on surface of wheel when not skidding $=350\times0{\cdot}13=45{\cdot}5$ tons.

Maximum resistance on surface of wheel when skidding $=350\times0{\cdot}05=17{\cdot}5$ tons.

(i) Pressure on brake shoes when not skidding $\Big\}=\dfrac{45{\cdot}5}{0{\cdot}26}=175$ tons.

The pressure must be greater than this in order to start skidding.

After skidding commences the pressure need not be greater than $\Big\}\dfrac{17{\cdot}5}{0{\cdot}26}=$ **67·3 tons.**

(ii) Kinetic energy of train $=\dfrac{350\times88^2}{2\times32{\cdot}2}=42{,}100$ foot-tons.

Distance run (*a*) $\dfrac{42100}{45{\cdot}5}=$ **925 feet** when not skidding.

(*b*) $\dfrac{42100}{17{\cdot}5}=$ **2406 feet** when skidding.

7. *Find the time required to bring a locomotive to rest and the distance* S *it will travel after the application of the brakes. Weight of engine and tender, exclusive of wheels,* 90 *tons; pressure on each of the brake blocks on the four wheels,* 7,090 *lb.; coefficient of friction,* 0·3; *weight of each wheel,* 1 *ton; radius of gyration of wheels,* 2·5 *feet; rolling and other resistances,* 8 *lb. per ton; speed,* 60 *m.p.h.; diameter of wheels,* 6 *ft.*

Braking force on each wheel $=0{\cdot}3\times7{,}090=2{,}127$ lb.

Kinetic energy of engine and tender $=\dfrac{WV^2}{2g}+\frac{1}{2}I\omega^2$

$$=\frac{(90+4)88^2}{64{\cdot}4}+\frac{1}{2}\left(\frac{4\times2{\cdot}5^2}{32{\cdot}2}\right)\left(\frac{88}{3}\right)^2$$

$=11{,}300+334=11{,}630$ foot-tons.

Work done in braking and overcoming resistances = kinetic energy.

$$S\left(\frac{2127}{2240}\times4+\frac{8\times94}{2240}\right)=11{,}630$$

$4{\cdot}13S=11{,}630$ $\qquad$ $S=2{,}820$ feet.

Mean speed=44 feet per second.

$$\text{Time}=\tfrac{2420}{44}=\mathbf{64\ seconds.}$$

8. *Twelve railway trucks, each weighing* 13 *tons, are hauled up an incline of* 1 *in* 40. *The rolling coefficient of the wheels on the rails is* 0·014 *in inch units, and the coefficient of friction between the axles and bearings is* 0·03. *The diameter of the wheels is* 3′ 6″ *and that of the axles* 3·25″. *Find the horse-power required at* 20 *miles per hour.*

$$\text{Rolling resistance per truck}=\frac{WK}{R}\quad\text{(see p. 298, Vol. I.)}$$

$$=\frac{13\times2240\times0\cdot014}{21}=19\cdot4\text{ lb.}$$

Resistance per truck due to axle friction referred to circumference of wheel

$$=\frac{13\times2240\times0\cdot03\times3\cdot25}{42}=67\cdot6\text{ lb.}$$

Total resistance=12(67·6+19·4)=1,044 lb.

Work done in lifting trucks

$$=\frac{12\times13\times2240\times20\times5280}{40\times60}=15{,}400{,}000\text{ ft.-lb. per min.}$$

Work done in overcoming resistances

$$=\frac{1044\times20\times5280}{60}=1{,}840{,}000\text{ ft.-lb. per min.}$$

Total work per minute=17,240,000 ft.-lb.

$$\text{H.P.}=\frac{17240000}{33000}=\mathbf{522}.$$

9. *Find the efficiency of a worm and wheel in which the end-thrust is taken by an ordinary collar bearing. Angle of worm,* 30°; *angle of friction for worm and collar,* 12°; *diameter of collar, external,* 4″; *diameter of collar, internal,* 2·5″; *mean diameter of worm,* 3″; *revolutions per minute,* 200; *horse-power on worm wheel,* 6.

$$\text{Efficiency of worm (see p. 295, Vol. I.)}=\frac{\tan 30^\circ}{\tan 42^\circ}=\frac{0\cdot577}{0\cdot9}=64\text{ per cent.}$$

$$\text{Pitch of worm}\quad p=\frac{3\times3\cdot14\tan30^\circ}{12}=0\cdot453\text{ feet.}$$

Horse-power transmitted to worm wheel $= \frac{W\rho N}{33000}$

$$W = \frac{\text{H.P.} \times 33000}{200 \times 0{\cdot}453} = 2{,}200 \text{ lb.}$$

Horse-power wasted in collar friction (see p. 330, Vol. I.)

$$= \frac{\pi N}{198000} \times \frac{2\mu W(R_1{}^3 - R_2{}^3)}{3(R_1{}^2 - R_2{}^2)}$$

$$= \frac{3{\cdot}14 \times 2 \times 0{\cdot}213(2^3 - 1{\cdot}25^3)WN}{198000 \times 3(2^2 - 1{\cdot}25^2)}$$

$$= \frac{WN}{1800000} = \frac{2200 \times 200}{1800000} = 2{\cdot}5 \text{ nearly.}$$

H.P. wasted in worm $= 6(1 - 0{\cdot}64) = 2{\cdot}16$

„ „ collar $= 2{\cdot}50$

Total . . . 4·66

$$\text{Efficiency} = \frac{6}{6 + 4{\cdot}66} = \textbf{57 per cent.}$$

10. *A load of* 10 *tons is lifted* 16 *inches by a screw jack and then lowered to its original position. Find the work done in lifting and lowering* (a) *when the load rotates with the screw*, (b) *when the load rests on a loose head which does not rotate with the screw. Mean diameter of threads*, 4·25″ ; *pitch of threads*, 1·5″ ; *coefficient of friction* (μ), 0·11 ; *friction angle* (ϕ), 6° 20′ ; *effective diameter of collar*, 1·7 × *diam. of threads.*

Angle of threads $\tan \alpha = \frac{1{\cdot}5}{4{\cdot}25 \times 3{\cdot}14} = 0{\cdot}112 \quad \alpha = 6° 25'$

$\tan(\alpha + \phi) = 0{\cdot}226 = \frac{\tan \alpha}{\tan(\alpha + \phi)}$ (see p. 295, Vol. I.)

(*a*) Efficiency of screw when weight rotates $= \frac{0{\cdot}112}{0{\cdot}226} = 49{\cdot}5$ per cent.

$$\text{Work done in lifting} = \frac{10 \times 16}{0{\cdot}497} = \textbf{322 inch-tons.}$$

$$\text{Reversed efficiency} = \frac{0{\cdot}112 - 0{\cdot}11}{0{\cdot}112} = 1{\cdot}8 \text{ per cent.}$$

$$\text{Work done in lowering} = -\frac{10 \times 16 \times 1{\cdot}8}{100} = \textbf{-2·9 inch-tons.}$$

(*b*) Efficiency when wt. does not rotate $= \dfrac{0{\cdot}112}{0{\cdot}112+0{\cdot}11\times 2{\cdot}7} =$ 27·4 per cent. nearly

Work done in lifting $= \dfrac{10\times 16}{0{\cdot}274} =$ **584 inch-tons.**

Reversed efficiency $= \dfrac{0{\cdot}112-0{\cdot}11\times 2{\cdot}7}{0{\cdot}112} = -$ 165 p.c. nearly

Work done in lowering $= \dfrac{10\times 16\times 165}{100} =$ **264 inch-tons.**

11. *A weight is lifted by a vertical square threaded screw having* 3 *threads per inch. Mean diameter of threads,* 1·35″. *Mean diameter of collar taking the thrust* = 2 × *mean diameter of screw threads. The screw is driven by a worm and wheel, the angle of the worm is* 10°. *The mean diameter of the worm collar is* 0·8 *mean diameter of the worm thread. Friction angle* ϕ *for all surfaces,* 5°. *Find the efficiency when raising the weight.*

Screw and collar $\tan\alpha = \dfrac{1}{3\times 1{\cdot}35\pi} = 0{\cdot}0786 \quad \alpha = 4^\circ\ 30'$

$\alpha+3\phi = 4^\circ\ 30'+15^\circ = 19^\circ\ 30'$ (see p. 296, Vol. I.)

Efficiency when raising $= \dfrac{\tan\alpha}{\tan(\alpha+3\phi)}$ approx. $= \dfrac{0{\cdot}0786}{0{\cdot}354} = 22{\cdot}2$ p.c.

Worm and collar—

Efficiency when raising $= \dfrac{\tan\alpha}{\tan(\alpha+1{\cdot}8\phi)} = \dfrac{\tan 10^\circ}{\tan 19^\circ} = \dfrac{0{\cdot}176}{0{\cdot}344} = 51$ p.c.

Efficiency of screw and worm when raising = 0·22 × 0·51 = **11 per cent.**

In the actual machine referred to the weight moves in vertical guides which introduces a further amount of friction. When thoroughly oiled the efficiency when raising is found by experiment to be about 7 per cent., but under normal working conditions it is just over 3 per cent.

12. *A load of* 3 *tons is lifted by a screw jack which is turned by a bar passing through the screw. The effort is applied equally at each end of the bar at a distance of* 20 *inches from the axis of the screw. Find the effort and the efficiency. Diameter of screw,* 3″ ; *pitch,* 0·5″ ; *depth of thread,* 0·25″ ; *coefficient of friction,* 0·08. *The load rotates with the screw.*

Mean diam. of threads, 2·75″.

Mean circumference, 8·64″, angle of thread $\alpha = 3^\circ\ 20'$

$$\tan\phi = \mu = 0{\cdot}08 \qquad \phi = 4^\circ\ 35'$$

$$\text{Efficiency} = \frac{\tan 3^\circ\ 20'}{\tan 7^\circ\ 55'} = \frac{0{\cdot}0582}{0{\cdot}1391} = 41{\cdot}8 \text{ per cent.}$$

(See p. 296, Vol. I.)

Distance moved through by each end of bar per revolution

$$= 2 \times 20 \times 3{\cdot}14 = 125{\cdot}6 \text{ inches}$$

$$\text{Effort} = \frac{3 \times 2240 \times 0{\cdot}5}{2 \times 0{\cdot}418 \times 125{\cdot}6} = \left\{\begin{array}{l}\textbf{32 pounds} \text{ at each end of} \\ \text{the bar.}\end{array}\right.$$

Find the effort applied at one end of the bar only if the same load were lifted by a turnbuckle having a right- and left-handed thread. Length of bar, 24″ ; *diameter of screw*, 1″ ; *threads per inch*, 10 ; *angle of* V *threads*, 60° ; *tan* ϕ, 0·14.

Since the screw is right- and left-handed the load will be raised $2 \times \frac{1}{10}$ inch per revolution. (See p. 276, Vol. I.)

$$\text{Force} = \frac{2W \tan(\alpha + 1{\cdot}15 \times \phi)}{48}$$

$$\tan\alpha = \frac{1}{31{\cdot}4} = 0{\cdot}032, \qquad \alpha = 1^\circ\ 50', \qquad \phi = 8^\circ.$$

$$\text{Force} = \frac{2 \times 3 \times 2240 \tan(1^\circ\ 50' + 1{\cdot}15 \times 8^\circ)}{48} = \textbf{53 lb.}$$

13. *In an Archimedian drill the angle of the threads is* 45° *and the friction angle* 5°. *Find the reversed efficiency also the maximum efficiency.*

$$\text{Reversed efficiency} = \frac{\tan(45 - 5)}{\tan 45} = \frac{0{\cdot}839}{1{\cdot}00} = \textbf{83·9 per cent.}$$

Coefficient of friction $\mu = \tan 5^\circ = 0{\cdot}0875$.

The maximum efficiency will be obtained if the angle of the threads is $45^\circ - 2{\cdot}5^\circ$.

Maximum efficiency (approximate expression)

$$= 1 - 2 \times 0{\cdot}0875 \quad \text{(see p. 296, Vol. I.)}$$

$$= 82{\cdot}5 \text{ per cent. (nearly)}$$

$$\textit{or} \text{ max. effy.} = \left(\frac{1 - \tan 2{\cdot}5}{1 + \tan 2{\cdot}5}\right)^2 = \textbf{83·9 per cent.}$$

14. *A bicycle is placed on a smooth sloping beam which rests on supports at each end. A frame suspended from the saddle has a platform beneath the beam on which weights are placed. One crank is removed and is replaced by a grooved pulley around which a cord is wrapped and a scale pan is attached to its lower end. Weights are added to the scale pan until the bicycle moves up the slope without acceleration. Find the mechanical efficiency of the bicycle from the following data: Gear,* 64″; *diameter of pulley on crank axle,* 14″; *weight on saddle,* 180 *lb.; weight of bicycle,* 28 *lb.; weight in scale pan when the bicycle is mounting the slope,* 300 *lb.; slope,* 1 *in* 8. *What horse-power is exerted by a cyclist when travelling up this slope at* 12 *miles per hour? Neglect wind resistance.*

Distance travelled by cycle for one rev. of crank

$$=64\times3\cdot14=201''.$$

Height cycle and weight raised per rev. of road wheel

$$=\frac{64\times3\cdot14}{8}=25\cdot1''.$$

Vertical descent of weight suspended from crank pulley

$$=14\times3\cdot14-25\cdot1=18\cdot9''.$$

Work done in lifting cycle and weight on saddle per rev. of road wheel $=\frac{208\times64\times3\cdot14}{8}=5{,}220$ inch-lb.

Work done on pulley $=300\times18\cdot9=5{,}670$ inch-lb.

Efficiency $=\frac{5220}{5670}=$ **92 per cent.**

Resistance referred to road surface $=\frac{300\times14}{64}=65\cdot6$ lb.

R.p.m. of road wheel $=\frac{12\times5280\times12}{60\times201}=63.$

Horse-power exerted at 12 m.p.h. $=\frac{5670\times63}{12\times33000}=$ **0·9.**

15. *Rolling Resistance.*—When a loaded $\left\{\begin{matrix}\text{ball or}\\\text{roller}\end{matrix}\right\}$ rolls over a flat or cylindrical surface the $\left\{\begin{matrix}\text{ball or}\\\text{roller}\end{matrix}\right\}$ is flattened in the neighbourhood of the contact and the surface of the track

is distorted. If the two surfaces in contact are not of exactly the same form they do not spread exactly alike, consequently there is slipping between some parts of the areas in contact, the amount of sliding being a maximum at the outside edges and becoming zero over an inner area of the surfaces in contact. The proportion between the non-slip area and the whole area in contact depends upon various properties to be discussed later.

The friction between the sliding surfaces and the distance through which the sliding takes place determines the amount of the rolling resistance. The work done against the rolling resistance therefore depends upon: (i) the area of the slipping surfaces, (ii) the mean distance they slide over one another, (iii) the coefficient of friction between the surfaces, (iv) the pressure.

(i) and (ii) depend upon the hardness of the surfaces in contact and the normal pressure between the roller and its track. It is obvious that the harder the surfaces the smaller will be their area in contact, which will increase as the load increases.

If a lubricant be interposed it will reduce the coefficient of friction, but it will also reduce the non-slip area thereby increasing the area of the outer ring of slipping surfaces. Thus the one effect of the lubricant is to reduce and the other to increase the rolling resistance, with the result that the one partially neutralizes the other. In ball and roller bearings the friction of the cage is often quite an appreciable fraction of the whole resistance of the bearing. In such a case lubrication may make a very material difference to the friction.

If a cylindrical roller rolls on another of the same size and material each will be distorted at the area in contact to exactly the same extent and manner, with the result that no slipping will take place between the surfaces.

It is believed that, apart from internal molecular lag of the materials, no frictional resistance would be experienced in such a case; unfortunately it appears to be impossible to test this by experiment. Likewise when one cylinder rolls on another larger than itself the spread of the surfaces in contact will be less than when it rolls on a flat surface—with a corresponding smaller frictional resistance.

A ball rolling in a grooved track which nearly fits it will have a much larger surface in contact than when it rolls on a flat surface; hence the frictional resistance in a grooved

track is greater than on a flat surface, but the carrying capacity of the ball is also greater.

When a ball of diam. d rolls on a flat surface the diameter of the contact circle is about $\frac{d}{20}$ when the ball is loaded up to its safe limit; but balls thus loaded must only be allowed to run at low speeds because the heat generated per revolution of the ball would seriously overheat the bearing if it were repeated many times per minute. With a reduced load the area of the contact circle is also reduced with a corresponding reduction in the frictional resistance of the ball, and consequently there is less heat generated per revolution of the ball.

For the same heat generated per minute in the two cases we have

$$\left.\begin{array}{l}\text{High load and high frictional}\\ \text{resistance} \times \text{low speed}\end{array}\right\} = \left\{\begin{array}{l}\text{Low load and low frictional}\\ \text{resistance} \times \text{high speed.}\end{array}\right.$$

The determining factor is therefore not the r.p.m. of the bearing so much as the r.p.m. of the balls themselves relatively to the tracks.

The author's expression for the safe load for a ball bearing was arrived at thus—

It was assumed that the load-carrying capacity of a ball is proportional to the sectional area of the ball through the diameter, viz. $0{\cdot}785d^2$, where d is the diameter in inches.

Let N = number of revs. per min. of bearing.

D = diameter of ball track to middle of balls.

m = number of balls in bearing.

n = number of r.p.m. of balls $n \propto \frac{\text{ND}}{d}$.

$$\left.\begin{array}{l}\text{Safe}\\ \text{load}\end{array}\right\} = \frac{\left\{\begin{array}{l}\text{Constant depending on material and}\\ \text{whether races are plain or grooved}\end{array}\right\} \times m \times 0{\cdot}785d^2}{\frac{\text{ND}}{d} + \left\{\begin{array}{l}\text{A constant } c \text{ to cover the case in which the}\\ \text{speed is practically zero}\end{array}\right.}$$

$$= \frac{\text{K}md^3}{\text{ND}+cd}$$

A large number of bearings were tested to find suitable values for K and C, which will be found on p. 310, Vol. I.

Professor Osborne Reynolds measured the coefficient of rolling resistance by tilting a plane until a roller rolled

down without acceleration. The mean values he obtained were as follows :—

	Coefficient of resistance,	
	Clean.	Oiled or blackleaded.
Cast-iron roller on cast-iron . .	0·00051	0·00047
,, ,, glass	0·00038	0·00051
,, ,, brass	0·00041	0·00052
,, ,, boxwood . .	0·00114	0·00047
,, ,, rubber . . .	0·0064	0·0056

It will be noticed that in some cases the lubricant actually increased the rolling resistance. Apart from the cage resistance in a roller bearing the above-mentioned coefficients are about one-half as great as those found by friction tests on roller bearings. In a roller bearing the rollers are loaded along two lines of contact, whereas in Reynolds' tests there was only one line of contact, the only load on the roller was its own weight.

Experiments on bicycle wheels with pneumatic tyres show that the tangential rolling resistance, including the friction on the ball bearings when running on hard smooth roads, is given by $T = \frac{W}{2(p+15)}$, where

T is the tangential resistance in pounds at the surface of the tyre.

W is the load on the wheel in pounds.

p is the air pressure in the tyres, in lb. sq. inch.

The speed does not appear to have any marked effect upon the resistance.

Owing to the stretching of the track on which a roller travels a roller or wheel on a rubber road never travels its geometrical distance. The following figures are due to Reynolds.

Cast-iron roller on steel track			Slip not perceptible.
,,	,,	rubber 0·015″ thick glued to wood	1 in 80
,,	,,	do. 0·08″	1 in 41
,,	,,	do. 0·36″	1 in 70

Roller with rubber tyre 0·75″ thick on steel bar, 1 in 450.
" " " clean rubber track, 1 in 202.
" " " blackleaded rubber track, 1 in 270.
Balls in a ball bearing (Goodman) from 1 in 570 to 1 in 750.

For experimental data on the behaviour of balls and rollers, *see* articles in *Engineering:*

" The Contact Area of an Elastic Sphere when compressed between Flat Elastic Plates," August 3, 1923.

" The Approach of Flat Elastic Plates under Load when separated by a Ball of Similar Material," August 24, 1923.

" The Distribution of Load on the Balls and Rollers of Ball and Roller Bearings," June 19, 1925.

16. *A shaft is mounted on anti-friction wheels as shown on p.* 299, *Vol. I. Find the friction referred to the surface of the shaft and compare it with that of a plain bearing. Diameter of shaft,* 5″; *diameter of anti-friction wheels,* 24″; *diameter of friction wheel axles,* 3″; *angle* θ, 30°; *load,* 4 *tons;* K (*see p.* 298, *Vol. I.*), 0·01; *coefficient of friction for axles,* 0·012.

Normal pressure on both axles $= \frac{W}{\cos\theta} = \frac{4 \times 2240}{0{\cdot}866}$

Rolling resistance on the surface of anti-friction wheels $= \frac{4 \times 2240 \times 0{\cdot}01}{0{\cdot}866 \times 2{\cdot}5} = 41{\cdot}4$ lb.

Axle friction referred to surface of anti-friction wheels $= \frac{4 \times 2240 \times 0{\cdot}012 \times 1{\cdot}5}{0{\cdot}866 \times 12} = 15{\cdot}5$ lb.

Total friction=**56·9 lb.**

For a plain bearing—

Friction=4×2,240×0·012=**108 lb.**

17. *A radial ball bearing having grooved races has* 10 *balls* 1·1″ *diam. The diameter of the inner ball track is* 2·5″. *Find the safe load for the bearing when running at* 1, 50, 500, 1,000, 2,000 *r.p.m. Also find the load* P *on the most highly stressed ball.*

(See p. 310, Vol. I.)

Diam. of ball race to middle of balls=2·5+1·1=3·6″.

$W = \frac{2500000md^3}{ND+Cd} = \frac{2500000 \times 10 \times 1{\cdot}331}{N \times 3{\cdot}6 + 2000 \times 1{\cdot}1}$

$W = \frac{33300000}{3{\cdot}6N+2200}$ $\quad P = \frac{5W}{m}$ (see *Engineering,* June 19, 1925)

In round numbers—

W_1 =**15,000 lb.**	P_1 =**7,550 lb.**
W_{50} =**14,000 lb.**	P_{50} =**7,000 lb.**
W_{500} =**8,400 lb.**	P_{500} =**4,200 lb.**
W_{1000}=**5,800 lb.**	P_{1000}=**2,900 lb.**
W_{2000}=**3,500 lb.**	P_{2000}=**1,750 lb.**

There is considerable difference of opinion as to the safe load for ball bearings: if they run for only a few hours daily they may be much more highly loaded than if they run night and day. Readers interested in the question should consult Macaulay's *Ball and Roller Bearings*, published by Sir Isaac Pitman & Son.

18. *A thrust ball bearing with grooved races has* 14 *balls* 1·5 *inch diameter. Diameter of ball path,* 7 *inches. Find the safe load for* 1, 100, 1,000, 2,000 *r.p.m.*

$$W = \frac{1250000 \times 14 \times 3{\cdot}375}{7N + 200 \times 1{\cdot}5} = \frac{59000000}{7N + 300}$$

In round numbers.	*Mean load per ball.*
W_1 =**190,000 lb.**	13,000 lb.
W_{100} =**59,000 lb.**	4,200 lb.
W_{1000}=**8,000 lb.**	570 lb.
W_{2000}=**4,000 lb.**	290 lb.

The load per ball at which skidding will occur (see Example 8, Chap. VIII., Vol. II.) at 2,000 r.p.m. is

$$P = \frac{0{\cdot}00000165 \times 0{\cdot}75^3 \times 2000^2 \times 3{\cdot}5}{0{\cdot}33} = 30 \text{ lb.}$$

Hence there is an ample margin against this contingency.

The safe loads given above are according to Goodman's formula; there is a considerable divergence between different authorities on this question.

19. *A thrust ball bearing with grooved races consists of two concentric rings as shown in Fig.* 299, *p.* 304, *Vol. I. The inner ring has* 10 *balls* 1·75 *inch diameter; diameter of ball path,* 6·5 *inches. Outer ring* 20 *balls* 1·75 *inch diameter; diameter of ball path,* 12 *inches. Find the safe load for* 1, 100, 1,000, 2,000 *r.p.m.*

$$W = \frac{1250000 \times 10 \times 5{\cdot}36}{6{\cdot}5N + 200 \times 1{\cdot}75} + \frac{1250000 \times 20 \times 5{\cdot}36}{12N + 200 \times 1{\cdot}75}$$

$$W = 67{,}000{,}000\left(\frac{1}{6{\cdot}5N + 350} + \frac{2}{12N + 350}\right)$$

In round numbers—	*Mean load per ball—*
W_1 = **560,000 lb.**	19,000 lb.
W_{100} = **153,000 lb.**	5,100 lb.
W_{1000} = **21,000 lb.**	700 lb.
W_{2000} = **11,000 lb.**	370 lb.

The load per ball at which skidding will occur at 2,000 r.p.m. is

$$P = \frac{0{\cdot}00000165 \times 0{\cdot}875^3 \times 2000^2 \times 3{\cdot}25}{0{\cdot}33} = \mathbf{43\ lb.}$$

20. *Find the friction moment on the ball bearings referred to in Q.* 17, 18, 19, *when running at* 1,000 *r.p.m.* $\mu = 0{\cdot}001$.

(17) $W = 5{,}800$ lb.

$$\text{Friction moment} = 5800 \times 0{\cdot}001 \times \frac{2{\cdot}5}{2} = \mathbf{7{\cdot}25\ inch\text{-}lb.}$$

(18) $W = 8{,}000$ lb.

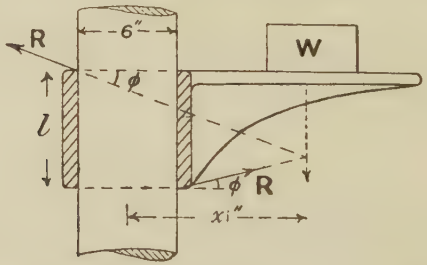
$$\text{Friction moment} = 8000 \times 0{\cdot}001 \times \tfrac{7}{2} = \mathbf{28\ inch\text{-}lb.}$$

(19) $W = 21{,}000$ lb. of which about 7,000 lb. is taken by the inner race and 14,000 lb. by the outer race.

$$\text{Friction moment} = 7000 \times 0{\cdot}001 \times \frac{6{\cdot}5}{2} + 14{,}000 \times 0{\cdot}001 \times \tfrac{12}{2}$$

$$= 23 + 84 = \mathbf{107\ inch\text{-}lb.}$$

21. *A bracket fits loosely on a vertical shaft* 6 *ins. diameter.*



The bracket weighs 50 *lb. and its centre of gravity is* 15 *ins. from the centre of the shaft. If the coefficient of friction is*

0·3, *find the maximum length l of the sleeve so that the bracket shall not slide down.*

The reactions of the shaft on the bracket will occur at the points indicated in the diagram, and at an angle ϕ upwards from the horizontal. So long as the bracket does not slide

$$(15-3)\tan\phi+(15+3)\tan\phi=l \qquad \therefore \tan\phi=\frac{l}{30}$$

Hence, if tan ϕ cannot exceed 0·3, the length l must not exceed **10 ins.**

22. *The shaft in the last example rotates at* 60 *r.p.m. The length* $l=6''$. *The bracket is prevented from rotating. Find the speed at which it will slide down.* $x=20''$

$$\text{Peripheral velocity of shaft} = \frac{\pi\times 6\times 60}{60} = 6\pi \text{ ins. per sec.}$$

The friction acts against the direction of *relative* motion. Hence, if the bracket slides v inches per sec., the effective coefficient of friction against its axial motion is $\mu\times\frac{v}{\sqrt{v^2+36\pi^2}}$

When steady sliding is attained :

$$2\times 20\times\frac{\mu v}{\sqrt{v^2+36\pi^2}}=6 \qquad \therefore v=\mathbf{21\cdot3 \text{ ins. per sec.}}$$

23. *Find the length of bearing required for a continuous running shaft* 6 *ins. diam. Load* 7 *tons. R.p.m.* 200. *Ring lubrication* $\mu=0{\cdot}006$ (*working in a cool place*). *What horse-power will be required ?*

$$\text{Surface speed } S=3{\cdot}14\times\tfrac{6}{12}\times 200=314 \text{ ft.-min.}$$

Nominal area of bearing surface

$$=\frac{0{\cdot}006\times 7\times 2240\times 314}{773\times 0{\cdot}5}=76 \text{ sq. ins.} \quad \text{(see p 328, Vol. I.)}$$

$$\text{Length}=\tfrac{76}{6}=12{\cdot}7 \text{ ins.}$$

$$\text{This gives a pressure of } \frac{7\times 2240}{76}=206 \text{ lb. per sq. in.}$$

$$\left.\begin{matrix}\text{The horse-power absorbed}\\ \text{per bearing (p 329, Vol. I.)}\end{matrix}\right\}=\frac{0{\cdot}006\times 7\times 2240\times 6\times 200}{126000}=\mathbf{0{\cdot}9.}$$

As a check take the frictional resistance F (p. 329, Vol. I.) for ring oiling as 1·5 lb. per sq. in.

$$\text{Horse-power} = \frac{3{\cdot}14 \times 6^2 \times 12{\cdot}5 \times 1{\cdot}5 \times 200}{12 \times 33000} = \mathbf{1 \text{ nearly.}}$$

If a ball or good roller bearing had been used the horse-power required would only be about one-sixth of the above.

24. *Find the horse-power required to drive a* 6″ *diam. vertical shaft at* 100 *r.p.m. when resting on* (i) *a flat pivot footstep;* (ii) *conical; angle* $\alpha = 30°$. $\mu = 0{\cdot}02$. *Taking the maximum intensity of bearing pressure at* 40 *lb. sq. in.*

Area of flat pivot $= 28{\cdot}3$ sq. ins.

Permissible load $= 28{\cdot}3 \times 40 = 1130$ lb.

$$\text{(i) H.P.} = \frac{0{\cdot}02 \times 1130 \times 6 \times 100}{189000} = \mathbf{0{\cdot}07}$$ (See p. 330, Vol. I.)

$$\text{(ii) H.P.} = \frac{0{\cdot}07}{\sin 30} = \frac{0{\cdot}07}{0{\cdot}5} = \mathbf{0{\cdot}14.}$$

25. *Find the horse-power required to drive a vertical shaft which rests on a collar bearing on the assumption that the pressure is evenly distributed over the whole of the surfaces in contact. Load on shaft* (W), 4 *tons; diam. of shaft* ($2R_2$), 7″; *outside diam. of collar* ($2R_1$), 10″; *r.p.m.* (N), 84; *coefficient of friction* (μ), 0·073.

Let T be the twisting moment due to the friction.

$$T = \frac{2\mu W(R_1^3 - R_2^3)}{3(R_1^2 - R_2^2)}$$ (See p. 331, Vol. I.)

$$T = \frac{2 \times 0{\cdot}073 \times 4 \times 2240(5^3 - 3{\cdot}5^3)}{3(5^2 - 3{\cdot}5^2)} = 2810 \text{ inch-pounds}$$

$$\text{H.P.} = \frac{2T\pi N}{12 \times 33000} = \frac{2810 \times 3{\cdot}14 \times 84}{198000} = \mathbf{3{\cdot}75.}$$

26. *Find the horse-power required to drive a Michell bearing having* 8 *pads each* 9 *sq. ins. area. Diam. of shaft* 6″; *outside diam. of collar,* 14″; *radius to centre of pads,* 5·5″; *pressure on pads,* 300 *lb. sq. in.; r.p.m.,* 1000; *coefficient of friction,* 0·002.

Friction per sq. in. of pad pressure $= 300 \times 0{\cdot}002 = 0{\cdot}6$ lb.

Total load on bearing $= 9 \times 8 \times 300 = 21{,}600$ lb.

Total frictional resistance $= 0{\cdot}6 \times 8 \times 9 = 43{\cdot}2$ lb.

$$\text{Mean rubbing speed} = \frac{1000 \times 5{\cdot}5 \times 2 \times 3{\cdot}14}{12} = 2880 \text{ f.m.}$$

$$\text{Horse-power absorbed} = \frac{2800 \times 43{\cdot}2}{33000} = \mathbf{3{\cdot}8}.$$

27. *Find the H.P. required to drive a double ring (grooved) thrust ball bearing. Load,* 20,000 *lb. Coefficient of friction,* 0·001. *Speed,* 100 *r.p.m.*

Take the inner ring as 6″ internal and 10·25″ external diameter, the outer ring 10·25″ internal and $14\frac{1}{2}$″ external. Balls 1·75″ diameter. We can get 13 balls in the inner and 19 in the outer ring. Then the safe load

$$P = \frac{1250000 \times 13 \times 1{\cdot}75^3}{1000 \times 8 + 350} + \frac{1250000 \times 19 \times 1{\cdot}75^3}{1000 \times 12 + 350}$$

$$P = 10{,}500 + 10{,}300 = 20{,}800 \text{ lb.}$$

$$\text{H.P.} = \frac{0{\cdot}001 \times 3{\cdot}14 \times 1000}{12 \times 33000}(10{,}500 \times 8 + 10{,}300 \times 12) = \mathbf{1{\cdot}7} \text{ (nrly.)}$$

The two rings should be bedded on a washer of yielding material such as vulcanised fibre, linoleum or hard wood in order to distribute the load on both rings.

If a single row bearing were used with a mean diameter of ball race of 10″, having 12 $2\frac{1}{2}$″ diam. balls, the safe load is 22,500 lb., and the horse-power 1·8. Such large balls, however, are much more costly than the smaller ones proposed for the double ring bearing, and probably the cost of the bearing would be much greater.

28. *The plunger of a hydraulic press is* 8 *ins. diameter, and the total load to be lifted is* 100 *tons. Find the water pressure if the press is in good condition.*

Area of plunger 50·26 sq. ins.

$$\left.\begin{matrix}\text{Water pressure} \\ \text{if no friction}\end{matrix}\right\} = \frac{2240 \times 100}{50{\cdot}26} = 4500 \text{ lb. sq. in. (nearly).}$$

For purposes of calculating friction assume $p = 5000$ lb. sq. in.

Then the friction pressure $F = 0{\cdot}08 \times 5000 + \frac{100}{8}$
(See p. 334, Vol. I.) $= 410$ lb. sq. in.

Thus the water pressure when lifting the load would be about **4900** lb. sq. in.

This formula for the friction of cup leathers has been much criticised. Some affirm that it gives too low and others that it gives too high values. Much depends upon whether the plunger is centrally loaded or not; if the load is out of the centre it tilts the plunger and sets up an abnormal amount of friction.

29. *A friction clutch on a line of shafting consists of a pair of flat discs pressed together to obtain the necessary friction for the drive. Find the horse-power which can be transmitted at* 200 *r.p.m. Diam. of discs*, 18″; *coefficient of friction*, 0·28; *pressure on face of discs*, 400 *lb.*

On the assumption that the pressure is evenly distributed over the face of the discs (see p. 330, Vol. I.)—

$$\text{H.P.} = \frac{0{\cdot}28 \times 400 \times 18 \times 200}{189000} = \mathbf{2{\cdot}13.}$$

Such an arrangement is poor from a mechanical point of view owing to the large end pressure required to produce sufficient friction between the faces of the discs.

30. *Find the horse-power which can be transmitted through a friction clutch having conical surfaces. Coefficient of friction* (μ), 0·4; *apex angle of cone* (2α), 48°; *axial pressure on cone*, 250 *lb.; r.p.m.*, 250; *mean diam. of coned surfaces* (2R), 16″.

(i) Normal pressure on surface of cone $= \frac{250}{\sin 24^\circ} = 615$ lb.

Friction (F) $0{\cdot}4 \times 615 = 246$ lb.

$$\left.\begin{matrix}\text{Horse-power} \\ \text{transmitted}\end{matrix}\right\} = \frac{2\pi \text{RNF}}{12 \times 33000} = \frac{3{\cdot}14 \times 16 \times 250 \times 246}{12 \times 33000} = \mathbf{7{\cdot}8.}$$

31. *A flywheel having a radius of gyration of* 18 *ins. is mounted on a horizontal shaft* 3 *ins. diameter and when rotating at a high speed the driving effort is cut off. The speed drops at the rate of* 8 *r.p.m. per minute. Find the mean coefficient of friction on the bearings.*

The change in the kinetic energy in a given time is equal to the work done against friction in the same time. Hence over a period of one minute, during which the speed drops from N_1 to N_2 r.p.m.—

$$\frac{W}{2g}\frac{2^2\pi^2\rho^2}{60^2}(N_1^2-N_2^2)=\mu W\pi d\left(\frac{N_1+N_2}{2}\right)$$

$$\frac{2\pi}{32\cdot2}\times\left(\frac{18}{12}\right)^2(N_1-N_2)=\frac{\mu\times0\cdot25\times60^2}{2}$$

$N_1-N_2=8$ whence $\mu=\dfrac{4\times3\cdot14\times2\cdot25\times8}{32\cdot2\times60^2\times0\cdot25}=$ **0·0078.**

This method has been used in the laboratory for determining the coefficient of friction of bearings, but it has not been a success, probably because the coefficient varies greatly with the speed.

32. *The following results were obtained from a test of a pulley block :—*

The vertical distance moved by the effort was 27·6 *ins. while raising the weight* 1·84 *ins.*

Effort	13	24	41	77	112	147
Load	100	250	500	1,000	1,500	2,000

Find the friction at each load, also the mechanical efficiency.

Velocity ratio $\dfrac{27\cdot6}{1\cdot84}=15$.

Effort if there were no friction.	Friction.	Efficiency per cent.
$\frac{100}{15}=6\cdot67$	$13-6\cdot67=6\cdot33$	$\frac{6\cdot67}{13}=51\cdot3$
$\frac{250}{15}=16\cdot7$	$24-16\cdot7=7\cdot3$	$\frac{16\cdot7}{24}=69\cdot6$
$\frac{500}{15}=33\cdot3$	$41-33\cdot3=7\cdot7$	$\frac{33\cdot3}{41}=81\cdot2$
$\frac{1000}{15}=66\cdot7$	$77-66\cdot7=10\cdot3$	$\frac{66\cdot7}{77}=86\cdot6$
$\frac{1500}{15}=100$	$112-100=12$	$\frac{100}{112}=89\cdot3$
$\frac{2000}{15}=133\cdot3$	$147-133\cdot3=13\cdot7$	$\frac{133\cdot3}{147}=90\cdot7$

The relation between the effort F and the load W can be expressed thus—

$F=6+0\cdot07W$ (nearly) (see p. 342, Vol. I.)

Hence the efficiency can approach but never exceed

$$\eta_{max}=\frac{100}{0{\cdot}07\times 15}=95{\cdot}2 \text{ per cent.}$$

33. *The initial frictional resistance of an unloaded pulley block is* 2·4 *lb. The friction increases at the rate of* 1·9 *lb. per* 100 *lb. lifted by the block. The velocity ratio is* 22. *Find the effort required to lift a load of* 1300 *lb. Will the weight run down when the block is left alone?*

$$\text{Friction}=2{\cdot}4+\frac{1{\cdot}9W}{100}=2{\cdot}4+\frac{1{\cdot}9\times 1300}{100}=27{\cdot}1 \text{ lb. for } 1300 \text{ lb.}$$

$$\text{Effort, if no friction}=\frac{1300}{22}=59{\cdot}1 \text{ lb.}$$

$$\text{Total effort}=86{\cdot}2 \text{ lb.}$$

$$\text{Efficiency}=\frac{59{\cdot}1}{86{\cdot}2}=68{\cdot}6 \text{ per cent.}$$

The weight will run down when left alone unless some special safety device such as that shown in Fig. 332 is fitted.

In the above instance we have (see p. 342, Vol. I.)—

$F=K+xW \qquad 86{\cdot}2=2{\cdot}4+x\times 1300 \qquad x=0{\cdot}0645$

The efficiency may approach but not exceed $\left\}\frac{100}{0{\cdot}0645\times 22}\right.=$ **70 per cent.**

34. *In a Weston pulley block the radius* r *of the smaller wheel is* 0·94R (*larger*). *The efficiency* η *is* 43 *per cent. Find the load lifted when the effort is* 50 *lb. Also find the reversed efficiency and the pull necessary to lower this load.*

(See p. 338, Vol. I.)

$$\frac{\text{Weight lifted } (W_u)}{\text{Effort } (W)}=\frac{2R}{R-r}\eta=\frac{2R\times 0{\cdot}43}{0{\cdot}06R}=33{\cdot}3\times 0{\cdot}43=14{\cdot}3$$

$$\text{Weight lifted } (W_u)=14{\cdot}3W=14{\cdot}3\times 50=715 \text{ lb.}$$

$$\text{Efficiency}=0{\cdot}43=\frac{W_u h_u}{W_u h_u+Wh}$$

and

$$Wh=1{\cdot}325W_u h_u=949h_u$$

$$\text{Reversed efficiency}=\frac{W_u h_u-Wh}{W_u h_u}=\frac{W_u h_u-1{\cdot}325W_u h_u}{W_u h_u}$$

$$=-32{\cdot}5 \text{ per cent.}$$

Let W_1=effort required to lower W_u

Then $$h_1=0{\cdot}94h=0{\cdot}94\times33{\cdot}3h_u=31{\cdot}3h_u$$

$$W_1h_1=Wh-W_uh_u=949h_u-715h_u=234h_u$$

$$\therefore W_1=\frac{234}{31{\cdot}3}=\mathbf{7{\cdot}5\ lb.}$$

If we assume that the friction on the blocks varies in the same proportion as the load, we have

$$\frac{W'h'}{Wh}=\frac{7{\cdot}5+715}{50+715}=0{\cdot}945$$

$$W'h'=0{\cdot}945\times1{\cdot}325W_uh_u=1{\cdot}25W_uh_u$$

$$\left.\begin{matrix}\text{Corrected reversed}\\ \text{efficiency}\end{matrix}\right\}=\frac{W_uh_u-W'h'}{W_uh_u}=1-1{\cdot}25=\mathbf{-25\ per\ cent.}$$

The assumption is not altogether justifiable. The correct result is probably between 32·5 and 25 per cent.

35. *Find the efficiency of power transmission by means of a solid cylindrical shaft when there is no side pull on the shaft and the bearings are in perfect alignment. Length,* 3000′; *coefficient of friction,* 0·05; *weight per foot run per sq. in. section,* 3·36 *lb.; torsional stress,* 8960 *lb. sq. in.*

$$\text{Efficiency}=1-\frac{2\mu WL}{f}\quad\text{(see p. 346, Vol. I.)}$$

$$=1-\frac{2\times0{\cdot}05\times3{\cdot}36\times3000}{8960}=\mathbf{0{\cdot}89.}$$

The coefficient of friction given is about what might be expected from ordinary ring oiled bearings. Such a high efficiency could not be obtained unless the bearings were in perfect alignment, and there was no side pull on the shaft.

36. *Find an expression for the efficiency of a cylindrical or parallel shaft of length* L *feet, of diameter* d *inches, coefficient of friction* μ, *weight per cubic inch* w *lb.*

(a) *With no pulleys or other side pulls.*

(b) *When the horizontal belt pull amounts to* 30 *lb. per foot run and the weight of the pulleys to* 25 *lb. per foot run.*

When there is no side pull the efficiency (see p. 346, Vol. I.)

$$\eta_1=1-\frac{2\mu WL}{f_s}$$

in which case the friction on the bearings is entirely due to

the weight of the shaft, but when there is a side pull the friction is increased (approximately) in direct proportion to the load on the bearings.

Weight of shaft per foot run $=12\times0{\cdot}785d^2w=9{\cdot}42d^2w$

With side pull resultant pressure due to belts, etc. $\Big\}=R=\sqrt{30^2+(25+9{\cdot}42d^2w)^2}$

$$\eta_2=1-\frac{2\mu Rl}{f_s\times0{\cdot}785d^2}$$ with side pull,

where l is the length in inches.

Let $\mu=0{\cdot}05$ $\quad w=0{\cdot}28$ lb. $\quad d=2''$

$l=1200''$ $\quad f_s=12{,}000$ lb. per sq. in.

Then $$\eta_1=1-\frac{2\times0{\cdot}05\times12\times0{\cdot}785\times0{\cdot}28\times100}{12000}=\mathbf{0{\cdot}998}$$

$$\eta_2=1-\frac{2\times0{\cdot}05\times46{\cdot}5\times1200}{12000\times0{\cdot}785\times4}=\mathbf{0{\cdot}985.}$$

37. *It is required to transmit* 50 *horse-power by means of a direct coupled hollow shaft through a distance of* 2000 *ft. Find the useful h.p. delivered at the far end; the efficiency; and the size of shaft required. Stress* (f_s), 6000 *lb. per sq. in.; coefficient of friction* (μ), 0·0015 (*ball bearings*); *r.p.m.*, 120; *internal diam.*=0·8 *exter. diam. of shaft; weight of shaft per foot long and* 1 *sq. inch section*=3·36 *pounds.*

See p. 346, Vol. I.

Useful moment = max. moment — friction moment

$$M=\frac{\pi f_s(R^4-r^4)}{2R}-W\pi(R^2-r^2)L\mu R$$

$$=\frac{3{\cdot}14\times6000(R^4-0{\cdot}8^4R^4)}{2R}-3{\cdot}26\times3{\cdot}14(R^2-0{\cdot}64R^2)R\times2000\times0{\cdot}0015$$

$$=5560R^3-11R^3=5549R^3 \text{ pound-inches.}$$

$$\text{H.P.}=\frac{2M\pi N}{12\times33000}=\frac{2\times5560R^3\times3{\cdot}14\times120}{12\times33000}=50$$

$R^3=4{\cdot}75$ and $R=1{\cdot}68$ inches.

$$\text{H.P. wasted in friction}=50\times\frac{11}{5560}=0{\cdot}1$$

Useful H.P. $=50-0{\cdot}1=\mathbf{49{\cdot}9}$

Efficiency $=\mathbf{99{\cdot}8}$ **per cent.**

This efficiency appears to be absurdly high, but with care in erection such a result can be obtained. The pitch of the hangers must be arranged so that there is no danger of the shaft whirling.

38. *The mechanical efficiency of a petrol engine may be estimated by short circuiting each of the sparking plugs in turn and measuring the B.H.P. at a pre-arranged speed. From the readings given below, which were obtained from a four-cylinder engine, deduce the I.H.P. and mechanical efficiency.*

B.H.P., all four cylinders firing, 14·8 ; three cylinders firing, 10·0, 10·6, 9·8, 9·5.

Let B_1 be B.H.P. when cylinder No. 1 is not firing.
Let I_1 be I.H.P. of cylinder No. 1.
Let F be friction horse-power.
Then assuming the friction horse-power is constant :—

$$B_1=I_2+I_3+I_4-F$$
$$B_2=I_1+I_3+I_4-F$$
$$B_3=I_1+I_2+I_4-F$$
$$B_4=I_1+I_2+I_3-F$$

$$B_1+B_2+B_3+B_4=3I-4F$$

also $B=I-F$

Hence $F=3B-(B_1+B_2+B_3+B_4)=44{\cdot}4-39{\cdot}9=4{\cdot}5$

$I=14{\cdot}8+4{\cdot}5=19{\cdot}3$. Mechanical efficiency $=\frac{14{\cdot}8}{19{\cdot}3}=$ **77 p.c.**

39. *A factory engine transmits to lines of shafting* 500 *H.P. by means of* 25 *hemp ropes. Determine from the data below the maximum tension in any rope, assuming that each rope transmits the same power.* (1) *Diam. of grooved flywheel,* 18′ ; (2) *angle of groove,* 60° ; (3) *angle of contact of rope with flywheel rim,* 210° ; (4) *coefficient of friction,* 0·20 ; *r.p.m.,* 60.

Horse-power of each rope $\frac{500}{25}=20$

$$(T_2-T_1)2\pi RN=20\times 33{,}000$$

$$T_2-T_1=\frac{20\times 33000}{2\times 3{\cdot}14\times 9\times 60}=195 \text{ lb.}$$

The effective coefficient of friction is $\frac{\mu}{\sin\theta}$, where $2\theta=60°$.

$$T_2=T_1e^{\frac{\mu a}{\sin\theta}}=T_1e^{\frac{0\cdot2\times3\cdot66}{0\cdot5}}=T_1e^{1\cdot464}$$

$$T_2=T_1\times2\cdot72^{1\cdot464}=4\cdot32T_1$$

$$T_2-\frac{T_2}{4\cdot32}=195 \qquad T_2=\mathbf{254\ lb.}$$

40. *Two pulleys on shafts* 8 *ft. apart are connected by an open belt* 6 *ins. wide. The diameters of the two pulleys are* 3 *ft. and* 1 *ft. respectively, and the larger pulley makes* 100 *revolutions per minute. If the coefficient of friction of the belt on the pulley rim is* 0·27 *and if the safe stress in the leather is* 80 *lb. per inch width, find the horse-power which this belt can transmit.*

It is necessary in this case to find the angle of contact of the belt on the smaller pulley. The angle the belt makes to the centre line is 7° 10′. It therefore embraces 165°. (See p. 352, Vol. I.)

$$\log\frac{T_1}{T_2}=\log_e\times\mu a.$$

$$\log\frac{T_1}{T_2}=0\cdot434\times0\cdot27\times\frac{165\times3\cdot14}{180}=0\cdot34$$

$$T_1=2\cdot19T_2. \quad \text{But } T_1=6\times80=480 \text{ lb.}$$

$$T_2=\frac{480}{2\cdot19}=220 \text{ lb.} \qquad T_1-T_2=260 \text{ lb.}$$

$$\text{H.P.}=\frac{260\times2\times3\cdot14\times1\cdot5\times100}{33000}=\mathbf{7\cdot4.}$$

If possible the tight side of the belt should be at the bottom in order to make the angle embraced as large as possible, it will then be greater than the calculated value, but if the tight side be at the top it will be less

41. *A belt runs over a pulley embracing an angle of* 150 *degrees. Find the maximum horse-power which can be transmitted when the centrifugal tension is* (a) *neglected*, (b) *taken into account. Coefficient of friction*, 0·4 ; *width of belt*, 8″ ; *thickness of belt*, 0·25″ ; *safe stress in belt*, 300 *lb. per sq. in. ; weight of belt per cubic inch*, 0·04 lb. ; *diameter of pulley*, 4′ ; *r.p.m.*, 300

(*a*) $\log \frac{T_1}{T_2} = 0{\cdot}434 \times 0{\cdot}4 \times 3{\cdot}14 \times \frac{150}{180} = 0{\cdot}455$

$T_1 = 2{\cdot}85\ T_2$

$$\text{H.P.} = \frac{(T_1 - T_2)V}{33000} = \frac{T_1\left(1 - \frac{1}{2{\cdot}85}\right)V}{33000}$$

$$= \frac{300 \times 8 \times 0{\cdot}25 \times 0{\cdot}65 \times 4 \times 3{\cdot}14 \times 300}{33000} = \mathbf{44{\cdot}5.}$$

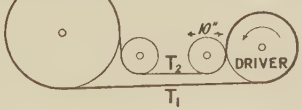
(*b*) Centrifugal stress $f = \frac{0{\cdot}04 \times 12V^2}{32{\cdot}2 \times 3600} = 59$ lb. sq. in.

$$\frac{T_1 - f}{T_2 - f} = 2{\cdot}85 \qquad T_2 = \frac{T_1}{2{\cdot}85} + 38$$

$$\text{H.P.} = \frac{\left(T_1 - \frac{T_1}{2{\cdot}85} - 38\right)3768}{33000} = \frac{(0{\cdot}65 \times 300 - 38)3768}{33000}$$

$= 18$ per sq. in. of belt section, or **36** H.P. for a belt $8'' \times \frac{1}{4}''$.

42. *Calculate the horse-power that can be transmitted by the pulleys arranged as shown in the figure. Width of belt,* 8″ ; *speed of belt,* 2,000 *f.p.m. ; coefficient of friction,* 0·3 ; *angle embraced,* 270° ; *maximum stress per inch width of belt,* 70 *lb. The guide pulleys are* 10 *inches diameter, the*



shafts are $1\frac{1}{2}''$ *diameter, the coefficient of friction* 0·001 (*ball-bearings*). *The work wasted in bending the belt to and fro at each bend is* 11 *foot-pounds per sq. foot of belt passed over the pulleys. Calculate the mechanical efficiency of the drive.*

$$\text{Log}\ \frac{T_1}{T_2} = 0{\cdot}434 \times 0{\cdot}3 \times \frac{270 \times 3{\cdot}14}{180} = 0{\cdot}61 \qquad T_1 = 4{\cdot}07T_2$$

$$\text{H.P.} = \frac{2000\left(T_1 - \frac{T_1}{4{\cdot}07}\right)}{33000} = \frac{T_1}{22} = \frac{70 \times 8}{22} = 25{\cdot}5.$$

H.P. wasted in bending belt to and fro four times

$$= \frac{2000 \times 8 \times 11 \times 4}{12 \times 33000} = 1{\cdot}78$$

H.P. wasted in friction of guide pulleys

$$= \frac{1{\cdot}5 \times 2000 \times 195 \times 0{\cdot}001 \times 2}{10 \times 33000} = 0{\cdot}0035$$

$$\text{Efficiency} = \frac{25{\cdot}5 - 1{\cdot}78 - 0{\cdot}0035}{25{\cdot}5} = \mathbf{93\ per\ cent.}$$

43. *Find the "creep" in the case of a belt (crossed) which embraces an angle of* 200 *degrees. Modulus of elasticity* (E), 8,000 *lb. sq. in.; tension on tight side*, 300 *lb. sq. in.; tension on slack side*, 80 *lb. sq. in.; coefficient of friction* (μ), 0·4.

$$\frac{N_1}{N_2} = \frac{(8000+300)d_2}{(8000+80)d_1} = 1{\cdot}027 \quad \text{or} \quad \mathbf{2{\cdot}7}\ \text{per cent. creep.}$$

The necessary angle of embrace is given by

$$T_1 = T_2 e^{\mu\alpha} \qquad \frac{300}{80} = 2{\cdot}72^{0{\cdot}4\alpha} = 3{\cdot}75$$

$$\alpha = 3{\cdot}3 \text{ radians} = 190 \text{ degrees.}$$

Hence the belt is just safe against slipping.

44. *In a given instance a belt can just transmit* 25 *horse-power without slipping when the tension on the tight side is* 400 *lb. and* 160 *lb. on the slack side. What horse-power can it transmit at the same speed when the initial tension on the belt is increased by* 200 *lb.?*

In the first case the pressure between the belt and pulley is due to a load of 400+160=560 lb.; in the second case to 560+2×200=960 lb.

The H.P. that can be transmitted when there is initial tension is therefore—

$$\frac{25 \times 960}{560} = \mathbf{43.}$$

In the solution given it is assumed that the belt is truly elastic.

45. *A belt is hung over a fixed pulley and embraces an angle of* 180°, *coefficient of friction* 0·4. *A weight of* 120 *lb. is suspended from one end of the belt. Find the pull required to* (i) *raise the weight*, (ii) *to lower it gently.*

(i) $T_1 = T_2 e^{\mu a}$ $\quad \log \dfrac{T_1}{T_2} = 0{\cdot}434 \times 0{\cdot}4 \times 3{\cdot}14 = 0{\cdot}545$

$T_1 = 3{\cdot}51 T_2 = 3{\cdot}51 \times 120 = \mathbf{420\ lb.}$ to raise weight.

(ii) $T_1 = T_2 e^{-\mu a}$

$T_2 = \dfrac{T_1}{3{\cdot}51} = \dfrac{120}{3{\cdot}51} = \mathbf{34\ lb.}$ to lower weight.

Thus if the applied pull drops below 34 lb. the weight will move downwards.

46. *Find the maximum horse-power which can be transmitted per sq. in. of belt section in the case of a thin sheet steel belt running under following conditions: Maximum tensile stress in belt,* 11,000 *lb. sq. in.; angle embraced,* 180°; *coefficient of friction,* 0·3; *weight per cubic in. of belt,* 0·28 *lb.*

$T_1 = T_2 e^{\mu a}$

$\log \dfrac{T_1}{T_2} = 0{\cdot}434 \times 0{\cdot}3 \times 3{\cdot}14 = 0{\cdot}409$, hence $\dfrac{T_1}{T_2} = 2{\cdot}56$

$$\text{H.P.} = \frac{T_1\left(1 - \frac{1}{2{\cdot}56}\right)V}{33000} = \frac{T_1 V}{54000}$$

$T_1 = 11{,}000 tw$

$$\text{H.P.} = \frac{11000 tw V}{54000} = \frac{tw V}{5}$$ nearly, neglecting centrifugal stress

where t is the thickness in inches, w is the width in inches, and V the velocity of the belt in feet per minute.

$$\text{Centrifugal stress } f = \frac{0{\cdot}28 \times 12 V_s^2}{32{\cdot}2} = \frac{V_s^2}{9{\cdot}6} = \frac{V^2}{34600}$$

$$\text{Effective tension} = T - \frac{V^2}{34600}$$

$$\text{Power transmitted } P = \left(TV - \frac{V^3}{34600}\right)\left(\frac{e^{\mu a} - 1}{e^{\mu a}}\right)$$

(See p. 355, Vol. I.)

$$\frac{dP}{dV} = \left(T - \frac{3V^2}{34600}\right)\left(\frac{e^{\mu a} - 1}{e^{\mu a}}\right)$$

This is a maximum when $T = \dfrac{V^2}{11530} = 11{,}000$

and when V=11,200 feet per min., which is too high for a cast-iron pulley (see p. 201, Vol. I.).

Effective tension per sq. in for maximum horse-power

$$=T-\frac{11200^2}{34600}=T-3{,}700$$

$$=11{,}000-3{,}700=7{,}300 \text{ lb. sq. in.}$$

Max. H.P. per sq. inch of belt section if the above-mentioned stress is not exceeded

$$=\frac{7300\times 11200}{54000}\times\frac{1{\cdot}56}{2{\cdot}56}=\mathbf{920.}$$

47. *The following case of a steel belt transmission is quoted in Kempe's "Engineer's Year Book": Width, 32 mm.; thickness, 0·3 mm.; H.P. at a speed of 15·7 m. per second=26·3. Find the stress in the belt assuming a coefficient of friction of 0·3 and angle of contact* 180° (i) *neglecting centrifugal force*, (ii) *taking it into account.*

Converting to British units :

32 mm.=32×0·0394=1·26".

0·3 mm.=0·3×0·0394=0·012" nearly.

15·7 m.=15·7×3·28=51·5 ft. per sec.=3,090 f.p.m.

$$\text{H.P.}=\frac{T_1V}{54000} \text{ (see Example 46)}$$

$$\frac{26{\cdot}3\times 54000}{3090}=T_1=460 \text{ lb.}$$

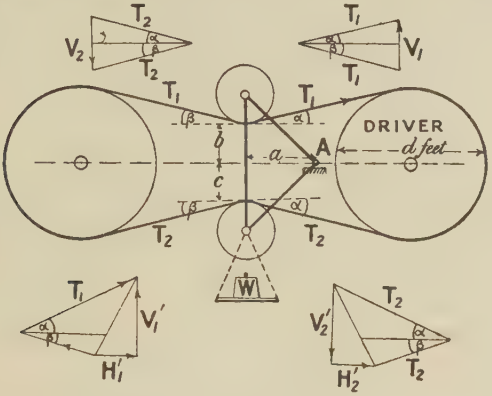
$$\text{Stress}=\frac{460}{1{\cdot}26\times 0{\cdot}012}=\begin{cases}\mathbf{30{,}000} \text{ lb. sq. inch. neglecting centrifugal stress.}\end{cases}$$

$$\text{Centrifugal stress for steel belting}=\frac{V^2}{34600}=\mathbf{280} \text{ lb. sq. in.}$$

To transmit the horse-power an effective stress of 30,000 lb. sq. in. is required, hence the total stress is 30,280 lb. sq. in.=13·5 tons per sq. inch, which is probably about $\frac{1}{5}$th of its ultimate strength.

48. *The diagram shows a belt transmission dynamometer. The tight side of the belt is on the top. The frame carrying the tightening pulleys is pivoted at* A, *the tight belt tends to lift the frame but this is prevented by adding weights* W

to the scale pan. Find an expression for the horse-power transmitted in terms of W *and* N *the revols. per minute.*



Let $\alpha=\beta$

$$V_1=T_1 \sin \alpha+T_1 \sin \beta=2T_1 \sin \alpha$$

$$V_2=2T_2 \sin \beta$$

Resultant force $=V_1-V_2=2 \sin \alpha(T_1-T_2)=W$

$$\text{H.P.}=\frac{(T_1-T_2)\pi dN}{33000}=\frac{W\pi dN}{2 \sin \alpha \times 33000}$$

When α is not equal to β we have—

The resultant of the two T_1's passes through the centre of the tightening pulley. The resultant force on the frame due to the two tensions T_1 passes through the centre of the upper tightening pulley and has a vertical component

$$V_1'=T_1(\sin \alpha+\sin \beta)$$

and a horizontal component $H_1'=T_1(\cos \alpha-\cos \beta)$.

Similarly for the two tensions T_2 on the lower pulley. Taking moments about A for the equilibrium of the frame we have—

$$W_a=(V_1-V_2)a+(H_1-H_2)(b+c)$$

$$W=(T_1-T_2)\left\{(\sin \alpha+\sin \beta)+\frac{b+c}{a}(\cos \alpha-\cos \beta)\right\}$$

$$\text{H.P.}=(T_1-T_2)\frac{V}{33000}$$

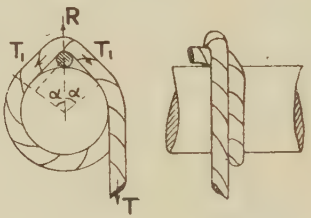
$$\text{H.P.}=\frac{WN}{\pi D \times 33000\left\{(\sin \alpha+\sin \beta)+\frac{b+c}{a}(\cos \alpha-\cos \beta)\right\}}$$

But α, β, a, b, c, π, D are constant for any given case, hence H.P. = Constant × WN.

Certain errors are neglected in the above treatment, viz. the power required to bend the belt to and fro, the friction of the tightening pulley bearings.

The weight of the frame and tightening pulleys should be balanced or be taken into account in evaluating W.

49. *A scaffold cord $\frac{3}{4}$ inch in diameter is suspended from a horizontal pole 3 inches in diameter by means of the simple hitch shown in the figure. Neglecting any compression of the cord at the point of crossing, find the least coefficient of friction which will ensure safety.*



$\cos \alpha = \frac{D-d}{D+d} = \frac{3}{5}$ $\quad \sin \alpha = \frac{4}{5}$ $\quad \alpha = 1$ radian, nearly.

Let T_2 be tension in loose end of cord at point of pinning.

T_1 be tension at point where it crosses the pinned end.

Then $\frac{T_1}{T_2} = e^{\mu 2\pi - \alpha}$ for coil friction.

Also, from the statical conditions at the point of crossing

$$\left.\begin{aligned} R &= 2T_1 \sin \alpha \\ T_2 &= \mu R \end{aligned}\right\} \qquad \frac{T_1}{T_2} = \frac{1}{2\mu \sin \alpha}$$

Hence $e^{\mu(2\pi - \alpha)} 2\mu \sin \alpha = 1$.

Solving by trial and error $\mu = \mathbf{0{\cdot}21}$.

Hence no slipping will occur if the coefficient of friction is greater than this value.

Note that since the question of slipping depends only on T_1 and T_2 the coil friction on the rope between T_1 and T does not affect the answer.

50. *The winding drum on a crane is fitted with a brake wheel and band brake When lowering the load pressure is*

applied to the band by means of a lever. Find the force required on the long arm of the lever to gently lower a load of 10 *tons. Diameter of brake drum to centre of rope,* 10″ ; *diameter of brake wheel,* 24″ ; *leverage,* 6 *to* 1 ; *coefficient of friction* (μ), 0·22 ; *angle embraced by band,* $\frac{5}{6}$ *circumference.*

Friction required on surface of brake drum $= \frac{10 \times 10}{24} = 4{\cdot}17$ tons.

Friction $= T_2 - T_1 = T_1 e^{\mu\alpha} - T_1$.

Hence $4{\cdot}17 = T_1(2{\cdot}72^{0{\cdot}22 \times 5\ 24} - 1) = 2{\cdot}17 T_1$

$$T_1 = \frac{4{\cdot}17}{2{\cdot}17} = 1{\cdot}92 \text{ tons.}$$

Pressure on end of lever when the tight end of band is fixed to fulcrum

$$= \frac{1{\cdot}92}{6} = 0{\cdot}32 \text{ ton} = \mathbf{720 \text{ pounds.}}$$

If the slack end of band is fixed to fulcrum the pressure is

$$\frac{1{\cdot}92 \times 3{\cdot}17}{6} = 1{\cdot}01 \text{ ton} = \mathbf{2{,}270 \text{ pounds.}}$$

CHAPTER X

STRESS, STRAIN, AND ELASTICITY

1. *An engine has a piston* 20″ *diam. Rod* $2\frac{1}{2}$″ *diam.* 80″ *long. Find the actual length of the rod during each stroke.* P=120 *lb. sq. in.* E=12,000 *tons sq. in.*

Forward. Area of rod 4·9 sq. ins. Piston 314 sq. ins.

$$\text{Thrust} = \frac{120 \times 314}{2240} = 16{\cdot}8 \text{ tons}$$

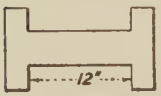
$$x = \frac{Wl}{aE} = \frac{16{\cdot}8 \times 80}{4{\cdot}9 \times 12000} = 0{\cdot}023''. \quad \textbf{Length 79·977″.}$$

Back. Piston area=314−4·9=309 sq. ins.

$$\text{Load} = \frac{120 \times 309}{2240} = 16{\cdot}6 \text{ tons.}$$

$$x = \frac{16{\cdot}6 \times 80}{4{\cdot}9 \times 12000} = 0{\cdot}023''. \quad \textbf{Length 80·023″.}$$

2. *The bar as shown in the figure is* 12″ *long between the flanges at* 60° *F. and is heated until two pieces of metal* 12·02″ *long can be slipped in on both sides. Find the necessary temperature of the bar. What will be the stress when it is cooled to* 60° *F.? Assume the pieces slipped in to be rigid.* E=12,000 *tons/in.*^{2}. *Coefficient of expansion,* 0·000007.



For a rise of temp. $t°$. Expansion on 12″=12×0·000007t

$$\therefore t = \frac{0{\cdot}02}{12 \times 0{\cdot}000007} = 238° \text{ F.} \quad \therefore \text{ temp.} = 238 + 60 = \textbf{298° F.}$$

$$f = \frac{Ex}{l} = \frac{12000 \times 0{\cdot}02}{12} = \textbf{20 tons/in.}^2.$$

(This only holds if the elastic limit is not exceeded.)

3. *A thin steel ring* 24·98″ *internal diameter is heated and placed on a shaft* 25″ *diam. Assuming that the shaft is rigid, find the stress in the ring when both are at the same temperature.* E=14,000 *tons sq. in.*

Find the lowest temperature to which the ring must be raised before being slipped on to the shaft. Steel expands 0·0000067 *of its own length for* 1° *F. Atmospheric temp.* 60° *F.*

$$\frac{x}{l}=\frac{f}{E} \qquad \frac{0{\cdot}02}{25}=\frac{f}{14000} \qquad f=\frac{14000\times 0{\cdot}02}{25}=\left\{\begin{matrix}\textbf{11·2 tons}\\ \textbf{sq. in.}\end{matrix}\right.$$

$$\text{Rise in temp.}=\frac{0{\cdot}02}{25\times 0{\cdot}0000067}=120^\circ \text{ F.}$$

$$\therefore \text{ Temp. of ring}=120+60=\mathbf{180^\circ\ F.}$$

In practice the temperature would have to be somewhat higher than this to give clearance when slipping on the ring.

4. *Two bars* A *and* B *are attached to one another at both ends when at a temperature* t_1° F. *They are then heated to a temperature* t_2° F. *Find the stress in each if both remain straight.*

Coefficient of free expansion of A=C_A
" " " B=C_B (greater than C_A)
Sectional area of A =A_A
" " B =A_B
Young's Modulus of Elasticity of A=E_A
" " " B=E_B
Let $t_2-t_1=t$.

Difference of length when free per unit length=$t(C_B-C_A)$.

The bar B with the higher coefficient of expansion will be in compression, and the bar A in tension.

The load (W_A) on A=the load (W_B) on B.

x_A=the stretch of A per unit length.
x_B=the compression of B per unit length.

$$x_A=\frac{W_A}{A_AE_A}=\frac{f_A}{E_A} \quad \text{and} \quad x_B=\frac{W_B}{A_BE_B}=\frac{f_B}{E_B}$$

Then $$x_A=\frac{A_BE_B}{A_AE_A}x_B$$

also $x_A+x_B=t(C_B-C_A)$

$$x_B\left(1+\frac{A_BE_B}{A_AE_A}\right)=t(C_B-C_A) \qquad x_B=\frac{t(C_B-C_A)A_AE_A}{A_BE_B+A_AE_A}=\frac{f_B}{E_B}$$

$$f_B=\frac{tE_B(C_B-C_A)A_AE_A}{A_BE_B+A_AE_A} \quad \text{and} \quad f_A=\frac{A_Bf_B}{A_A}.$$

5. *A flat steel bar has two similar bars of copper attached to it at both ends (sandwich fashion). When they are all at* 60° *Fahr. they are free from stress. When raised to a temperature of* 560° *Fahr. it is found that they have expanded* 0·05 *in. Find the stress in each bar at this temperature. What was their original length?* E (*steel*), 14,000 *tons per sq. in.* E (*copper*), 7,000 *tons per sq. in.*
Coefficient of expansion per degree Fahr. (*steel*), 0·0000069
" " " (*copper*), 0·00001.

Difference in temperature = 500° F.

Expansion of steel due to change of temperature $\left.\right\}$ $500\times0\cdot0000069l=0\cdot00345l$

Ditto. copper $500\times0\cdot00001l=0\cdot005l$

Extension of steel due to stress $=0\cdot05-0\cdot00345l$

Contraction of copper due to stress $=0\cdot005l-0\cdot05$

$$f_s=\frac{E_sx_s}{l}=\frac{14000(0\cdot05-0\cdot00345l)}{l}$$

$$f_c=\frac{E_cx_c}{l}=\frac{7000(0\cdot005l-0\cdot05)}{l}$$

But $x_s=x_c \qquad \therefore f_s=2f_c$

$0\cdot05-0\cdot00345l=0\cdot005l-0\cdot05 \qquad l=11\cdot84''$

$$f_s=\frac{14000(0\cdot05-0\cdot00345\times11\cdot84)}{11\cdot84}=\mathbf{10\cdot8\ tons\ sq.\ in.}$$

$f_c=\mathbf{5\cdot4\ tons\ sq.\ in.}$

6. *A bolt passes longitudinally through the middle of a gun-metal pipe. The head and nut of the bolt bed on thick washers which do not appreciably distort under load. The nut is just tight when the temperature is* 60° *F. Find the intensity of stress on the bolt and pipe, also the total load on each when the temperature is* 340° *F. Diam. of shank of bolt* (A), 1·25"; *diam. of pipe* (B), 4·0" and 4·75"; E_A (*bolt*), 11,000 *tons sq. in.*; E_B (*pipe*), 5,000 *tons sq. in.*; *coefficient*

of expansion C_A (*bolt*), 0·000007 ; *coefficient of expansion* C_B (*pipe*), 0·000010.

Difference in temperature $=280°$ F.
Area of bolt section A_A $=1·227$ sq. ins.
„ pipe „ A_B $=5·15$ sq. ins.
The load on the bolt = the load on the pipe.

$$f_A \times 1·227 = f_B \times 5·15 \qquad f_A = \frac{f_B \times 5·15}{1·227}$$

Difference in length of bolt and pipe when heated $= 280(0·000010 - 0·000007)$

$$280 \times 0·000003 = \frac{f_B}{5000} + \frac{f_A}{11000} = f_B\left(\frac{1}{5000} + \frac{5·15}{1·227 \times 11000}\right)$$

$$f_B = \frac{280 \times 0·000003}{\frac{1}{5000} + \frac{5·15}{1·227 \times 11000}} = \mathbf{1·44\ tons\ sq.\ in.}\ \text{compression in pipe.}$$

Load on pipe $= 1·44 \times 5·15 = \mathbf{7·44\ tons.}$

$$f_A = \frac{1·44 \times 5·15}{1·227} = \mathbf{6·06\ tons\ sq.\ in.}\ \text{tension in bolt.}$$

If the pipe were of cast iron having a coefficient of expansion of 0·000006 there would be no stress in either the pipe or bolt, because the bolt would expand more than the pipe. If, however, the bolt had been screwed into a cover plate attached to each end of the pipe the bolt would have been in compression and the pipe in tension, in which case the bolt would probably buckle.

If the cast-iron pipe and bolt had been cooled to a temperature below 60° F., the bolt would contract more than the pipe and would consequently be in tension.

Let the temperature be reduced to $-100°$ F.
E cast iron = 8000 tons sq. in.
Then $t = 160°$ F., and by similar reasoning

$$f_B = \frac{160 \times 0·000001}{\frac{1}{8000} + \frac{5·15}{1·227 \times 11000}} = \mathbf{0·32\ ton\ sq.\ in.}\ \text{compression in pipe.}$$

Load on pipe $= 0·32 \times 5·15 = 1·65$ tons.

$$f_A = \frac{0·32 \times 5·15}{1·227} = \mathbf{1·34\ tons\ sq.\ in.}\ \text{tension in bolt.}$$

7. *A copper bar 2″ diam. is rigidly connected at both ends to the inside of an iron pipe. Diams. 3″ and 2½″. The combination is then subjected to a load of 8 tons. Find the stress in each and the stretch in a length of 5 feet.* E *copper* $=7000$ *tons sq. in.* E *iron* $=12{,}000$ *tons sq. in.*

Since the strains are equal $\dfrac{f_c}{E_c}=\dfrac{f_i}{E_i}$

$$\frac{W_c}{A_cE_c}=\frac{8-W_c}{A_iE_i} \qquad \frac{W_c}{3{\cdot}14\times7000}=\frac{8-W_c}{2{\cdot}16\times12000}$$

$W_c=3{\cdot}67$ tons $\qquad W_i=4{\cdot}33$ tons.

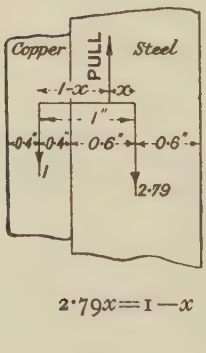
Stress in copper $=\dfrac{3{\cdot}67}{3{\cdot}14}=$ **1·17** tons sq. in.

„ iron $=\dfrac{4{\cdot}33}{2{\cdot}16}=$ **2·0** „ „

Stretch $=\dfrac{Wl}{AE}=\dfrac{1{\cdot}17\times60}{7000}=$ **0·01″.**

8. *A flat bar of steel is fastened at both ends to a similar bar of copper: they are then jointly subjected to a pull of 60 tons. Find* (a) *the load supported by each bar;* (b) *the stress in each bar. Where should the line of pull cut the cross-section in order to secure the stated distribution of stress? Width of both bars, 5″; thickness of steel bar, 1·2″; thickness of copper bar, 0·8″; modulus of elasticity—steel, 13,000 tons per sq. in.; modulus of elasticity—copper, 7,000 tons per sq. in.*

The extension of the steel is the same as that of the copper, hence—



$$\frac{W_s}{A_sE_s}=\frac{W_c}{A_cE_c}$$

$$W_s=W_c\frac{A_sE_s}{A_cE_c}=W_c\frac{1{\cdot}2\times13000}{0{\cdot}8\times7000}=2{\cdot}79W_c$$

$W_s+W_c=60$ $\qquad 3{\cdot}79\,W_c=60$

$\therefore$ $W_c=$ **15·8 tons** and $W_s=$ **44·2 tons.**

From the diagram we have by moments—

$2{\cdot}79x=1-x$ $\qquad x=$ **0·264″** from middle of steel bar.

9. *In a wire rope, one-seventh of the wires are of soft wrought iron, and the remainder are of hard steel; what proportion of the load is taken by the iron and the steel wires respectively? They are all of the same diameter. For the steel wires,* E=14,000 *tons sq. in., and for the iron wires,* E=11,000 *tons sq. in.*

Assume the extension of all the wires to be the same.

$$\frac{W_s}{A_sE_s} = \frac{W_i}{A_iE_i} \qquad \tfrac{1}{7}\text{th of the wires are soft}$$

$$\frac{W_s}{W_i} = \frac{A_sE_s}{A_iE_i} = \frac{6\times 14000}{1\times 11000} = \frac{84}{11}.$$

10. *The following results were obtained from a tension test. Find* (i) *Young's Modulus of Elasticity;* (ii) *the elastic limit;* (iii) *the yield point;* (iv) *the tensile strength;* (v) *the stress at fracture;* (vi) *the percentage of extension on* 10 *inches;* (vii) *the reduction in area per cent.;* (viii) *the elastic resilience;* (ix) *the work done in inch-tons per cubic inch in fracturing the bar.*

Load in tons .	1	2	3	4	5	6	7	8
Extensometer reading in ins. on 10" length	0·0026	0·0032	0·0039	0·0044	0·0051	0·0056	0·0062	0·0069
Load in tons .	9	10	11	12	13	14	15	16
Extensometer reading . .	0·0075	0·0082	0·0088	0·0094	0·0101	0·0108	0·0115	0·0122
Load in tons .	17	18	19	20	25	26·42	22·10	
Extensometer reading . .	0·0128	0·0140	0·0156	0·23	1·26	2·48	2·90	

Original diameter of bar 1·25 inches.
Final diameter at point of fracture 0·89 inch.

In order to get the average extension the results should be dealt with thus—

0·0094	0·0088	0·0082	0·0075	0·0069	0·0062
0·0056	0·0051	0·0044	0·0039	0·0032	0·0026
0·0038	0·0037	0·0038	0·0036	0·0037	0·0036

Average difference for 6 tons, 0·0037″

(i) $E = \frac{6 \times 10}{0{\cdot}0037 \times 1{\cdot}227} = \mathbf{13{,}200}$ tons sq. in.

(ii) Elastic limit $= \frac{18}{1{\cdot}227} = \mathbf{14{\cdot}7}$ tons sq. in. approx.

(iii) Yield $= \frac{20}{1{\cdot}227} = \mathbf{16{\cdot}3}$ tons sq. in. approx.

(iv) Maximum stress $= \frac{26{\cdot}42}{1{\cdot}227} = \mathbf{21{\cdot}5}$ tons sq. in. approx.

(v) Stress at fracture $= \frac{22{\cdot}10}{0{\cdot}618} = \mathbf{35{\cdot}8}$ tons sq. in. approx.

(vi) **29** per cent. extension on 10″.

(vii) $\frac{1{\cdot}227 - 0{\cdot}618}{1{\cdot}227} = \mathbf{49{\cdot}5}$ per cent. reduction in area.

(viii) Elastic resilience $(0{\cdot}0128 - 0{\cdot}0020)\frac{17}{2} = \mathbf{0{\cdot}092}$ in.-ton.

(ix) $\frac{x}{300}(L + 2M) = \frac{29}{300}(16{\cdot}3 + 2 \times 21{\cdot}5) = \mathbf{5{\cdot}7}$ inch-tons.

See p. 383, Vol. I.

11. *The following results were obtained from a tension test of a wire rope.*

Number of wires in rope, 36; *diam. of wires*, 0·072″; *test length*, 30″.

Load in tons .	0·5	1	1·5	2	2·5	3	3·5	4
Extension in inches . .	0·075	0·100	0·115	0·133	0·150	0·170	0·185	0·200
Load in tons .	4·5	5	5·5	6	6·5	7	7·5	8
Extension in inches . .	0·220	0·240	0·270	0·300	0·35	0·39	0·44	0·49
Load in tons .	8·5	9	9·5	10	10·5	10·8	(Broke)	
Extension in inches . .	0·54	0·59	0·65	0·70	0·75	0·80		

Find (i) *an approximate value for Young's Modulus of Elasticity of the rope. Why is it lower than the value obtained from a single wire?* (ii) *The elastic limit;* (iii) *the tensile strength of the rope;* (iv) *the efficiency of the*

rope if each single wire breaks at a load of 692 *pounds;* (v) *the percentage of extension.*

(i) The average elastic extension can be best obtained by plotting the loads and extensions, or by taking differences; thus for a range of 3 tons—

0·300	0·270	0·240	0·220	0·200	0·185
0·170	0·150	0·133	0·115	0·100	0·075
0·130	0·120	0·107	0·105	0·100	0·110

The rope was evidently approaching its elastic limit after 5 tons—so reject the first two differences.

Mean of remainder=0·1055″ for 3 tons.

Sectional area=$0{\cdot}785\times0{\cdot}072^2\times36=0{\cdot}146$ sq. ins.

$$E=\frac{3\times30}{0{\cdot}146\times0{\cdot}1055}=\mathbf{5{,}850}\text{ tons sq. in.}$$

If the load had been kept on for some hours this would have been higher, the tightening up of the strands is the main cause of the modulus being lower than that found for single wires.

(ii) Elastic limit $\frac{5{\cdot}5}{0{\cdot}146}=\mathbf{37{\cdot}7}$ tons sq. inch (approx.).

(iii) Tensile strength $\frac{10{\cdot}8}{0{\cdot}146}=\mathbf{74}$ tons sq. inch.

(iv) Breaking load of wires $\frac{692\times36}{2240}=11{\cdot}1$ tons.

Efficiency $\frac{10{\cdot}8\times100}{11{\cdot}1}=\mathbf{97{\cdot}3}$ per cent.

(v) Extension $\frac{0{\cdot}8\times100}{30}=\mathbf{2{\cdot}7}$ per cent.

12. *Calculate the breaking load of a bolt* $1\frac{7}{16}$ *ins. diameter; tensile strength of iron,* 25 *tons sq. in. When broken in the ordinary way the bolt stretched* 0·62 *in. on a length of* 12 *ins., but when the shank of a similar bolt was turned down to the diameter at the bottom of the thread, it stretched* 2·83 *ins. The elastic limit in the first case occurred at* 18·3 *tons, and in the second case,* 17·8 *tons. Find the relative capacity of the two bolts to resist shocks.*

(See p. 384, Vol. I.)

$1\frac{7}{16}''=11{\cdot}5$ (in $\frac{1}{8}''$ units).

Section at bottom of thread $= \frac{11\cdot5 \times 10\cdot5}{100} = 1\cdot21$ sq. ins.

Breaking load $= 1\cdot21 \times 25 =$ **30·2 tons.**

(See p. 382, Vol. I.)

Work done in fracturing—

$$\frac{0\cdot62}{3}(18\cdot3+2\times30\cdot2)=16\cdot3 \text{ inch-tons}$$

$$\frac{2\cdot83}{3}(17\cdot8+2\times30\cdot2)=73\cdot8 \text{ inch-tons}$$

Relative capacity to resist shock $= \frac{73\cdot8}{16\cdot3} =$ **4·5.**

13. *Find the load required to punch a hole 1·2″ diam. through a plate 0·75″ thick. Tensile strength of plate 22·5 tons sq. in.*

The shearing strength $= \frac{4}{5} \times 22\cdot5$. (See p. 391, Vol. I.)

Load $= 1\cdot2 \times 3\cdot14 \times 0\cdot75 \times 22\cdot5 \times \frac{4}{5} =$ **51 tons.**

14. *Bolt* $1\frac{1}{2}$*″ diam. Length from neck to first thread 9″. A hole is drilled down the shank to reduce its area to that at the bottom of thread. Find the diam. of hole. Find work done in fracturing bolt* (a) *when the shank is drilled;* (b) *not drilled. Total extension when drilled 2·4″, not drilled 0·31″. Tensile stress 22 tons sq. in. Elastic limit 14 tons sq. in.*

Area at bot. thread $\frac{D(D-1)}{100} = \frac{12\times11}{100} = 1\cdot32$ sq. ins.

(See p. 384, Vol. I.)

Net area of shank $= 0\cdot785\ (1\cdot5^2 - d^2) = 1\cdot32$. $\quad d = 0\cdot75″$.

(*a*) Work done when drilled (see p. 382, Vol. I.) $\left.\right\} \frac{x}{3}(2M+L)\times A = \frac{2\cdot4}{3}(44+14)1\cdot32$

$= $ **61·4** inch-tons.

(*b*) When not drilled, stress in shank $\left.\right\} = \frac{22\times1\cdot32}{0\cdot785\times1\cdot5^2} = 16\cdot4$ tons per sq. in.

Work done (approx.) $= \frac{61\cdot4\times0\cdot31}{2\cdot4} =$ **7·93 inch-tons.**

15. *A hard drawn steel tube is fitted into a socket and is secured by steel pins passing diametrically through the socket and tube. Find the least number of pins required to fulfil*

the conditions that the shearing stress in the pins shall not exceed 8 *tons per square inch and the bearing pressure on the pins shall not exceed* 12 *tons per sq. in. when the tensile stress in the tube is* 20 *tons per sq. in. You may assume that the load is evenly distributed over all the pins. External diam. of tube,* 1·88″; *internal,* 1·79″; *diam. of pins,* 0·25″.

Section of tube 0·259 sq. in.
Load on tube $0{\cdot}259 \times 20 = 5{\cdot}2$ tons.
Shearing area of each pin 0·098 sq. in.
Shearing resistance per pin $8 \times 0{\cdot}098 = 0{\cdot}78$ ton.

Number of pins required for shear $\frac{5{\cdot}2}{0{\cdot}78}$ =say 7.

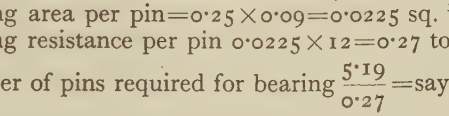
Bearing area per pin $= 0{\cdot}25 \times 0{\cdot}09 = 0{\cdot}0225$ sq. in.
Bearing resistance per pin $0{\cdot}0225 \times 12 = 0{\cdot}27$ ton.

Number of pins required for bearing $\frac{5{\cdot}19}{0{\cdot}27}$ =say **19.**

The section of the tube allowing for the pin holes is less than the above. When taken into account 6 pins are required for shear and 18 for bearing.

Such joints as the above nearly always fail through the bearing stress on the pins being excessive.

16. *Calculate the breaking load of the faulty cottered joint plug shown in the figure. Tensile strength of material* 30 *tons sq. in.*



The joint will break through at *a* and *b*.

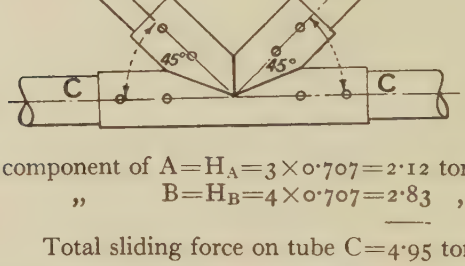
Section at $b = \frac{0{\cdot}785 \times 2^2 - 2 \times \frac{1}{2}}{2} = 1{\cdot}07$ sq. ins. approx.

Section at *a*. The distance of the short edge of the cotterway from the edge of the circle is 0·5 inch, and the same distance from the diameter. The length of the short edge is $\frac{\sqrt{3}}{2}d$. Hence the area is—

$$\tfrac{2}{3}\left(\frac{\sqrt{3}}{2}d \times \frac{d}{4}\right) \text{ nearly} = 0{\cdot}144d^2 = 0{\cdot}58 \text{ sq. in.}$$

Total area of section = 1·65 sq. ins.
Breaking load = 30 × 1·65 = **49·5 tons.**
If the cotterway had been cut fair the breaking load would have been = 30 × 2·14 = **64·2 tons.**

17. *The Fig. represents a socketed joint in a tubular frame; the bottom tube* CC *is continuous. There is a compressive force of* 3 *tons acting on* A, *and a tensile force of* 4 *tons on* B. *Find the number of pins* 0·3 *inch in diameter required in the tube* CC *in order to prevent the joint from sliding on the tube.*
Shear stress in pins not to exceed 6 *tons per sq. inch.*
Bearing ditto 12 *tons per sq. inch.*
External diam. of tube, 1·5 *ins.; internal,* 1·3 *ins.*
What is the maximum tensile stress in the tube CC *neglecting bending?*



Hor. component of A = H_A = 3 × 0·707 = 2·12 tons
" " B = H_B = 4 × 0·707 = 2·83 "

Total sliding force on tube C = 4·95 tons.

Shearing area of each pin = 2 × 0·071 = 0·142 sq. in.
" resistance of each pin = 0·142 × 6 = 0·852 ton

Number of pins for shear $= \frac{4 \cdot 95}{0 \cdot 852}$ = say 6.

Bearing area of each pin = 2 × 0·3 × 0·1 = 0·06 sq. in.
" resistance of each pin = 12 × 0·06 = 0·72 ton

Number of pins for bearing $= \frac{4 \cdot 95}{0 \cdot 72}$ = **7**

Net sectional area of tube = 3·14 × 1·4 × 0·1 − 2 × 0·3 × 0·1
" " = 0·44 − 0·06 = 0·38 sq. in.

Max stress in tube $= \frac{4 \cdot 95}{0 \cdot 38}$ = **13 tons sq. in.**

18. *Three wires support a load of* 1000 *lb. The centre wire is of steel, the two outer ones of brass, each* 0·1″ *diameter, all of the same length when unloaded at* 60° *F. Find the stress in each wire on the assumption that they remain elastic.*
(1) *at* 60° F. (2) *at* 150° *F.*
E *for steel* = 12,000 *tons in.*^{2}. E *for brass* = 5000 *tons in.*^{2}
Coeff. of expansion : steel, 0·000006 ; *brass,* 0·00001.

(1) at 60° F.

Let load on *both* brass wires together = L_B, stress = f_B ; load on steel wire = L_s, stress = f_s.

$$\frac{L_B}{A_B \times E_B} = \frac{1000 - L_B}{A_s \times E_s} \qquad \frac{L_B}{2 \times 5000} = \frac{1000 - L_B}{1 \times 12000}$$

$$\therefore 6L_B = 5000 - 5L_B \qquad \therefore L_B = \tfrac{5000}{11} = 455 \text{ lb.}$$

or each wire takes 227·5 lb. $L_s = 1000 - 455 = 545$ lb.

$$f_s = \frac{545}{0{\cdot}00785 \times 2240} = \textbf{31 tons per sq. in.}$$

$$f_B = \frac{227{\cdot}5}{0{\cdot}00785 \times 2240} = \textbf{12·95 tons per sq. in.}$$

(2) At 150° F.
Suppose each wire of unit length at 60° F.
Then at 150° F. :—
length of brass wire = $1 + 90 \times 0{\cdot}00001 = 1{\cdot}00090$
length of steel wire = $1 + 90 \times 0{\cdot}000006 = 1{\cdot}00054$
The steel wire must be strained initially by $1{\cdot}00090 - 1{\cdot}00054 = 0{\cdot}00036$.

The load required to do this = $12{,}000 \times 0{\cdot}00036 \times 0{\cdot}00785 \times 2240 = 76$ lb. Then the remaining load of $(1000 - 76) = 924$ lb. will be distributed in the same ratio as in (1), or

$L_B = \frac{227{\cdot}5 \times 924}{1000} = 215$ lb. per wire. $L_s = (1000 - 430) = 570$ lb.

$$f_s = \frac{580}{0{\cdot}00785 \times 2240} = \textbf{32·4 tons per sq. in.}$$

$$f_B = \frac{215}{0{\cdot}00785 \times 2240} = \textbf{12·2 tons per sq. in.}$$

19. *A load of* 1,000 *lb. is suspended by three parallel wires of equal length, fastened together at each end, of steel, copper, and brass respectively. Find the stress in each wire.*

Steel section, 0·045 *sq. in.; copper section*, 0·079 *sq. in.; brass section*, 0·067 *sq. in.;* E, *steel*, 12,000 *tons per sq. in.; copper*, 7,000 *tons per sq. in.; brass*, 4,000 *tons per sq. in.*

Since they are fastened together, all the wires extend equally—

$$0{\cdot}045f_s+0{\cdot}079f_c+0{\cdot}067f_B=1000$$

$$f_c=f_s\frac{E_c}{E_s}=\tfrac{7}{12}f_s \quad \text{and} \quad f_B=\tfrac{4}{12}f_s$$

$$\therefore\ 0{\cdot}045f_s+\frac{0{\cdot}079\times7}{12}f_s+\frac{0{\cdot}067}{3}f_s=1000$$

$$f_s=\frac{1000}{0{\cdot}1133}=8825 \text{ lb. sq. in. } =\mathbf{3{\cdot}94} \text{ tons sq. in.}$$

$f_c=\frac{7}{12}f_s=5150$ lb. sq. in. $=\mathbf{2{\cdot}30}$ „ „

$f_B=\frac{4}{12}f_s=2942$ lb. sq. in. $=\mathbf{1{\cdot}31}$ „ „

20. *Three long parallel wires of equal length and in the same vertical plane, jointly support a load of* 3,000 *lb. The middle wire is steel and the two outer ones are brass, the sectional area of each is* 0·25 *sq. in. Each wire is suspended from a tightening screw, and after loading the screws are so adjusted that the load on each wire is the same; a further load of* 7,000 *lb. is then added. Find the stress in each wire, and the proportion of the total load supported by each.*

E *steel*=30,000,000 *lb. sq. in.; brass*=12,000,000 *lb. sq. in.*

See Morley's "Strength of Materials," p. 28.

$$\frac{W_s}{A_sE_s}=\frac{2W_B}{2A_BE_B} \qquad \frac{W_s}{W_B}=\frac{E_s}{E_b}$$

$W_s+2W_B=7000$ lb. $\qquad W_s=\frac{30}{12}W_B$

$$2{\cdot}5W_B+2W_B=7000$$

$W_B=1550$ lb. each $\qquad W_s=3900$ lb.

$$f_B=\frac{1550+1000}{0{\cdot}25}=10{,}220 \text{ lb. sq. in.}$$

$$f_s=\frac{3900+1000}{0{\cdot}25}=19{,}560 \text{ lb. sq. in.}$$

Loads: Brass **2,550** lb. each; Steel **4,900** lb.

21. *Three pillars each* 1·5″ *diam. arranged equidistant and in line, together support a rigid block weighing* 30 *tons.*

Find the load on each pillar when (a) *the two outer pillars are* 19·99″ *long, and the central one* 20″ *long ;* (b) *the two outer are* 20″ *and the central* 19·99″. E=12,000 *tons sq. in.*

Sectional area of each=1·767 sq. ins.

W= load on central pillar.

$\frac{30-W}{2}$ = load on outer pillar.

f (centre) $= \frac{W}{1\cdot767}$ tons sq. in.

f (outer) $= \frac{30-W}{2\times1\cdot767}$ tons sq. in.

x (centre) $= \frac{fL}{E} = \frac{20W}{1\cdot767\times12000} = 0\cdot000944W$ inches.

x (outer) $= \frac{19\cdot99(30-W)}{2\times1\cdot767\times12000} = 0\cdot000471(30-W)$.

For case (*a*)—but x (centre)=x(outer)+0·01″

$0\cdot000944W = 0\cdot000471(30-W)+0\cdot01$

$W = \frac{0\cdot02413}{0\cdot001415} =$ **17·05** tons on central pillar

$\frac{30-W}{2} =$ **6·475** tons on each outer pillar

x (centre) $=0\cdot000944\times17\cdot05=0\cdot0161''$

x (outer) $=0\cdot000471\times12\cdot95=0\cdot0061''$.

For case (*b*)— x (centre)=x (outer)−0·01″

Then W=**2·93** tons on central pillar.

$\frac{30-W}{2} =$ **13·53** tons on each outer pillar.

x (centre)=0·00277″ x (outer)=0·01277″.

22. *The ends of a piece of pipe are closed by rigid flat covers at each end, which are held in position by means of a central bolt which is enlarged at the screwed ends. Find by how much the bolt stretches when the nut is tightened up sufficiently to subject the pipe to a compressive stress of* 3 *tons per sq. in. Length of pipe to outside of covers,* 32″ ; *internal diam. of pipe,* 2·4″ ; *external diam. of pipe,* 3·2″ ; *diam. of*

bolt, 1·75″; *modulus of elasticity of bolt*, 13,000 *tons sq. in.*; *modulus of elasticity of pipe*, 6,000 *tons sq. in.*

Net area of section of pipe = 3·52 sq. ins.
Load on pipe $3 \times 3{\cdot}52 = 10{\cdot}56$ tons.
Section of bolt = 2·4 sq. ins.

$$\text{Stress in bolt} = \frac{10{\cdot}56}{2{\cdot}4} = 4{\cdot}4 \text{ tons sq. in.}$$

$$\text{Extension of bolt} = \frac{4{\cdot}4 \times 32}{13000} = 0{\cdot}0108 \text{ inch.}$$

$$\text{Compression of pipe} = \frac{3 \times 32}{6000} = 0{\cdot}016 \text{ inch.}$$

The bolt has to be tightened up by **0·0268 inch.**

23. *A* 14″ *pipe is closed by a flange. Pressure in pipe* = 150 *lb. sq. in. Find the necessary number and diameter of bolts in the flange, the pitch not exceeding* 5d.

Taking the safe load in tons for a bolt as $2{\cdot}5d^2 - 1{\cdot}2d$, *where* d *is the diameter of the bolt in inches.*

$$\text{Total pressure} = \frac{0{\cdot}785 \times 14^2 \times 150}{2240} = 10{\cdot}3 \text{ tons}$$

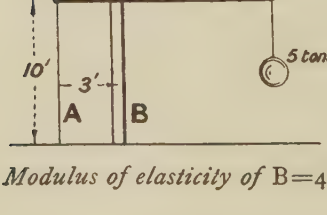
Assume dia. of bolt circle = 18″ circumference = $18\pi = 56{\cdot}6$″

Then $56{\cdot}6 = 5d \times n$ $\qquad \therefore n = \dfrac{56{\cdot}6}{5d}$

$$\frac{56{\cdot}6}{5d}(2{\cdot}5d^2 - 1{\cdot}2d) = 10{\cdot}3 \qquad d = 0{\cdot}85'' \text{ say } \tfrac{7}{8}''$$

$$n = \frac{56{\cdot}6}{5 \times \frac{7}{8}} = 13 \text{ or } \mathbf{14} \text{ for symmetry.}$$

24. *Find the amount* x *the free end of the lever shown in the figure will sink when a load of* 5 *tons is placed upon it as shown. The lever may be regarded as rigid. Sectional area of* A = 1·4 *sq. ins. Sectional area of* B = 9·4 *sq. ins. Modulus of elasticity of* A = 12,000 *tons sq. in. Modulus of elasticity of* B = 4,000 *tons sq. in.*



$W_A = 5 \times \frac{2}{3} = 15$ tons. $\quad W_B = \frac{5 \times 12}{3} = 20$ tons.

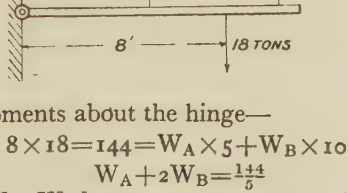
$\frac{x_A}{l} = \frac{W_A}{A_A E_A}$ $\quad x_A = \frac{15 \times 120}{1{\cdot}4 \times 12000} = \frac{3}{28}''$ $\quad x_1 = \frac{9}{28}''$

$x_B = \frac{20 \times 120}{9{\cdot}4 \times 4001} = \frac{6}{94}''$ $\quad x = \frac{24}{94}''$ at end.

$x = \frac{9}{28} + \frac{24}{94} = \mathbf{0{\cdot}576}''$ at end of lever.

25. *Find the load on each of the vertical ties* A *and* B. *The modulus of elasticity is the same for both. The beam may be regarded as rigid.*

Sectional area of A = 1·5 sq. ins. B = 1·8 sq. ins.



Taking moments about the hinge—

$$8 \times 18 = 144 = W_A \times 5 + W_B \times 10$$

$$W_A + 2W_B = \tfrac{144}{5}$$

$\frac{2W_A l}{A_A E} = \frac{W_B l}{A_B E}$ $\quad W_A = \frac{A_A W_B}{2A_B}$

$\frac{A_A W_B}{2A_B} + 2W_B = \frac{144}{5}$ $\quad \frac{1{\cdot}5 W_B}{3{\cdot}6} + 2W_B = 28{\cdot}8$

$W_B = \mathbf{11{\cdot}9}$ **tons.** $\quad W_A = 28{\cdot}8 - 2 \times 11{\cdot}9 = \mathbf{5}$ **tons.**

Check $\quad 5 \times 5 + 11{\cdot}9 \times 10 = 144 = 8 \times 18.$

26. *A bar* 24 *inches long is suspended centrally in a horizontal position by a vertical wire* a, *another vertical wire* b *is attached to the underside of one end of the bar. The length of each wire is* 10 *feet and the section* 0·1 *sq. inch. Find the deflection of the free end* c *of the bar when loaded with* 250 *lb.* E = 30,000,000 *lb. per sq. inch.*

Load on a = 500 lb., on b = 250 lb.

$$x_a = \frac{120 \times 500}{0{\cdot}1 \times 30{,}000{,}000} = 0{\cdot}02'' \text{ and } x_b = 0{\cdot}01''$$

Deflection at C = 0·01 + 2 × 0·02 = **0·05**″.

27. *Find the pitch of the outer row of rivets in a combined lap and cover plate joint for a boiler; thickness of plates,* $\frac{5}{8}$ *in., punched holes; tensile strength of plates,* 26 *tons per sq. in.; tensile strength of rivets,* 24 *tons per sq. in.; also find the bearing pressure on the rivets and the efficiency of the joint.* (See Vol. I., p. 416.)

$$d=1{\cdot}1\sqrt{\tfrac{5}{8}}=0{\cdot}87'' \qquad c=0{\cdot}04'' \qquad K=0{\cdot}2''$$

$$(w-d-c-k)tf_t=3\{0{\cdot}785(d+c)^2\times\tfrac{4}{5}f_r\}$$

$$(w-0{\cdot}87-0{\cdot}04-0{\cdot}2)\tfrac{5}{8}\times 26=3\times 0{\cdot}785\times 0{\cdot}83\times\tfrac{4}{5}\times 24$$

$$w=\mathbf{3{\cdot}41''}$$

$$\text{Bearing pressure}=\frac{2{\cdot}30\times 26}{3\times 0{\cdot}91}=\mathbf{22\ tons\ sq.\ in.}$$

$$\text{Efficiency}=\frac{2{\cdot}3}{3{\cdot}41}=\mathbf{67\ per\ cent.}$$

28. *Find pitch of rivets—single row, lap joint. Thickness of plate* $\frac{1}{2}''$, *diameter of rivet* 1″, *clearance* 0·07″, *thickness of ring damaged by punching* 0·125″. *Tensile strength of plate* 22 *tons sq. inch, of rivets* 23 *tons sq. in.*

$$\{w-(d+c+k)\}tf_t=0{\cdot}785(1{\cdot}07^2)\tfrac{4}{5}\times 23$$

$$11w-14{\cdot}5=16{\cdot}5 \qquad w=\mathbf{2{\cdot}83''}.$$

29. *Find the thickness of plates required for a cylindrical receiver* 7 *ft. diameter for an internal pressure of* 180 *lb. per square inch; the longitudinal riveted joint is provided with double cover straps and two rows of rivets in each plate. The diameter of the rivets is* $1{\cdot}05\sqrt{t}$; *the clearance in the rivet holes is* 0·04d. *The working stress in the plates is* 5 *tons per square inch and the working shearing stress in the rivets* 4·5 *tons per square inch. The rivets are safe as regards bearing pressure.*

$$(d+c)^2=(d+0{\cdot}04d)^2=1{\cdot}08d^2 \text{ (see p. 412, Vol. I.)}$$

$$(d+c)^2=1{\cdot}08\times 1{\cdot}05^2t=1{\cdot}19t$$

$$w-d-c)t\times 5 \ =3{\cdot}14\times 1{\cdot}19t\times 4{\cdot}5=16{\cdot}8t \qquad w=3{\cdot}36+1{\cdot}04d$$

$$\text{Efficiency}=\frac{3{\cdot}36}{4{\cdot}4\ \text{(approx.)}}=76\text{ per cent.}$$

$$pd=2fte \text{ (see p. 420, Vol. I.)} \qquad t=\frac{180\times 84}{2\times 5\times 2240\times 0{\cdot}76}=\mathbf{0{\cdot}89''}$$

$$d=1{\cdot}05\sqrt{0{\cdot}89}=0{\cdot}99'', \text{ say, } 1'' \text{ rivets.}$$

30. *Calculate the pitch of the rivets for a double cover plate riveted joint with diamond riveting, the outer cover plate being wide enough to take three rows of rivets (in each plate) but the inner cover is only wide enough for two rows; thickness of plates,* $\frac{7}{8}$ *inch; drilled holes; material, steel; the pitch of the outer row of the inner cover plate must not exceed six times the diameter of the rivet. What is the efficiency of the joint and the bearing pressure on the rivets? Tensile strength of the plates and rivets,* 28 *tons per sq. inch.*

There are 9 rivet areas to be sheared through.

Take $d=1{\cdot}25\sqrt{0{\cdot}875}=1{\cdot}17''$ $\quad c=0{\cdot}05$

Tearing through middle row of holes.

$$(w-2d-2c)tf+(d+c)^2\tfrac{\pi}{4}\tfrac{4}{5}f=9(d+c)^2\tfrac{\pi}{4}\tfrac{4}{5}f$$

$w=\mathbf{10{\cdot}98}''$ (outer row) $\mathbf{5{\cdot}49}$ (inner row)

$$\text{Efficiency} = \frac{9(d+c)^2\pi\times 8}{10{\cdot}98\times 7\times 5} = \mathbf{88\ per\ cent.}$$

Bearing pressure when the rivets are about to shear.

$$\frac{\pi(d+c)^2 28\times 4\times 8\times 11}{4(d+c)7\times 5\times 6} = \mathbf{45\ tons\ sq.\ in.}$$

31. *Find the pitch of rivets required for a double-cover-plate riveted joint with four rows of rivets in all. The holes are punched, the clearance amounts to* 0·07 *inch, the damage done by punching extends to a depth of* 0·12 *inch all round the holes. The tensile strength of the plates is* 22 *tons per square inch and the tensile strength of the rivet material is* 24 *tons per square inch. Thickness of plates,* $\frac{5}{8}$ *inch. What is the efficiency of the joint? and what is the bearing pressure on the rivets when the joint is stressed to one-fourth of its breaking load?*

Taking $d=1{\cdot}25\sqrt{0{\cdot}625}=0{\cdot}99''$

$$(w-d-c-k)tf_t = \frac{4\times 3{\cdot}14}{4}(d+c)^2 \quad \tfrac{4}{5}f_r$$

$$w-d-c-k = \frac{3{\cdot}14\times 1{\cdot}12\times 4\times 24\times 8}{5\times 5\times 22}$$

$$w=4{\cdot}95+1{\cdot}3=\mathbf{6{\cdot}25}''$$

Bearing area $=(0{\cdot}99+0{\cdot}07)2\times\frac{5}{8}=1{\cdot}32$ sq. ins.

Load on a strip of plate $=(6{\cdot}25-0{\cdot}99-0{\cdot}07-0{\cdot}24)\frac{5}{8}\times 22$
$=68{\cdot}2$ tons

Bearing stress at ultimate load $\frac{68{\cdot}2}{1{\cdot}32}=51$ tons sq. in.

or **12·8 tons sq. in.** working stress.

$$\text{Efficiency}=\frac{6{\cdot}25-0{\cdot}99-0{\cdot}07-0{\cdot}24}{6{\cdot}25}=\textbf{79 per cent.}$$

The rivets, however, are too far apart to make a satisfactory joint. Taking the maximum pitch as $5d$ or say $5''$ we get the bearing pressure 38 tons sq. inch, and the efficiency

$$\frac{5-1{\cdot}3}{5}=\textbf{74 per cent.}$$

32. *Diamond riveted joint for boiler. Plates $\frac{3}{4}''$ thick. Holes drilled. The inside cover plate only takes two rows of rivets, the outer one three rows in each plate. Find the pitch of the outer row of rivets. Strength of material tension and shear, 28 tons sq. in.*

NOTE.—*The pitch of the outside row of rivets is so great that the plates would pucker when caulked, hence the inside cover is made narrower with the caulking edge close to a row of rivets of only one-half the pitch of the outer.*

Let $d=1{\cdot}25\sqrt{0{\cdot}75}=1{\cdot}08''$, or if desired a smaller rivet can be used (see p. 413, Vol. I.), $d+c=1{\cdot}13''$.

Strength of one rivet in single shear $=0{\cdot}785\times 1{\cdot}13^2\times\frac{4}{5}\times 28$
$=22{\cdot}5$ tons.

Load required to shear all rivets in one group, *i.e.* 4 in double and 1 in single shear $=9\times 22{\cdot}5=202{\cdot}5$ tons.

Load required to tear at outer row

$$(w-1{\cdot}13)0{\cdot}75\times 28=21w-23{\cdot}7$$

Load required to tear at second row

$$(w-2{\cdot}26)0{\cdot}75\times 28+22{\cdot}5=21w-24{\cdot}9$$

The joint will tear along the second row.

Hence $21w-24{\cdot}9=202{\cdot}5$ $\qquad w=\textbf{10·8}''$.

If the bearing stress (ultimate) were limited to 50 tons sq. inch we get

$(4\times\frac{3}{4}\times 1{\cdot}08+\frac{3}{8}\times 1{\cdot}08)50=(w-1{\cdot}13)\frac{3}{4}\times 28$ $\qquad w=\textbf{9·8}''$.

33. *A treble riveted diamond pattern butt joint for the longitudinal seam of a boiler has inside cover plates to take four rows of rivets and outside covers to take six rows.*

Find (a) *the diameter of the rivets for a* $\frac{3}{4}$*-inch plate,* (b) *the pitch of the rivets,* (c) *the efficiency of the joint, and* (*d*) *the bearing pressure on the rivets. Working stress in tension,* 5 *tons sq. inch ; shear stress on rivets,* 4·5 *tons sq. inch ; initial clearance in rivet holes* (*about*), 0·05 *inch.*

(*a*) $d=1{\cdot}2\sqrt{t}=\mathbf{1''}$.

Diam. when closed, 1·05″. Area, 0·866 sq. in.

(*b*) $(w-1{\cdot}05)\frac{3}{4}\times 5=0{\cdot}866\times 4{\cdot}5\times 9 \qquad w=\mathbf{10{\cdot}4''}$.

(*c*) $\dfrac{10{\cdot}4-1{\cdot}05}{10{\cdot}4}=\mathbf{90\ per\ cent.}$

(*d*) $(10{\cdot}4-1{\cdot}05)\frac{3}{4}\times 5=35{\cdot}1$ tons on a unit strip of plate.

$$\text{Bearing stress}=\frac{35{\cdot}1}{(4\times\frac{3}{4}\times 1{\cdot}05+\frac{3}{8}\times 1{\cdot}05)}=\mathbf{9{\cdot}9\ tons\ sq.\ in.}$$

34. *Design a lozenge riveted joint* (*see Fig.* 400, *Vol. I.*) *for a flat tie bar. Double cover plates ; load,* 80 *tons ; rivets,* $1\frac{1}{8}''$ *diam. ; clearance in rivet holes,* 0·05″. *The following stresses are not to be exceeded : tensile,* 5 *tons sq. in. ; shear,* 4 *tons sq. in. ; bearing,* 6 *tons sq. in. Width of bar about* 15 *times thickness.*

Net section $\frac{80}{5}=16$ sq. ins.

Effective diam. rivets $1{\cdot}125+0{\cdot}05=1{\cdot}175''$.

Effective area of rivets, 1·084 sq. ins. each.

Shear resistance per rivet section, $4\times 1{\cdot}084=4{\cdot}34$ tons.

Number of rivet sections required, $\dfrac{80}{4{\cdot}34}$, say 20.

Number of rivets in double shear, say 10.

Width of bar, say $15\frac{1}{2}''$.

Thickness, say $1\frac{1}{8}''$.

Net section $(15{\cdot}5-1{\cdot}175)1{\cdot}125=16{\cdot}1$ sq. ins., which will do, since 16 sq. ins. are required.

Bearing area of each rivet, $1{\cdot}125\times 1{\cdot}175=1{\cdot}32$ sq. ins.

Bearing resistance, $1{\cdot}32\times 6=7{\cdot}92$ tons.

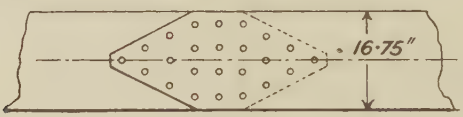
Number of rivets required for bearing $=\dfrac{80}{7{\cdot}92}$, say 10, which is the same as found for shear.

Efficiency (first row) $\left\{\frac{15{\cdot}5-1{\cdot}175}{15{\cdot}5}\right.=92{\cdot}4$ per cent.

Efficiency (second row) $\left\{\frac{(15{\cdot}5-2{\cdot}35)5\times1{\cdot}125+4{\cdot}34}{15{\cdot}5\times5\times1{\cdot}125}\right.=89{\cdot}7$ per cent.

The efficiency through the third and fourth rows is higher.

35. *Design a lap riveted joint for connecting two lengths of tie bar. Thickness,* $\frac{1}{2}''$; *load,* 40 *tons; rivets,* $\frac{3}{4}''$. *The stress must not exceed* 5 *tons sq. in. in tension or* 4 *tons sq. in. in shear. Find the bearing pressure on the rivets.*



Rivet section, 0·442 sq. in.
Load per rivet, $0{\cdot}442\times4=1{\cdot}77$ tons for shear.

Number of rivets required, $\frac{40}{1{\cdot}77}$, say 24.

Net section through first rivet $\frac{40}{5}=8$ sq. ins.
Net width of bars, $8\times2=16''$.
Gross width, $16''+0{\cdot}75''=16{\cdot}75''$.

Efficiency (first row) $\frac{16}{16{\cdot}75}=95{\cdot}5$ per cent.

Efficiency (second row) $\frac{(16{\cdot}75-1{\cdot}5)5+1{\cdot}77}{16{\cdot}75\times5}=\left\{\begin{matrix}93{\cdot}1\text{ per}\\ \text{cent.}\end{matrix}\right.$

The efficiency is higher through the other rows.
Bearing area per rivet, $\frac{1}{2}\times\frac{3}{4}=\frac{3}{8}$ sq. in.

Bearing pressure on rivets (mean) $\frac{40\times8}{3\times24}=$ **4·4** tons sq. in.

When a rivet is subjected to its maximum permissible shear the bearing pressure is

$$\frac{1{\cdot}77\times8}{3}=\mathbf{4{\cdot}7}\text{ tons sq. in.}$$

36. *A formula in use by boiler designers is*

$$S=\frac{75{,}000t^2}{p}$$

where S *is the area in sq. inches of the largest circle that can be drawn on an unstayed flat surface;* t *is the thickness of*

plates in inches; p *is the working pressure in pounds per sq. inch. What intensity of stress in the plates does this represent?*

$$f=\frac{pd^2}{4t^2} \quad \text{(see p. 407, Vol. I.)}$$

$$S=\frac{75000t^2}{p}=0{\cdot}785d^2$$

$$75{,}000t^2=0{\cdot}785pd^2=3{\cdot}14t^2f$$

$$f=\frac{75000}{3{\cdot}14}=\mathbf{23{,}900}\text{ lb. sq. in.}$$

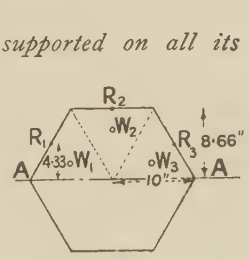
37. *A plain rectangular steam chest cover* 18″×12″ *is subjected to a steam pressure of* 150 *lb. per sq. inch. Stress,* 5,000 *lb. sq. in. Find the thickness* (a) *when edges are freely supported,* (b) *when rigidly held.*

(*a*) $f=\frac{a^2b^2}{2(a^2+b^2)}\cdot\frac{p}{t^2}$ (see p. 406, Vol. I.)

$$t^2=\frac{a^2b^2p}{2(a^2+b^2)f}=\frac{324\times144\times150}{2\times468\times5000}=1{\cdot}5 \qquad t=\mathbf{1{\cdot}22}''.$$

(*b*) $t^2=\frac{1{\cdot}5\times2}{3}=1 \qquad t=\mathbf{1}''.$

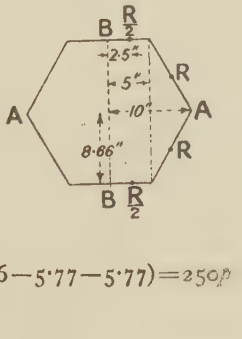
38. *A hexagonal cover is freely supported on all its edges and is subjected to fluid pressure on one side. Find the pressure required to produce a stress of* 5 *tons sq. in. in the cover. Thickness,* 1″*; length of each edge,* 10″.



Area=6×5×8·66=260 sq. ins.

Circumference, 60″.

Mean reaction of edges tons per inch=$\frac{260p}{60}=4{\cdot}33p$.



Assuming this is evenly distributed—

Bending moment about AA

$$=2\times4{\cdot}33R_1+8{\cdot}66R_2-2\times\frac{8{\cdot}66}{3}W_1-2\frac{8{\cdot}66}{3}W_2$$

But R=W=10×4·33p=43·3p.

Bending moment=43·3p(8·66+8·66−5·77−5·77)=250p

Modulus of section AA$=\frac{20}{6}$.

$$250p=\frac{5\times 20\times 2240}{6} \qquad p=150 \text{ lb. sq. in.}$$

Bending moment about BB, making the same assumption as in AA.

$$M=2\frac{R}{2}\times 2{\cdot}5+2R\times 7{\cdot}5-2\times 5\times 8{\cdot}66\times 2{\cdot}5p$$
$$-2\times(5+\tfrac{5}{3})8{\cdot}66\times 2{\cdot}5p=252p.$$

Modulus of section BB $=\frac{17{\cdot}32}{6}$

$$252p=\frac{5\times 17{\cdot}32\times 2240}{6} \qquad p=\mathbf{128} \text{ lb. sq. inch.}$$

Thus the cover is weaker across BB than across AA. The pressures are for *average* skin stresses of 5 tons sq. in. across AA and BB respectively.

39. *A bar is tested in tension and afterwards in torsion with results as below. Find* (i) *the modulus of elasticity* E ; (ii) *the modulus of rigidity* G ; (iii) *Poisson's ratio. Why is this an unsatisfactory method of finding* (iii) *? Diameter,* 0·991 *inch ; test length,* 10 *inches.*

Tension readings—

Load (tons) .	1	2	3	4	5	6	7	8
Extension . .	0·0014	0·0025	0·0035	0·0046	0·0055	0·0064	0·0075	0·0086

Torsion readings—

Twisting moment inch-tons	0·2	0·4	0·6	0·8	1·0	1·2	1·4	1·6
Twist (degrees)	0·240	0·476	0·710	0·943	1·175	1·408	1·642	1·877

(i) Area of section, 0·771 sq. in.
Mean extension for 4 tons = 0·0040″ (by differences) (see Q. 11).

$$E=\frac{4\times 10}{0{\cdot}771\times 0{\cdot}004}=\mathbf{12{,}970 \text{ tons sq. in.}}$$

(ii) $$Z_p=\frac{d^3}{5{\cdot}1}=0{\cdot}191.$$

Mean angle of twist for 0·8 inch-tons=0·933 degree (see p. 578, Vol. I.).

$$G=\frac{584Tl}{\theta D^4}=\frac{584\times0\cdot8\times10}{0\cdot933\times0\cdot964}=\mathbf{5{,}190\ tons\ sq.\ in.}$$

$$G=\tfrac{5190}{12970}E=0\cdot4E \text{ (see p. 578, Vol. I.)}$$

(iii) $n=\dfrac{2G}{E-2G}=\dfrac{2\times5190}{12970-2\times5190}=\mathbf{4\cdot02}$ (see p. 399, Vol. I.)

This is an unsatisfactory method because a small error in determining either E or G makes a large difference in *n*.

40. *The walls of a gas cylinder are 0·25″ thick, internal diameter 4″. The cylinder is fitted with an internal strut between the two ends the load on which is 25 tons. Find the relation between the circumferential strain in the walls and the simple ring strain. Internal pressure, 1,500 lb. sq. in.; Poisson's ratio, 4.*

Cross sectional area of cylinder walls=3·33 sq. ins.

$$f_y=\frac{25}{3\cdot33}=7\cdot5 \text{ tons sq. in. due to strut load.}$$

$$f'_y=\frac{1500\times0\cdot785\times16}{2240\times3\cdot33}=2\cdot5 \text{ tons sq. in. due to gas pressure.}$$

$$f_x=\frac{pd}{2t}=\frac{1500\times4}{0\cdot5\times2240}=5\cdot36 \text{ tons sq. in.}$$

Circumferential strain due to ring stress$=\dfrac{5\cdot36}{E}$.

,, ,, ,, strut$=-\dfrac{10}{nE}=-\dfrac{2\cdot5}{E}$.

,, ,, ,, both$=\dfrac{1}{E}(5\cdot36-2\cdot5)=\dfrac{2\cdot86}{E}$.

Ratio$=\dfrac{2\cdot86}{5\cdot36}=\mathbf{0\cdot53}$.

41. *Find expressions for the normal and tangential components of the stress on any plane whose normal lies in the plane of two principal stresses* p_x *and* p_y *and which makes an angle* θ *with* p_x. *If* p_x *and* p_y *are equal and both of the same sign, show that the resultant stress on any plane* θ *is invariable.*

See p. 395, Vol. I.

Resolving perpendicular to face AC

$$n \cdot AC = p_x AB \cos\theta + p_y BC \sin\theta$$

$$n = p_x \cos^2\theta + p_y \sin^2\theta.$$

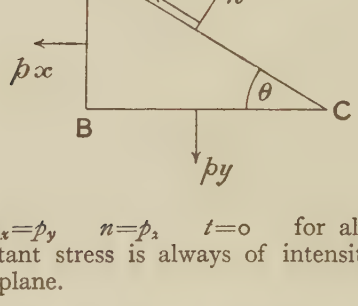
Resolving along face AC

$$t \cdot AC = p_y BC \cos\theta - p_x AB \sin\theta$$

$$t = (p_y - p_x) \sin\theta \cos\theta$$

Resultant stress $= \sqrt{p_x^2 \cos^2\theta + p_y^2 \sin^2\theta}$

inclined to p_x at an angle $\tan^{-1}\dfrac{p_x \cdot AB}{p_y \cdot BC} = \tan^{-1}\dfrac{p_x}{p_y}\tan\theta$



When $p_x = p_y$ $n = p_x$ $t = 0$ for all values of θ, and the resultant stress is always of intensity $p_x = p_y$ and normal to the plane.

42. *A hydraulic machine is suspended by its own water supply pipe. Find the maximum longitudinal and hoop strain per unit length in the material of the pipe regarded as* (a) *a thin cylinder*, (b) *thick cylinder. External diameter of pipe*, 2 *inches; internal diameter of pipe*, $1\frac{1}{8}$ *inches; weight of machine*, 10 *tons; water pressure*, 800 *lb. per sq. in.*; E, 30,000,000 *lb. per sq. in.*; *Poisson's ratio*, 4.

Section of pipe $= 2{\cdot}148$ sq. ins.

$$\text{Tensile stress} = \frac{22400 + 800}{2{\cdot}148} = 10{,}800 \text{ lb. sq. in.}$$

$$\text{Ring stress (thin cylinder)} = \frac{800 \times 1{\cdot}125 \times 16}{2 \times 7} = 1030 \text{ lb. sq. in.}$$

$$\text{(or by Barlow's theory of thick cylinders)} = \frac{800 \times 2 \times 16}{2 \times 7} = 1830 \text{ lb. sq. in.}$$

$$\text{Longitudinal strain per unit length} = \frac{10800}{30{,}000{,}000} - \frac{1030}{4\times 30{,}000{,}000} = \frac{1}{2710}$$

or by Barlow $=\frac{1}{2900}$.

$$\text{Hoop strain} = \frac{1030}{30{,}000{,}000} - \frac{10800}{4\times 30{,}000{,}000} = -\frac{1}{18000}$$

or by Barlow, Hoop strain $=-\frac{1}{34500}$.

43. *Obtain an expression for the elastic energy stored in a plate subjected to principal stresses* p *and* q. *Find the elastic energy stored in the cylindrical shell of a boiler* 90 *ins. in diameter and* 30 *ft. long, with plates* $\frac{1}{2}$ *in. thick, if the boiler pressure is* 160 *lb. per square inch. Take* E=30,000,000 *lb. per square inch and Poisson's ratio*=4.

Let P=circumferential stress$=2p$.

p=longitudinal stress.

$$x_P = \frac{P}{E} - \frac{p}{nE} \qquad x_p = \frac{p}{E} - \frac{P}{nE}$$

Energy stored per cubic inch

$$= \frac{P}{2}\left(\frac{P}{E} - \frac{p}{nE}\right) + \frac{p}{2}\left(\frac{p}{E} - \frac{P}{nE}\right) = \frac{1}{2E}\left(P^2 - \frac{2Pp}{n} + p^2\right)$$

$$= \frac{1}{2E}\left(P^2 - \frac{P^2}{4} + \frac{P^2}{4}\right) = \frac{P^2}{2E}$$

$$P = \frac{160\times 90}{2\times 0{\cdot}5} = 14{,}400 \text{ lb. per sq. inch.}$$

$$\text{Energy stored} = \frac{14400^2}{2\times 30000000} = \mathbf{3{\cdot}45} \text{ inch-pounds per cu. in.}$$

44. *A thin cylinder is subjected to internal fluid pressure, and at the same time to axial load. Compare the hoop and longitudinal strains with those due to internal pressure only. Diameter of cylinder,* 8 *ins.; thickness of walls,* 0·2 *in.; fluid pressure,* 800 *lb. sq. in.; axial load,* 10,000 *lb.; Poisson's ratio,* 4*; modulus of elasticity,* 30,000,000 *lb. sq. in.*

Let S_x=proportional strain in a longitudinal direction.

S_y= " " " circumferential "

$$\text{Axial stress} = \frac{10000}{0{\cdot}785(8^2 - 7{\cdot}96^2)} = \frac{10000}{5{\cdot}01} = 1{,}996 \text{ lb. sq. in.}$$

Circumferential stress due to fluid pressure $= \frac{800 \times 8}{2 \times 0{\cdot}2} = 16{,}000$ lb. sq. in.

$S_z = \frac{f_z}{E} - \frac{f_x}{En} - \frac{1996}{En} = \frac{13500}{E} = 0{\cdot}00045$ for axial tension

$S_z = \frac{14500}{E} = 0{\cdot}00048$ for axial compression

$S_x = \frac{16000}{E} = 0{\cdot}00053$ for pure ring tension

$S_x = \frac{f_x}{E} - \frac{f_z}{En} + \frac{1996}{E} = \frac{8000}{E} - \frac{4000}{E} + \frac{1996}{E} = \frac{5996}{E}$

$= \mathbf{0{\cdot}0002}$ for axial tension $= \mathbf{0{\cdot}00007}$ for axial compression

$= \mathbf{-0{\cdot}00013}$ for pure ring tension.

45. *A cylindrical vessel,* 6 *inches internal diameter and* 6·5 *inches external, is subjected to a fluid pressure of* 1,100 *lb. per sq. inch. The vessel is also subjected to a longitudinal pull of* 10,000 *lb. total load. Find the circumferential and longitudinal strains in the material.* E = 12,000 *tons sq. inch; Poisson's ratio,* 4.

Circumferential stress due to internal pressure $= f_z = \frac{pd}{2t} = \frac{1100 \times 6}{2 \times 0{\cdot}25} = 13{,}200$ lb. sq. in.

Longitudinal do $f_x = 6600$ lb. sq. in.

Do. due to pull $= \frac{10000}{4{\cdot}91} = 2{,}040$ lb. sq. in.

Do. due to both $f_x' = 8{,}640$ lb. sq. in.

Circumferential strain $= \frac{f_z}{E} - \frac{f_x'}{nE} = \frac{1}{12000 \times 2240}\left(13200 - \frac{8640}{4}\right) = \mathbf{\frac{1}{2440}}$

Longitudinal strain $= \frac{8640}{E} - \frac{13200}{4E} = \mathbf{\frac{1}{5030}}$.

46. *A temporary water-pipe line crosses a river and hangs in a curve which is approximately parabolic.*

Find (a) *the ring stress,* (b) *the longitudinal stress,* (c) *the circumferential strain as compared with the pure ring strain. External diameter of pipe,* 3″; *internal diameter of pipe,* 2·4″; *weight of metal in walls of pipe,* 0·28 *lb. per cu. in.; span,* 245 *ft.; dip,* 8 *ft.; water*

pressure, 1,500 *lb. sq. in.*; E, 30,000,000 *lb. sq. in.*; *Poisson's ratio*, 4.

Treat the pipe as a thin cylinder.

Section of waterway, 4·52 sq. ins.

$$\text{Ring stress} = \frac{1500 \times 3}{2 \times 0{\cdot}3} = \mathbf{7{,}500\ lb.}\ \text{per sq. in.}$$

Weight of water per foot run $= 4{\cdot}52 \times 0{\cdot}434 =$ 1·96 lb.
Weight of pipe $= (7{\cdot}07 - 4{\cdot}52) 0{\cdot}28 \times 12 =$ 8·55 lb.

Total weight per foot run . . . . . 10·51 lb.

$$\text{Tension in pipe} = \frac{10{\cdot}5 \times 245^2}{8 \times 8} = 9{,}800\ \text{lb.}$$

$$\text{Longitudinal stress in pipe} = \frac{9800}{2{\cdot}54} = \mathbf{3{,}850\ lb.}\ \text{sq. in.}$$

$$\text{Pure ring strain per unit length} = \frac{7500}{30{,}000{,}000} = \frac{1}{\mathbf{4000}}$$

$$\text{Circumferential strain} = \frac{7500}{30000000} - \frac{3850}{4 \times 30{,}000{,}000} = \frac{1}{\mathbf{4600}}.$$

47. *Find the bursting pressure of cast-iron pipes tested under water pressure when the ends are closed by the following methods:* (a) *By using two pistons in the pipe attached to the ends of a long rod, the water being supplied between the pistons.* (b) *By bolting covers to flanges on both ends of the pipe.* (c) *By using two large plates as covers with bolts outside the pipe so that when the bolts are tightened up the pipe is under longitudinal compression of* M *pounds per sq. inch. Diameter*, D=6 *inches; thickness*, t=0·5 *inch; circumferential ring stress*, f=20,000 *lb. sq. in.*; M=40,000 *lb. sq. in.*; *Poisson's ratio*, n=4.

(*a*) $p_a D = 2ft$

$$p_a = \frac{2 \times 20000 \times 0{\cdot}5}{6} = \mathbf{3{,}333\ lb.}\ \text{sq. in.}$$

$$\text{Circumferential strain} = \frac{f}{E}$$

(*b*) $$\text{Circumferential strain} = \frac{f}{E} - \frac{f}{2nE} = \frac{7}{8} \cdot \frac{f}{E}$$

$$p_b = \tfrac{8}{7} \times 3{,}333 = \mathbf{3{,}800\ lb.}\ \text{sq. in.}$$

(*c*) Circumferential strain $= \frac{f}{E} + \frac{M}{nE}$

$$p_c = \frac{\frac{f}{E}p_a}{\frac{f}{E} + \frac{M}{nE}} = \frac{fp_a}{f + \frac{M}{4}} = \frac{20000 \times 3333}{20000 + 10000}$$

p_c=**2,222 lb.** sq. inch.

48. *A compound tube is made by shrinking on to an inner cylinder an external ring, and the dimensions of the compound tube are as follows: Internal diameter,* 6 *ins.; common diameter,* 9 *ins.; external diameter,* 11 *ins. As a result of the shrinking process a radial pressure is produced at the* $4\frac{1}{2}$*-inch radius of* 4,000 *lb. per sq. in. Determine the radial and circumferential stresses at the inner and outer skins and at the common radius.* (Edinburgh University.)

When $x = 4{\cdot}5''$ $\quad p_x = 4{,}000$ lb. sq. in.

For inner tube $4{,}000 = \frac{b}{4{\cdot}5^2} - a$ (Lamé's theory)

$x = 3''$ $\quad p = 0$ $\quad 0 = \frac{b}{9} - a$ $\quad 4{,}000 = b\left(\frac{1}{20{\cdot}25} - \frac{1}{9}\right)$

$b = -64{,}830$ $\quad a = -7{,}203.$

Inner tube circumferential stresses

$x = 3''$ $\quad p_y = -\frac{64830}{9} - 7{,}203 = -14{,}407$ lb. sq. in.

$x = 4{\cdot}5''$ $\quad p_y = -\frac{64830}{20{\cdot}25} + 7{,}203 = -10{,}405$,,

Inner tube radial stresses—

When $x = 3''$ $\quad p_x = 0$. When $x = 4{\cdot}5''$ $\quad p_x = 4{,}000$ lb. sq. in.

Outer tube—

When $x = 4{\cdot}5''$ $\quad p_x = 4{,}000$ lb. sq. in. When $x = 5{\cdot}5''$ $\quad p_x = 0$.

For outer tube,

$$4{,}000 = \frac{b'}{20{\cdot}25} - a' \qquad 0 = \frac{b'}{30{\cdot}25} - a'$$

$$b' = 245{,}000 \qquad a' = 8{,}100$$

$$4{,}000 = b'\left(\frac{1}{20{\cdot}25} - \frac{1}{30{\cdot}25}\right).$$

Outer tube circumferential stresses—

When $x=4{\cdot}5''$ $\quad p_y=\frac{245000}{20{\cdot}25}+8{,}100=$ **20,200 lb.** sq. in.

$x=5{\cdot}5''$ $\quad p_y=\frac{245000}{30{\cdot}25}+8{,}100=$ **16,200 lb.** sq. in.

Outer tube radial stresses—

When $x=4{\cdot}5''$ $p_x=$ **4,000 lb.** sq. in. When $x=5{\cdot}5''$ $\mathbf{p_x=0.}$

49. *A compound tube is made by shrinking an external ring on to an inner cylinder, and it is known that this produces a radial pressure across the surfaces in contact of* 5,000 *lb. per sq. in. The dimensions of the compound tube are: external diameter,* 12 *ins.; internal diameter,* 6 *ins.; thickness of inner cylinder,* 2 *ins. If the compound tube is subjected to an internal pressure of* 10,000 *lb. per sq. in., calculate the circumferential and radial stresses in both tubes at their inner and outer skins.* (Edinburgh University.)

(See p. 424, Vol. I.)

(i) Stress due to shrinkage—

Inner tube. When $r=3''$ $\quad p=0.$

When $r=5''$ $\quad p=5{,}000$ lb. sq. in.

$$p=a+\frac{c}{r^2} \qquad 5{,}000=a+\frac{c}{25} \qquad 0=a+\frac{c}{9}$$

$$c=-70{,}300 \qquad a=7{,}810.$$

Outer tube. When $r=5''$ $\quad p=5{,}000$ lb. sq. in.

$r=6''$ $\quad p=0$

$$5{,}000=a'+\frac{c'}{25} \qquad c'=409{,}100 \qquad 0=a'+\frac{c'}{36} \qquad a'=-11{,}360.$$

Stress—Inner tube $\quad f=a-\frac{c}{r^2}$

$f_3=7{,}810+\frac{70300}{9}=15{,}510$ lb. sq. in. (compression)

$f_5=7{,}810+\frac{70300}{25}=10{,}620$,, ,,

Outer tube

$f_5=-11{,}360-\frac{409000}{25}=-27{,}720$ lb. sq. in. (tension)

$f_6=-11{,}360-\frac{409000}{36}=-22{,}720$,, ,,

(ii) Stress due to internal pressure.

When $r=3''$ $p=10{,}000$ lb. sq. in. $r=6''$ $p=0$

$$10{,}000=a+\frac{c}{9} \qquad c=120{,}000 \qquad 0=a+\frac{c}{36} \qquad a=-3{,}330$$

$f_3=-3{,}330-\frac{120000}{9}=-16{,}700$ lb. sq. in. (tension)

$f_5=-3{,}330-\frac{120000}{25}=-8{,}130$ „ „

$f_6=-3{,}330-\frac{120000}{36}=-6{,}700$ „ „

Resultant stresses—

Inner tube.

At r_3 $f=15{,}510-16{,}700=$ **−1,190 lb.** sq. in. (tension)

r_5 $f=10{,}620-8{,}130=$ **2,490** „ „ (comp.)

Outer tube.

At r_5 $f=-27{,}720-8{,}130=$ **−35,850 lb.** sq. in. (tension)

r_6 $f=-22{,}720-6{,}700=$ **−29,420** „ „

Radial stresses.

(i) Inner skin, $p=10{,}000$ lb. sq. in.

Radial stress at surface of contact due to internal pressure

$$p=a+\frac{c}{r^2}=-3{,}330+\frac{12000}{25}=\textbf{1,470 lb.}\text{ sq. in.}$$

At surface of contact—

total radial stress $=5{,}000+1{,}470=$ **6,470 lb.** sq. in.

At outer skin radial stress $=0$.

50. *A tube* 15·5″ *external diameter is shrunk on to another tube, their common diameter being* 13·5″. *The internal diameter is* 9″. *At the common surface there is a radial compression of* 4,000 *lb. sq. in. Find the circumferential stresses at the outer and inner skins when there is no fluid pressure acting.*

$$p=a+\frac{c}{r^2} \qquad f=a-\frac{c}{r^2}$$

$p=0$ at the inner and outer skins.

When $r=6{\cdot}75''$ $p=4{,}000$ lb. sq. in. $r=4{\cdot}5''$ $p=0$.

Hence for inner tube

$$4{,}000=a+\frac{c}{6{\cdot}75^2} \qquad 0=a+\frac{c}{4{\cdot}5^2} \qquad a=-\frac{c}{4{\cdot}5^2}$$

$$4{,}000=c\left(\frac{1}{45{\cdot}56}-\frac{1}{20{\cdot}25}\right) \qquad c=-145{,}000 \qquad a=7{,}160.$$

Inner tube. Circumferential stress

$r=4{\cdot}5''$ $f=-\frac{145000}{20{\cdot}25}-7{,}160=$ **−14,320 lb.** sq. in. (comp.)

$r=6{\cdot}75''$ $f=-\frac{145000}{45{\cdot}56}-7{,}160=$ **−10,340 lb.** sq. in. (comp.)

Radial stress.

When $r=4{\cdot}5''$ $p=0$. $r=6{\cdot}75''$ $p=4{,}000$ lb. sq. in.

Outer tube. Radial stresses.

When $r=4{\cdot}5''$ $p=4{,}000$ lb. sq. in. $r=7{\cdot}75''$ $p=0$

$$4{,}000=\frac{c'}{45{\cdot}56}+a' \qquad c'=755{,}000$$

$$0=\frac{c'}{60{\cdot}06}+a' \qquad a'=-12{,}580$$

$r=6{\cdot}75''$ $f=\frac{755000}{45{\cdot}56}+12{,}580=$ **29,200 lb.** sq. in. tension

$r=7{\cdot}75''$ $f=\frac{755000}{60{\cdot}06}+12{,}580=$ **25,160** ,, ,,

51. *A thick cylinder is built up of rings shrunk on one another in such a manner that the initial compressive hoop stress on the inner skin is 3 tons per sq. in. and the initial tensile hoop stress on the outer skin is 2·6 tons per sq. in. Find the resultant stress on both the outside and the inside when the cylinder is under fluid pressure of 4·5 tons per sq. inch. Internal diam.* d_1, 10 *ins.; external diam.* d_2, 20 *ins.*

Initially

When $r_1=5''$ $f_1=3$ tons sq. in. (compn.)

$r_2=10''$ $f_2=-2{\cdot}6$ tons sq. in. (tension).

Due to fluid pressure—

$r_1=5''$ $p_1=4{\cdot}5$ tons sq. in. $r_2=10''$ $p_2=0$

$4{\cdot}5=a+\frac{c}{25}$ $c=150$ $0=a+\frac{c}{100}$ $a=-1{\cdot}5$

$f_1=-1{\cdot}5-\frac{150}{25}=-7{\cdot}5$ tons sq. in (tension)

$f_2=-1{\cdot}5-\frac{150}{100}=-3{\cdot}0$,, ,,

Resultant stresses

$-7{\cdot}5+3=$ **−4·5** tons sq. in. on inside (tension).

$-2{\cdot}6-3=$ **−5·6** tons sq. in. on outside (tension)

52. *A thick cylinder is required for a water pressure of* 3 *tons per sq. inch, the internal diameter is* 8 *ins. Find the external diameter by* (a) *the ordinary ring theory,* (b) *Lamé's theory,* (c) *Barlow's theory. The maximum stress in the material must not exceed* 4 *tons per sq. in. Find the minimum circumferential stress.*

(*a*) $pd=2ft$ $\quad t=\frac{3\times 8}{2\times 4}=3''$ $\quad D_e=\mathbf{14''}$

(*b*) $\frac{f}{p}=\frac{r_e^2+r_i^2}{r_e^2-r_i^2}$ $\quad \frac{4}{3}=\frac{r_e^2+16}{r_e^2-16}$ $\quad r_e=10{\cdot}6''$ $\quad D_e=\mathbf{21{\cdot}2''}$

(*c*) $\frac{f}{p}=\frac{r_e^2+4r_e}{r_e^2-16}$ $\quad \frac{4}{3}=\frac{r_e^2+4r_e}{r_e^2-16}$ $\quad r_e=16''$ $\quad D_e=\mathbf{32''}.$

In this case

$pD_e=2ft$ $\quad 3\times 32=2\times 4\times t$ $\quad t=12''$ $\quad D_e=32''$

The minimum circumferential stress by Lamé's theory—

$3=\frac{c}{16}+a$ $\quad c=56$ $\quad -4=-\frac{c}{16}+a$ $\quad a=-0{\cdot}5$

$f_e=-0{\cdot}5-\frac{56}{112}=-1$ ton sq. in. (tension) $\quad p_e=\frac{56}{112}-0{\cdot}5=\mathbf{0}.$

53. *A cylinder is required for an internal pressure of* 1·5 *tons per sq. in. Find the required thickness by Lamé's theory. Internal diam.,* 10 *ins.; circumferential stress,* f=7,000 *lb. sq. in.*

When $r=5''$ $\quad p=1{\cdot}5\times 2{,}240+15=3{,}375$ lb. sq. in.

$3{,}375=\frac{c}{5^2}+a$ $\quad 7{,}000=\frac{c}{5^2}-a$ $\quad a=-1{,}812$ $\quad c=-129{,}670$

At the outside the pressure is 15 lb. sq. in.

$15=-\frac{129670}{R^2}-1{,}812$ $\quad R=8{\cdot}42''$ $\quad t=\mathbf{3{\cdot}42''}.$

Rough check by Barlow's theory—

$PD_e=2ft$ $\quad 3{,}375(10+2t)=14{,}000t$ $\quad t=4{\cdot}66''.$

From Fig. 406, p. 425, Vol. I., it will be seen that Barlow's theory exaggerates the stress on the inside skin of a thick cylinder.

54. *Find the thickness of metal for a hydraulic main to fulfil the following conditions: Internal diam.* 6 *ins.;*

internal pressure (p), 0·75 *ton sq. in.; maximum tensile stress in metal* (f), 1·5 *tons per sq. in. Determine the circumferential stress in the outer skin.* (See p. 424, Vol. I.)

$$f=\frac{p(R_e^2+R_i^2)}{R_e^2-R_i^2} \qquad 1{\cdot}5=\frac{0{\cdot}75\{(3+t)^2+3^2\}}{(3+t)^2-3^2}$$

$$2=\frac{(3+t)^2+3^2}{(3+t)^2-3^2} \qquad \therefore\ (3+t)^2=3\times 3^2=27$$

$$t+3=\sqrt{27}=\pm 5{\cdot}2 \qquad t=\mathbf{2{\cdot}2''}$$

$$p=X+\frac{Y}{R^2} \qquad f=-X+\frac{Y}{R^2}$$

$$0{\cdot}75=X+\frac{Y}{9} \qquad 0{\cdot}75=X+\frac{10{\cdot}125}{9}$$

$$1{\cdot}50=-X+\frac{Y}{9} \qquad X=-0{\cdot}375$$

$$2{\cdot}25=\frac{2Y}{9}$$

$$Y=10{\cdot}125 \qquad f_e=0{\cdot}375+\frac{10{\cdot}125}{27}$$

$$=0{\cdot}375+0{\cdot}375=\mathbf{0{\cdot}75}\text{ ton sq. in.}$$

55. *A hollow sphere is subjected to internal pressure and to a longitudinal force at the same time. Find the longitudinal and circumferential strains in the material. Internal diam.* (d), 6″; *external diam.*, 6·5″; *internal fluid pressure* (p), 1,100 *lb. sq. in.; longitudinal pull,* 10,000 *lb.; modulus of elasticity* (E), 12,000 *tons sq. in.; Poisson's ratio* (n), 4.

For a thin sphere such as this we have

$$pd=4ft \qquad f=\frac{1100\times 6}{4\times 0{\cdot}25}=6{,}600\text{ lb. per sq. inch}$$

Section of sphere $=33{\cdot}18-28{\cdot}27=4{\cdot}91$ sq. ins.

Longitudinal stress due to pull $=\frac{10000}{4{\cdot}91}=2{,}040$ lb. sq. in.

Total longitudinal stress $=6{,}600+2{,}040=8{,}640$ lb. sq. in.

Longitudinal strain $=\frac{1}{12000\times 2240}\left(8{,}640-\frac{6600}{4}\right)=\mathbf{0{\cdot}00026}$

Circumferential strain $=\frac{1}{12000\times 2240}\left(6{,}600-\frac{8640}{4}\right)=\mathbf{0{\cdot}000165.}$

56. *A sphere 4 ft. internal diam. and $\frac{1}{2}$" thick is subjected to an internal fluid pressure. The elastic limit of the material is 30,000 lb. per sq. in. Poisson's ratio*=0·33.

Find the intensity of fluid pressure required to produce permanent set in the walls of the sphere; also the ratio of the internal volume when under pressure and when open to the atmosphere. Young's modulus of elasticity, 13,000 tons per sq. in.

In a sphere $f_x = f_z$

Proportional strain $= \frac{f_x}{E} - \frac{f_z}{En} = \frac{f_x}{E}\left(1 - \frac{1}{n}\right) = \frac{2}{3} \cdot \frac{f_x}{E}$

$\frac{p\pi d^2}{4} = f_x \pi d t$, when lateral contraction is neglected.

Allowing for lateral contraction

$$p = \frac{4 f_x t}{d} \times \frac{3}{2} = \frac{4 \times 30000 \times 0{\cdot}5 \times 3}{48 \times 2} = 1{,}875 \text{ lb. sq. in.}$$

$$\text{Proportional strain} = \frac{2 \times 30000}{3 \times 13000 \times 2240} = \frac{1}{1456}$$

$$\frac{\text{Vol. under pressure}}{\text{Vol. when not under pressure}} = \frac{\left(1 + \frac{1}{1456}\right)^3}{1} = 1{\cdot}002.$$

CHAPTER XI

BEAMS

1. *Find the load and the forces acting on the blocks in the case of a three-block beam model* (*Fig.* 412, *Vol. I.*).

A_a=4 *sq. ins.* A_b=5 A_c=3
y_a=10″ y_b=8″ y_c=5″
E_a=12,000 *tons sq. in.* E_b=10,000 E_c=13,000
f_a=3 *tons sq. in.* L=16 *ft.*

The stress on the block b is :

$$f_b=\frac{E_b y_b}{E_a y_a} f_a=2 \text{ tons sq. in.}$$

and on the block c $\quad f_c=\dfrac{E_c y_c}{E_a y_a} f_a=1{\cdot}625$ tons sq. in.

Force acting on a $=4\times3=$**12 tons,** $\quad b=2\times5=$**10 tons,**
c $=3\times1{\cdot}625=$**4·87 tons.**

$WL=2(12\times10+10\times8+4{\cdot}87\times5)$ $\quad W=$**2·34 tons.**

2. *Compare the weights of beams of equal strength* (i) *solid circular ;* (ii) *hollow circular* (D=2*d*) ; (iii) *R.S.J. in which* H=1·6 B=12*t*.

(See p. 452, Vol. I.)

(i) $z=\dfrac{D^3}{10{\cdot}2}$ $\quad$ Area=0·785D²

(ii) $z=\dfrac{D_1{}^4-d^4}{10{\cdot}2D_1}=\dfrac{15}{16}\cdot\dfrac{D_1{}^3}{10{\cdot}2}$ } Where D_1 is the diam. of a beam of equal strength

D_1=1·022 D $\quad$ Area=0·615 D²

Area and weight ratio=$\dfrac{0{\cdot}615}{0{\cdot}785}=$**0·784**

(iii) $z=BtH$ approx.=$\dfrac{H^3}{19{\cdot}2}$ $\quad H=1{\cdot}235D$

B=0·773D $\quad t=0{\cdot}103D$ $\quad$ Area=0·265D²,

Area and weight ratio=$\dfrac{0{\cdot}265}{0{\cdot}785}=$**0·338**

Weight ratio : (i) 1 ; (ii) 0·784 ; (iii) 0·338.
or (i) 2·96 ; (ii) 2·32 ; (iii) 1.

3. *Compare the strength of a beam of circular section with that of the strongest rectangular beam which can be cut out of the same section.*

(See p. 442, Vol. I.)

Let the breadth of the rectangular beam be B and the depth H. The diameter of the circular section D.

$$\text{Rectangular } Z = \frac{BH^2}{6} = \frac{B}{6}(D^2 - B^2) = \frac{D^2B - B^3}{6}$$

$$\frac{dZ}{dB} = \frac{D^2 - 3B^2}{6}$$

which gives a maximum for Z when $D^2 = 3B^2$, or when $D = \sqrt{3}B$, or $B^2 = \frac{D^2}{3}$, we also have $H^2 = D^2 - \frac{D^2}{3} = \frac{2}{3}D^2$

$$\frac{\text{Z circle}}{\text{Z rect.}} = \frac{\frac{D^3}{10 \cdot 2}}{\left(\frac{D \times 2D^2}{\sqrt{3} \times 3}\right)^{\frac{1}{6}}} = 1 \cdot 53.$$

4. *Find the moment of inertia and modulus of the section of a beam whose section is an equilateral triangle of* 12 *in. sides. The neutral axis is parallel to the base.*

Consider a narrow strip of the section parallel to the base of breadth b, thickness dh distant h from the apex. The height of the triangle is H, breadth of base B. Then

$$\frac{b}{B} = \frac{h}{H} \quad \text{and} \quad b = \frac{B}{H}h$$

$$\text{I for strip} = b \,.\, dh \,.\, h^2 = \frac{B}{H}h^3 dh$$

I for triangle about a line parallel to the base and passing through the apex $= \frac{B}{H}\int h^3 dh = \frac{BH^4}{4H} = \frac{BH^3}{4}$

I about N.A. $= I - Ax^2$ (See p. 76, Vol. I.)

$$= \frac{BH^3}{4} - \frac{BH}{2}\left(\tfrac{2}{3}H\right)^2 = \frac{B^4}{55 \cdot 5}$$

When B=12″ I=374 H=10·4″

Z for unit stress at apex $=\frac{374\times3}{2\times10\cdot4}=$**54 inch units.**

Z „ „ base=**108** „

5. *Find the tension and compression moduli of the section for a Tee-shaped beam section. Breadth of flange,* 5 *ins.; total depth of section,* 10·5 *ins.; thickness throughout,* 1 *in.*

Taking moments to find the position of the N.A.

$$y=\frac{5\times10+9\cdot5\times4\cdot75}{5+9\cdot5}=6\cdot56 \text{ ins. from top of web}$$

or 10·5−6·56=3·94 ins. from outside of flange.

(See p. 446, Vol. I.)

$$Z_t=\frac{1\times6\cdot56^3+5\times3\cdot94^3-4\times2\cdot94^3}{3\times6\cdot56}=\mathbf{24\cdot7}$$

$$Z_c=\frac{486\cdot5}{3\times3\cdot94}=\mathbf{41\cdot2.}$$

6. *Find the two moduli of a Tee-shaped beam section having a very shallow web. Breadth of flange,* 5″; *total depth of section,* 2·5″; *thickness throughout,* 1″.

Centre of gravity of section from top of web

$$\frac{5\times1\times2+1\cdot5\times1\times0\cdot75}{5+1\cdot5}=1\cdot71''.$$

Thus the neutral axis passes through the flange, since the web is only 1·5″ deep.

The expression for Z (see p. 446, Vol. I.) then becomes:

$$\frac{B_1H_1^3+B_2H_2^3+bh^3}{3H_1 \text{ or } 3H_2}$$

$$\frac{1\times1\cdot71^3+5\times0\cdot79^3+4\times0\cdot21^3}{3\times1\cdot71 \text{ or } 3\times0\cdot79}=\mathbf{1\cdot46 \text{ or } 3\cdot16}$$

according to the edge at which the stress is taken.

7. *Find the ratio of the tensile and compressive stresses, also the* Z_t *for an* I *section beam having unequal flanges. Depth,* 20 *inches; breadth of compression flange,* 4 *inches; breadth of tension flange,* 16 *inches; thickness of compression flange,* 1·5 *inches; thickness of tension flange,* 2 *inches; thickness of web,* 1·25 *inches.*

Taking moments about the outer edge of the tension flange to find the c. of g. of section—

$$y=\frac{2\times16\times1+16\cdot5\times1\cdot25\times10\cdot25+6\times19\cdot25}{6+20\cdot62+32}=6\cdot12''$$

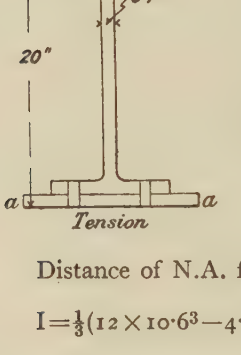
$\therefore\ h_t=6\cdot12''$ and $h_c=13\cdot88''$

Ratio of stresses $\frac{f_c}{f_t}=\frac{h_c}{h_t}=\frac{13\cdot88}{6\cdot12}=2\cdot27$

$$Z_t=\frac{4\times13\cdot88^3+16\times6\cdot12^3-2\cdot75\times12\cdot38^3-14\cdot75\times4\cdot12^3}{6\cdot12\times3}$$

$$=\mathbf{442\ inch\ units.}$$

8. *Find Z_c and Z_t for the girder section shown. Two $\frac{3}{4}''$ rivets in each flange. The area of rivet holes need not be deducted in the compression flange.*



Take moments about *a a* for finding position of neutral axis:

Section.	Area.	y	Moment.
Top plate .	$12''\times1\cdot25''=15$	21·87	328·5
Top flange	$7\cdot8''\times0\cdot95''=7\cdot41$	20·78	154·0
Web . .	$18\cdot1''\times0\cdot70''=12\cdot7$	11·25	142·4
Bot. flange	$6\cdot3''\times0\cdot95''=5\cdot97$	1·72	10·3
Bot. plate.	$10\cdot5''\times1\cdot25''=13\cdot13$	0·62	8·2
	54·2		643·4

Distance of N.A. from $aa=\frac{643\cdot4}{54\cdot2}=11\cdot9$

$$I=\tfrac{1}{3}(12\times10\cdot6^3-4\cdot2\times9\cdot35^3-7\cdot1\times8\cdot4^3+10\cdot5\times11\cdot9^3$$
$$-4\cdot2\times10\cdot65^3-5\cdot6\times9\cdot70^3)$$

$=4710$ (See p. 445, Vol. I.)

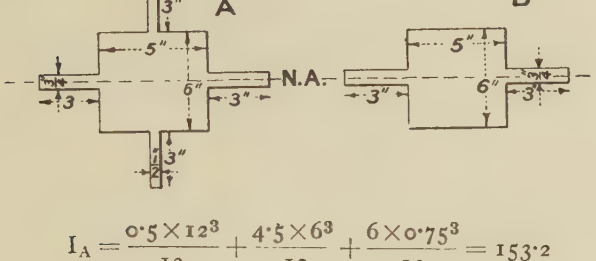
$Z_t=\frac{4710}{11\cdot9}=\mathbf{396}$ $\qquad$ $Z_c=\frac{4710}{10\cdot6}=\mathbf{444.}$

Approximation:

$$(12\times1\cdot25+7\cdot8\times0\cdot95)20=448=Z_c$$
$$(10\cdot5\times1\cdot25+6\cdot3\times0\cdot95)20=382=Z_t$$

which is sufficiently near for most practical purposes.

9. *Compare the strength and stiffness of the beam sections* A *and* B.



$$I_A = \frac{0{\cdot}5\times 12^3}{12} + \frac{4{\cdot}5\times 6^3}{12} + \frac{6\times 0{\cdot}75^3}{12} = 153{\cdot}2$$

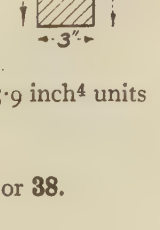
$$Z_A = \frac{153{\cdot}2}{6} = 25{\cdot}5 \qquad I_B = \frac{5\times 6^3}{12} + \frac{6\times 0{\cdot}75^3}{12} = 90{\cdot}2$$

$$Z_B = \frac{90{\cdot}2}{3} = 30{\cdot}1.$$

The stiffness is proportional to the moment of inertia I of the section (see p. 510, Vol. I.), but the strength is proportional to the modulus of the section Z.

Thus section A is stiffer but weaker than B. Outside webs always weaken flat covers, if webs are used they should be put on the pressure side of cast-iron covers.

10. *A rectangular beam* 3″ *wide*, 9″ *deep, has a circular hole* 2″ *diam. running lengthwise through it, and whose centre is* 2″ *below the top surface of beam. Find* I *and* Z.



Taking moments about the base to find the position of the N.A.

$y(27-3{\cdot}14) = 27\times 4{\cdot}5 - 3{\cdot}14\times 7 \quad y = 4{\cdot}17''$

$$\text{I of rectangle} = \frac{3(4{\cdot}83^3 + 4{\cdot}17^3)}{3}$$

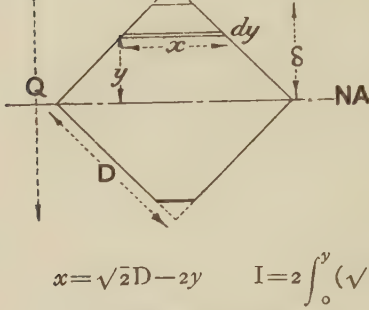
$$= 185{\cdot}2 \text{ inch}^4$$

$$\text{I of circle} = \frac{\pi D^4}{64} + \frac{\pi D^2}{4}\times 2{\cdot}83^2 = 25{\cdot}9 \text{ inch}^4 \text{ units}$$

$$\text{Nett I} = 185{\cdot}2 - 25{\cdot}9 = \mathbf{159{\cdot}3}$$

$$Z = \frac{159{\cdot}3}{4{\cdot}83} \text{ or } \frac{159{\cdot}3}{4{\cdot}17} = \mathbf{33} \text{ or } \mathbf{38}.$$

11. *A beam of square section is loaded with one diagonal horizontal and coinciding with the neutral axis. Show that if the top and bottom apexes of the beam section are removed the beam is stronger than before, the maximum strength being obtained when the vertical depth is $\frac{8}{9}$ of the original diagonal of the square.*



$$x=\sqrt{2}D-2y \qquad I=2\int_0^y(\sqrt{2}Dy^2-2y^3)dy \qquad Z=\frac{I}{\delta}$$

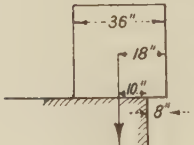
$$I=\left[\frac{2\sqrt{2}Dy^3}{3}+\frac{4y^4}{4}\right]_{y=0}^{y=\delta} \qquad I=\frac{2\sqrt{2}D\delta^3}{3}-\delta^4$$

$$Z=\frac{2\sqrt{2}D\delta^2}{3}-\delta^3 \qquad \frac{dz}{d\delta}=\frac{4\sqrt{2}D\delta}{3}-3\delta^2$$

This gives a maximum for Z when $\frac{4\sqrt{2}D}{3}=3\delta$

$$\delta=\frac{4\sqrt{2}}{9}D \qquad D=\tfrac{4}{3}Q \qquad 2\delta=\tfrac{8}{9}Q.$$

12. *A 3-feet cube of stone rests on a flat slab and one edge of the cube overhangs the edge of the slab by 8 inches. Find the maximum intensity of pressure between the cube and slab. Weight of stone 150 lb. per cubic foot.*



Weight of cube $150\times27=4{,}050$ lb.

The distribution of stress is shown by the shaded trapezium (see p. 60, Vol. I.).

c. of g. of shaded trapezium from edge

$$=\frac{H}{3}\left(\frac{2S_1+S_2}{S_1+S_2}\right)$$

The c. of g. of the cube is 10" from the edge of the slab.

Hence $$10=\tfrac{28}{3}\left(\frac{2S_1+36}{S_1+36}\right) \qquad S_1=2{\cdot}77''.$$

$$\text{Area of shaded figure} = \frac{2{\cdot}77+36}{2} \times 28 = 543 \text{ sq. ins.}$$

$$\text{Mean compressive stress} = \frac{4050}{36 \times 28} = 4{\cdot}02 \text{ lb. sq. in.}$$

$$\text{Maximum compressive stress} = \frac{4{\cdot}02 \times 36}{(36+{}^{1}2{\cdot}77)\frac{1}{2}} = \mathbf{7{\cdot}46 \text{ lb. sq. in.}}$$

$$\text{Minimum compressive stress} = \frac{4{\cdot}02 \times 2{\cdot}77}{19{\cdot}34} = 0{\cdot}58 \text{ lb. sq. in.}$$

13. *A temporary beam centrally loaded is constructed of two lengths of timber of* 1 *ft. square section having flat butt ends which are joined together in the middle by a single flat fish plate on the tension side rigidly bolted to the timber. Find the position of the neutral axis and the moment of resistance of the beam. Net section of plate,* $10'' \times \frac{1}{4}''$; *modulus of elasticity of timber,* 600 *tons per sq. in.; ditto in plate,* 12,000 *tons per sq. in.; stress in plate not to exceed* 5 *tons per sq. in.; stress in timber not to exceed* 0·5 *ton per sq. in.*

Let p = area of plate section = 2·5 sq. ins.

b = breadth of timber section = 12 ins.

h = { height of section measured from middle of plate } = 12·125 ins.

x = distance of N.A. measured from middle of plate.

Then equating tension and compression at joint:

$$\frac{b(h-x)^2}{2} = p \times \tfrac{12000}{600} x = 2{\cdot}5 \times 20x$$

$$(h-x)^2 = 8{\cdot}33x \qquad \mathbf{x = 5{\cdot}41''} \quad \text{and} \quad h-x = 6{\cdot}71''$$

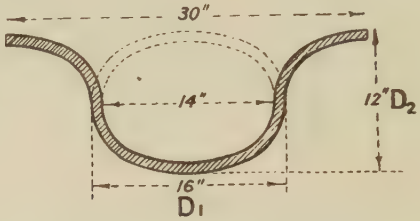
Moment of resistance of beam when stress in timber is 0·5 ton sq. in. (see p. 471, Vol. I.)

$$= b\left(\frac{h-x}{2}\right)\left\{\frac{(h-x)2}{3} + x\right\}0{\cdot}5 = 198{\cdot}9 \text{ tons-inches}$$

Moment of resistance of beam when stress in plate is 5 tons sq. in. $= 2{\cdot}5 \times 5(5{\cdot}41 + \frac{2}{3} \times 6{\cdot}71) = \mathbf{123{\cdot}5 \text{ tons-inches.}}$

Thus when the stress in the plate is 5 tons per sq. inch that in the timber is $\frac{0{\cdot}5 \times 123{\cdot}5}{198{\cdot}9} = 0{\cdot}31$ ton sq. in.

14. *Find the modulus of the section for a strip of elliptical corrugated flooring as shown in the figure. Thickness* 1 *in. Total width* 8 *ft.*



I for ellipse

$$= \frac{\pi D_1 D_2^3}{64} \qquad \text{(See p. 90, Vol. I.)}$$

$$Z = \frac{\pi}{32 D_2}\left(D_2^3 D_1 - d_2^3 d_1\right) \text{ for hollow ellipse}$$

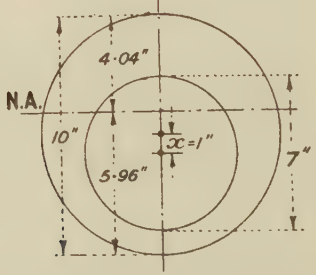
$$= \frac{3{\cdot}14}{32 \times 12}(12^3 \times 16 - 10^3 \times 14) = \frac{3{\cdot}14}{384}(27{,}640 - 14{,}000)$$

$$= 111{\cdot}6 \text{ for one complete corrugation.}$$

8 feet contains 3·2 complete corrugations.

Z for 8 ft. $= 111{\cdot}6 \times 3{\cdot}2 =$ **357 inch units.**

15. *Find the modulus of the section for a hollow circular beam with an eccentric core, the neutral axis being at right*



angles to the line passing through the centres. External diam. 10". *Internal diam.* 7". *Distance between centres* 1".

(See p. 90, Vol. I.)

$$I = \frac{3{\cdot}14}{64}\left(10^4 - 7^4 - \frac{16 \times 10^2 \times 7^2 \times 1}{10^2 - 7^2}\right) = 297$$

Distance of centre of gravity from a $\left\{ \frac{78{\cdot}54 \times 5 - 38{\cdot}48 \times 6}{78{\cdot}54 - 38{\cdot}48} = 4{\cdot}04'' \right.$

$$Z_1 = \frac{297}{4{\cdot}04} = \mathbf{73} \qquad Z_2 = \frac{297{\cdot}1}{5{\cdot}96} = \mathbf{50.}$$

16. *A* U *expansion bend on a 6-inch steam pipe is* $\frac{1}{2}''$ *thick and has a throw of 2′ 6″. The bend closes 0·25″ under a force of 100 lb. Find the maximum stress in the bend when under steam at 180 lb. per sq. in. gauge. Length of pipe line 40 feet, ends rigidly held. Temperature of pipe, 330° F. It is free from stress at 60° F. Coefficient of expansion (Fah.) 0·0000065.*

Expansion of pipe between 60° and 330° F.

$= 0{\cdot}0000065(330 - 60) \times 40 \times 12$

$= 0{\cdot}84$ in. which is taken up by the expansion bend.

$$\text{Effective thrust on bend} = \frac{0{\cdot}84}{0{\cdot}25} \times 100 = 336 \text{ lb.}$$

$$\text{Maximum bending moment} = 336 \times 30 = 10{,}080 \text{ in.-lb.}$$

$$\text{Maximum bending stress} = \frac{10080 \times 32 \times 7}{\pi(7^4 - 6^4)} = 650 \text{ lb. per sq. in.}$$

$$\text{Longitudinal stress } f = \frac{pd}{4t} = \frac{180 \times 6}{4 \times 0{\cdot}5} = 540 \text{ lb. sq. in.}$$

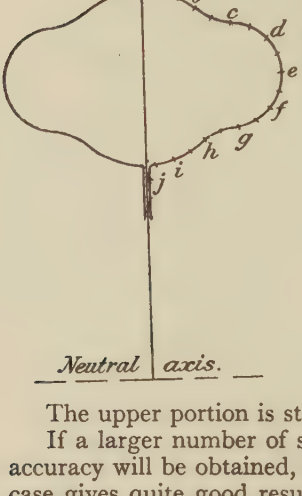
Maximum stress = **1190 lb. sq. in.**

17. *The diagram shows one-half of a section of a symmetrically shaped aeroplane spar built up of thin sheet steel. Find the moment of inertia and the modulus of the section. Thickness of sheet* = 0·012″.

It is obvious that none of the graphic or analytic methods in common use are applicable in this case. Starting from the top of the section step off short lengths as shown (in this case they were 0·4″ long before the diagram was reduced for printing). Letter alternate marks *a*, *b*, *c*, etc. Let the distance of *a* from the N.A. be y_a, and of *b*, y_b, etc.

Then the moment of inertia of the hollow flange (the web will be dealt with later) is :

$$I = 4 \times 0{\cdot}4 \times 0{\cdot}012(y_a^2 + y_b^2 + y_c^2 + \text{etc.})$$



The reason for the 4 is because we are only dealing with one-quarter of the symmetrical section :

$$I = 0{\cdot}0192(3{\cdot}99^2 + 3{\cdot}85^2 + 3{\cdot}68^2 + 3{\cdot}50^2 + 3{\cdot}12^2 + 2{\cdot}78^2 + 2{\cdot}60^2 + 2{\cdot}48^2 + 2{\cdot}28^2 + 2{\cdot}08^2) + 4 \times 0{\cdot}2 \times 0{\cdot}012 \times 1{\cdot}78^2 = 1{\cdot}882$$

Total depth of web 8″

$$I_{web} = \frac{0{\cdot}012 \times 8^3}{12} = 0{\cdot}512$$

I for whole section = **2·394**

$$Z = \frac{2{\cdot}394}{4} = \mathbf{0{\cdot}598.}$$

The upper portion is strip welded to the web.

If a larger number of steps be taken a greater degree of accuracy will be obtained, the proportion taken in the above case gives quite good results. In order to show the degree of accuracy obtainable with this method it is worth while trying it on a thin-walled hollow circle which can be checked by direct calculation.

Sections built up of very thin sheet metal never give the anticipated strength in the form of a beam because the compression flange always buckles locally before the tension flange reaches the tensile strength of the metal.

18. *A wooden beam* 10″ *deep* 6″ *wide is stiffened by firmly bolting a steel plate* ½″ *thick and* 10″ *deep to each side. Find the bending moment to satisfy the conditions:* E, *wood,* 600 *tons per sq. in.; steel,* 13,000 *tons per sq. in.; safe stress in wood,* 0·5 *ton per sq. in.; safe stress in steel,* 7·0 *tons per sq. in.*

I_w for wooden beam $= \frac{6 \times 10^3}{12} = 500$ $\qquad Z_w = 100$

I_s for steel beam $= \frac{1 \times 10^3}{12} = 83{\cdot}3$ $\qquad Z_s = 16{\cdot}6$

Bending moment $M = \frac{EI}{\rho}$, ρ is the same for both the wood and the steel (see p. 506, Vol. I.).

$$\frac{M_s}{M_w} = \frac{E_s I_s}{E_w I_w} = \frac{13000 \times 83{\cdot}3}{600 \times 500} = 3{\cdot}61$$

Hence the bending moment on the steel plates is $\frac{3{\cdot}61}{4{\cdot}61}$ of the whole, and on the wooden beam $\frac{1}{4{\cdot}61}$.

We also have $M = \frac{fI}{y}$, hence since y is the same for both

$$\frac{f_s I_s}{f_w I_w} = \frac{E_s I_s}{E_w I_w} \qquad \frac{f_s}{f_w} = \frac{E_s}{E_w} = \frac{13000}{600} = 21{\cdot}6$$

But the given ratio of permissible stress is 14. Hence the steel will reach its safe limit before the wood.

Safe M_w for the original beam $= \frac{0{\cdot}5 \times 6 \times 10^2}{6} = 50$ tons-inches.

M for composite beam $= \frac{f_s b h^2}{6} + \frac{f_s b_1 h^2}{21{\cdot}6 \times 6} =$ **149 tons-inches.**

Since the wooden beam only takes $\frac{1}{4{\cdot}61}$ of the total load, the deflection of the composite beam will be $\frac{1}{4{\cdot}61}$ of the deflection of the wooden beam without plates.

19. *Find the position of the neutral axis, also the moments of resistance of a reinforced concrete beam of rectangular section. Breadth of section* (B), 7″; *depth of section* (H), 16″; *depth to centre of rods* (d), 15″; *area of reinforcements* (a), 2 *per cent. of gross area of beam section;* $r = E/E_c = 11$.

Stress in concrete (f_c) *not to exceed* 500 *lb. per sq. in.*

Stress in rods (f) *not to exceed* 10,000 *lb. per sq. in.*

Area of beam section $= 7 \times 16 = 112$ sq. ins.

Area of reinforcement $(a) = 0{\cdot}02 \times 112 = 2{\cdot}24$ sq. ins.

Dist. of N.A. from compression skin (C) (see p. 470, Vol. I.)

$$= \sqrt{\frac{ar}{B}\left(\frac{ar}{B} + 2d\right)} - \frac{ar}{B}$$

$$C = \sqrt{\frac{2{\cdot}24 \times 11}{7}\left(\frac{2{\cdot}24 \times 11}{7} + 30\right)} - \frac{2{\cdot}24 \times 11}{7} = \mathbf{7{\cdot}3''}$$

$$Z_c f_c = \frac{BC}{2}(\tfrac{2}{3}c + d - g)500 = \mathbf{161{,}000\ in.\text{-}lb.}$$

$$Z_t f = 2{\cdot}24 \times 12{\cdot}57 \times 10{,}000 = \mathbf{281{,}000\ in.\text{-}lb.}$$

20. *Design a reinforced concrete beam of rectangular section to comply with the following: Effective depth* (d), 18″; *moment of resistance*, 350,000 *inch-lb.*; E/E_c, 15. *Compressive stress in concrete not to exceed* (f_c), 660 *lb. per sq. in.*; *tensile stress in steel not to exceed* (f), 13,600 *lb. sq. in.*

(See p. 470, Vol. I.) Area of rods $= a$.

$$\text{Moment of resistance} = fa\left(d - \frac{c}{3}\right) = 350{,}000$$

$$\frac{f}{f_c} = 15\left(\frac{1-n}{n}\right) \quad \text{where } n = \frac{c}{d} \qquad \frac{13600n}{660} = 15 - 15n \qquad n = 0{\cdot}421$$

Distance of N.A. from compression skin:

$$c = nd = 0{\cdot}421 \times 18 = 7{\cdot}58'' \qquad n = \sqrt{0{\cdot}0225p^2 + 0{\cdot}3p} - 0{\cdot}15p$$

$p = 1{\cdot}02$ per cent. of steel in section $\qquad a = \frac{pBd}{100} = 0{\cdot}0102Bd$

$$350{,}000 = 13{,}600 \times 0{\cdot}0102Bd \times 0{\cdot}859d \qquad B = 9'' \text{ approx.}$$

Area of rods (a) $= 0{\cdot}0102 \times 9 \times 18 = 1{\cdot}743$ sq. ins.

Say seven—$\frac{9}{16}$″ rods, which gives an area of 1·74 sq. inches.

21. *A concrete cantilever of rectangular section is reinforced with two* $\frac{3}{4}$″ *rods to take the tension. Find the bending moment required to produce a compressive stress of* 0·25 *ton per sq. in. in the concrete. Width of section* (B), 5″; *depth of section from compression skin to middle of rods* (d), 10″; E (*rods*), 12,000 *tons sq. in.*; E_c (*concrete*), 800 *tons sq. in.*

Sectional area of rods $= 0{\cdot}884$ sq. in. Tension in rods $= 0{\cdot}884f$

$$\text{Compression in concrete (C)} = \frac{5c}{2} \times 0{\cdot}25 = 0{\cdot}625c$$

(See p. 470, Vol. I.)

$$f = \frac{0{\cdot}625c}{0{\cdot}884} = \frac{15(10-c)}{4c} \qquad c = 5{\cdot}1$$

$$C = 0{\cdot}625 \times 5{\cdot}1 = 3{\cdot}2 \text{ tons}$$

$$\text{Arm of couple} = 10 - 5{\cdot}1 + \tfrac{2}{3} \times 5{\cdot}1 = 8{\cdot}3''$$

$$M = 3{\cdot}2 \times 8{\cdot}3 = \mathbf{26{\cdot}6\ inch\text{-}tons.}$$

This may be checked by the method given on p. 470, Vol. I., which gives precisely the same result.

22. *A hollow rectangular reinforced concrete conduit is used for conveying water over a roadway; it has a clear span of* 30 *feet. Find the stress in the concrete and in the reinforcements when the depth of water in the conduit is* 4 *feet* 6 *inches. External depth,* 8′; *internal depth,* 5′; *external width,* 6′; *internal width,* 4′. *The reinforcements consist of* 20 *rods* $\frac{3}{4}$ *inch diameter, the centres of which are* 2 *inches from the bottom of the section. The rods take the whole of the tension. Weight of concrete,* 140 *lb. per cubic ft.; ratio of moduli of elasticity,* 12.

Section of concrete $6\times8-5\times4=28$ sq. ft.

Weight per foot run $\frac{28\times140}{2240}=1{\cdot}75$ tons ft. run

Weight of water per foot run $\left.\frac{4\times4{\cdot}5\times62{\cdot}4}{2240}\right\}=0{\cdot}5$,, ,,

Total . . . $2{\cdot}25$,, ,,

Area of rods, 8·84 sq. ins.

Bending moment in middle of span $=\frac{2{\cdot}25\times30^2}{8}=253$ tons ft.

(See p. 470, Vol. I.)

The neutral axis is 5·93 feet above the bottom of the section. For method of calculation see other examples of Tee beams. The rods are $2''=0{\cdot}17'$ from the bottom. The neutral axis is 2·07 feet from the compression skin.

The arm of the couple $=(5{\cdot}93-0{\cdot}17)+\frac{2}{3}\times2{\cdot}07=7{\cdot}14$ feet.

Moment of resistance of concrete for 1 ton per sq. foot $\left.\frac{7{\cdot}14\times2{\cdot}07\times2}{2}\right\}=14{\cdot}8$ tons-ft.

Compression stress in concrete

$=\frac{253}{14{\cdot}8}=17{\cdot}1$ tons sq. ft. $=0{\cdot}12$ tons sq. in. $=$ **266 lb.** per sq. in.

Moment of resistance of rods for 1 ton per sq. inch $\left.\right\}=8{\cdot}84\times7{\cdot}14=$ **63·1 tons-ft.**

Tensile stress in rods $=\frac{253}{63{\cdot}1}=$ **4 tons sq. inch.**

23. *A concrete beam* 12 *inches deep,* 10 *inches broad,* 12 *feet span, is reinforced with four* $\frac{5}{8}$″ *square steel rods, the centres of which are* $1\frac{1}{2}$″ *from the underside of the beam. The ultimate compressive strength of the concrete is* 1 *ton per sq. inch, and the elastic limit of the steel* 16 *tons per sq. inch. You may assume that the steel bars take all the tension and that their modulus of elasticity is* 10 *times that of the concrete. Find the safe central load for the beam, taking a factor of safety of* 4.

For symbols employed see p. 470, Vol. I.

$$\frac{f_c}{f}=\frac{C}{d-c}\cdot\frac{E_c}{E}=\frac{C}{10(d-c)} \qquad \frac{f_c}{2}CB=fa$$

$a=1{\cdot}55$ sq. inches $\qquad d=10{\cdot}5$ inches

$$\frac{f_c}{f}=\frac{2a}{CB}=\frac{c}{10(d-c)} \qquad \frac{2\times1{\cdot}55}{10c}=\frac{c}{10(10{\cdot}5-c)}$$

$c=4{\cdot}36''$ $\qquad d-c=6{\cdot}14''$.

$$\text{Thrust on concrete}=\frac{2240\times10\times4{\cdot}36}{2}=49{,}000 \text{ lb.}$$

Tension on rods not to exceed $1{\cdot}55\times16\times2240=56{,}000$ lb.

Concrete moment of resistance must not exceed:

$$49{,}000(\tfrac{2}{3}\times4{\cdot}36+6{\cdot}14)=443{,}000 \text{ lb.-inches.}$$

Working moment of resistance for concrete $=110{,}500$ lb.-inches.

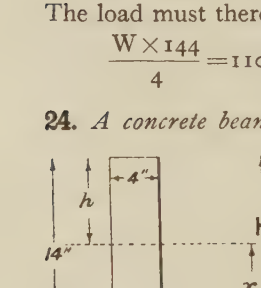
Working moment of resistance of rods

$$=\tfrac{56000}{4}(\tfrac{2}{3}\times4{\cdot}36+6{\cdot}14)=126{,}700 \text{ lb.-inches.}$$

The load must therefore be taken to suit the concrete:

$$\frac{W\times144}{4}=110{,}500 \qquad W=\mathbf{3{,}070 \text{ lb.}}$$

24. *A concrete beam of Tee section is reinforced in the flange of the Tee, the vertical web is in compression, find from the data given below the moment of resistance of the beam.*



Reinforcements, eight $\frac{3}{8}$ *in. round rods with their centres* 1 *in. from the tension skin.*

Permissible tensile stress in the rods, 10,000 *lb. per sq. in.*

Permissible compressive stress in the concrete, 500 *lb. per sq. in.*

Ratio of moduli of elasticity, 12. (Edinburgh University.)

Area of rods=0·883 sq. in.

$\frac{4(H-x)^2}{2}=a_s x\frac{E_s}{E_c}$ (see p. 470, Vol. I.)

$(H-x)^2=5\cdot3x$ $\quad x=9\cdot79''$ $\quad h=7\cdot21''$.

Moment of resistance—rods :

$0\cdot883\times10{,}000(9\cdot79+\frac{2}{3}\times7\cdot21)=$ **128,900** inch-lb.

Moment of resistance—concrete :

$\frac{4\times7\cdot21\times500}{2}(9\cdot79+\frac{2}{3}\times7\cdot21)=$ **105,300** inch-lb.

25. *A reinforced concrete beam of rectangular section has to support a load including its own weight of* 5 *cwt. per foot run on a span of* 24 *feet. The ends of the beam are built in. Breadth of beam* 10 *ins. If the limiting stresses are* 500 *lb. and* 13,000 *lb. per sq. in. respectively in the concrete and steel, and if* $E_s/E_c=15$, *determine a suitable cross section for the beam and reinforcement.* (Edinburgh University.)

Maximum bending moment

$$M=\frac{WL}{12}=\frac{120\times112\times24\times12}{12}=323{,}000 \text{ in.-lb.}$$

For symbols employed see p. 470, Vol. I.

$\frac{f_c}{f}\times15=\frac{c}{d-c}$ $\quad \frac{f_c cb}{2}=af$ $\quad af\left(d-\frac{c}{3}\right)=323{,}000$

$\frac{500\times15}{13000}=\frac{c}{d-c}$ $\quad c=0\cdot366d$

$\frac{500\times10\times0\cdot366d}{2}=13{,}000a$ $\quad a=0\cdot07d$

$0\cdot07d\times13{,}000\left(d-\frac{0\cdot366d}{3}\right)=323{,}000$ $\quad d=20''$ nearly.

The gross depth should be not less than **21·5″**.
Area of rods 1·4 sq. ins., or say **three $\frac{3}{4}$″ rods.**

26. *A reinforced concrete beam,* 5″ *wide,* 12″ *deep, is continuous over two spans of* 20 *ft. each. The beam is reinforced with four* $\frac{1}{2}$″ *rods with their centres* 2″ *from the skin of the concrete. Load* 0·1 *ton per ft. run*$=\frac{0\cdot1}{12}$ *ton per in. run. Ratio of moduli of elasticity* 10.

Calculate the stress in the steel and concrete over the central pier.

Position of N.A. $\frac{5(10-e)^2}{2} = 0{\cdot}785 \times 10e$ (See p. 470, Vol. I.)

Distance of N.A. from centre of rods $e = 5{\cdot}76''$, $10-e = 4{\cdot}24''$

Arm of internal resistance couple $= 5{\cdot}76 + \frac{2}{3} \times 4{\cdot}24 = 8{\cdot}59''$

Moment of resistance of concrete per sq. inch $= \frac{5 \times 4{\cdot}24}{2} \times 8{\cdot}59 = 91$

Moment of resistance of rods per square inch
$= 0{\cdot}785 \times 8{\cdot}59 = 6{\cdot}74$ (See p. 532, Vol. I.)

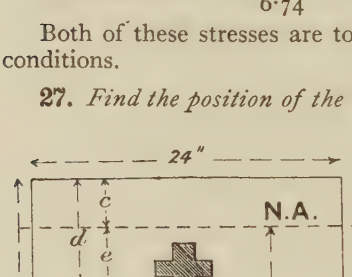
Bending moment over pier $= \frac{wl^2}{32} = \frac{0{\cdot}1 \times 480^2}{32 \times 12} = 60$ inch-tons

Stress in concrete $= \frac{60}{91} =$ **0·66** tons sq. inch.

Stress in rods $= \frac{60}{6{\cdot}74} =$ **8·9** tons sq. inch.

Both of these stresses are too high for ordinary working conditions.

27. *Find the position of the neutral axis and the moment of resistance of the section of a reinforced concrete beam of the section shown in the figure. Ratio of moduli of elasticity,* 15; *sectional area of steel,* 1·23 *sq. ins.; moment of inertia of section about its own centre of gravity,* 1·413; *stress in concrete not to exceed* 400 *lb. per sq. in.; stress in steel not to exceed* 10,000 *lb. per sq. in.*



To find the position of the neutral axis :

$c \times 24 \times \frac{c}{2} = 1{\cdot}23 \times 15 \times e$ $\qquad 12c^2 = 18{\cdot}4(d-c)$

$c = 1{\cdot}83''$ $\qquad \therefore e = 4 - 1{\cdot}83 = 2{\cdot}17''$ $\qquad d = 4''$

Moment of inertia I_s of steel section about N.A
$= (1{\cdot}413 + 1{\cdot}23 \times 2{\cdot}17^2) = 7{\cdot}2$ in.^{4}

Moment of inertia I_c of concrete $= \frac{24 \times 1{\cdot}83^3}{3} = 49$ in.^{4}.

In this case the stress cannot be assumed to be evenly distributed over the steel section; it is greatest at the skin most remote from the neutral axis—distant y. Then

$$\frac{f_c}{f}=\frac{cE_c}{yE}=\frac{c}{15y} \qquad f_c=\frac{fc}{15y}$$

The moment of resistance of the beam

$$=\frac{I_s f}{y}+\frac{I_c f_c}{c}=\frac{f}{y}\left(I_s+\frac{I_c}{15}\right) \quad \text{or} \quad =\frac{f_c}{c}(I_c+15I_s)$$

When the stress in the concrete is 400 lb. per sq. in.

$$M_c=\frac{400}{1{\cdot}83}(49+15\times 7{\cdot}2)=\mathbf{34{,}300\ lb.\text{-}ins.}$$

When the stress in the steel is 10,000 lb. per sq. in.

$$M_s=\frac{10000}{3{\cdot}67}(7{\cdot}2+\tfrac{49}{15})=\mathbf{28{,}500\ lb.\text{-}ins.}$$

If this problem had been treated as an ordinary reinforced beam and the arm of the couple taken as the distance from the c. of g. of the concrete "triangle" to the c. of g. of the steel section (which is incorrect in this case), we get:

M_c=**29,800 lb.-ins.** M_s=**41,700 lb.-ins.**

CHAPTER XII

BENDING MOMENTS AND SHEAR

1. *A timber beam* 20 *ft. long supported at each end is loaded with* 1 *ton in the middle and* 2 *tons at a point* 4 *ft. from the right-hand support. Find the maximum bending moment. Taking the stress at* 0·6 *ton per sq. in., find the size of square beam required. The weight of the beam itself may be neglected.*

Left-hand reaction, $R_l = \frac{1\times 10+2\times 4}{20} = 0{\cdot}9$ ton

Right-hand reaction, $R_r = 3-0{\cdot}9 = 2{\cdot}1$ tons

Bending moment in middle, $M_1 = 0{\cdot}9\times 10 = 9$ tons-ft.

B.M. under 2 tons load, $M_2 = 2{\cdot}1\times 4 = 8{\cdot}4$ tons-ft.

$$9\times 12 = f\frac{S^3}{6} \qquad \therefore S^3 = 1080 \qquad S = 10{\cdot}26 \text{ ins.}$$

Thus a beam $10\frac{1}{4}''$ square will meet the requirements.

2. *A rolled joist* $20''\times 9''\times 1\frac{1}{2}''$ *rests on supports* 40 *ft. apart. Find* (i) *the bending stress in the flanges due to its own weight. The vertical web is* $\frac{3}{4}''$ *thick, the material weighs* 0·28 *lb. per cubic inch.* (ii) *The additional evenly distributed load per ft. run required to produce a stress of* 6 *tons per sq. inch in the flanges.*

Weight of beam—

$$(18\times 1{\cdot}5+17\times 0{\cdot}75)0{\cdot}28\times 12 = 133 \text{ lb. per ft.}$$

or 11 lb. per inch run (nearly) (see p. 492, Vol. I.)

$$\frac{wl^2}{8} = fbth \text{ adopting the usual approximation}$$

$$\frac{11\times 480^2}{8} = f\times 9\times 1{\cdot}5\times 20$$

(i) f=**1,170** lb. per sq. in. due to its own weight.

(ii) $\frac{w_1 \times 480^2}{8 \times 9 \times 1{\cdot}5 \times 20} = 6 \times 2{,}240$ $\quad w_1 =$ **126 lb.** per in. run

or $\frac{126 \times 480}{2240} =$ **27 tons** additional evenly distributed load.

NOTE.—To avoid errors the load should always be expressed per *inch* run when the other dimensions of the beam are taken in inches.

3. *A steel water main is* 36″ *outside diameter and* 34″ *inside. Find the unsupported span* (*when it is full of water*) *to produce a stress of* 5 *tons sq. in. Weight of pipe per inch run,* 30·5 *lb.; weight of water per inch run,* 32·8 *lb.; total weight per inch run,* 63·3 *lb.*

$$Z = \frac{36^4 - 34^4}{10{\cdot}2 \times 36} = 935 \quad \text{(See p. 453, Vol. I.)}$$

$$\frac{wl^2}{8} = fZ \quad \text{(Freely supported at ends.)}$$

$$\frac{63{\cdot}3 l^2}{8} = 5 \times 2240 \times 935 \qquad l = 1150'' = \mathbf{95{\cdot}8'}.$$

The above calculation neglects the effect of the circumferential joints in the main. If the efficiency be taken at 80 per cent. (see p. 417, Vol. I.), the effective Z becomes $940 \times 0{\cdot}8 = 752$ approx.

Then $\frac{Wl^2}{4} = f \times 752$ and $l = 1031'' =$ **86 feet.**

4. *A beam* 10 *ft. long is loaded for one-half its length with* 0·75 *ton per foot run. Find the position and magnitude of the maximum bending moment.*

The bending moment is a maximum where the shear is zero (see pp. 484 and 502, Vol. I.).

Load on beam, $5 \times 0{\cdot}75 = 3{\cdot}75$ tons

Reaction at unloaded end $= R_A$

$$R_A \times 10 = 3{\cdot}75 \times 2{\cdot}5 = 0{\cdot}938 \text{ ton}$$

$$R_B = 3{\cdot}75 - 0{\cdot}938 = 2{\cdot}812 \text{ tons}$$

Zero shear occurs at $\frac{2{\cdot}812}{0{\cdot}75} = 3{\cdot}75$ ft. from B

Max. ben. momt. = Area of shear diagram to this point
$= \frac{1}{2} \times 2{\cdot}812 \times 3{\cdot}75 = \mathbf{5{\cdot}27}$ tons-ft.

or, by taking moments about the section where the b.m. is a maximum

Max. ben. momt. $= 2{\cdot}812 \times 3{\cdot}75 - \frac{0{\cdot}75 \times 3{\cdot}75}{2} = \mathbf{5{\cdot}27}$ **tons-ft.**

5. *The load on an overhung crank pin is* 75,000 *lb. The intensity of pressure on the projected area* (*diameter* × *length*) *is* 700 *lb. per sq. inch. The bending stress in the pin is* 8,000 *lb. per sq. inch. The shearing stress may be neglected. Find the diameter and length of the pin.*

$700dl = 75{,}000 \qquad dl = 107$ (See p. 453, Vol. I.)

$$M = \frac{fd^3}{10{\cdot}2} = \frac{8000d^3}{10{\cdot}2} = \frac{75000l}{2}$$

$$\frac{75000l \times 10{\cdot}2}{2 \times 8000} = d^3 = 47{\cdot}8l = \frac{47{\cdot}8 \times 107}{d}$$

$d^4 = 5116 \qquad d = \mathbf{8{\cdot}46''} \qquad l = \frac{107}{8{\cdot}46} = \mathbf{12{\cdot}7''}.$

6. *A vertical crank shaft rotates at* 1200 *r.p.m.; the crank pin is* 8 *inches long and has a weight of* 20 *lb. at the free end of the pin. Radius of crank,* 16". *Bending stress in pin,* 15,000 *lb. per sq. inch. Weight of material in pin,* 0·28 *lb. per cubic inch. Find the diameter of the pin.*

Weight of pin $= 0{\cdot}785 \times d^2 \times 8 \times 0{\cdot}28 = 1{\cdot}76d^2$
Centrifugal force acting on pin $= 0{\cdot}00034 \times 1{\cdot}76d^2 \times \frac{16}{12} \times 1200^2$
$= 1148d^2$ lb.

Ditto on weight $= 0{\cdot}00034 \times 20 \times \frac{16}{12} \times 1200^2 = 13{,}050$ lb.

Bending moment on pin $= 1148d^2 \times \frac{8}{2} + 13{,}050 \times 8$
$= 4592d^2 + 104{,}400$ inch-lb.

$$M = f_s \frac{d^3}{10{\cdot}2} \qquad 10{\cdot}2\left(\frac{4592d^2 + 104{,}400}{15{,}000}\right) = d^3$$

$3{\cdot}12d^2 + 71 = d^3 \qquad d = \mathbf{5{\cdot}5''}$ (Solving by trial and error.)

7. *A brick wall, triangular in elevation with one end vertical, rests on a beam which is freely supported at each end. Find a suitable section for a rolled joist.*

Weight of brickwork, 125 *lb. per cu. ft.; span of beam* (*l*),

20 ft.; height of wall (h), 10 ft.; thickness of wall, 18 ins.; stress in flanges, 5 tons sq. in. The base of the triangle covers the joist.

The maximum bending moment occurs at a distance from the apex of triangle of

$$\frac{l}{\sqrt{3}}$$ (See pp. 209 and 502, Vol. I.)

$$\text{Reaction at apex, } R_1 = \frac{125 \times 1{\cdot}5 \times 20 \times 10}{2 \times 3} = 6{,}250 \text{ lb.}$$

$$\text{Max. ben. moment} = \frac{6250 \times 20}{1{\cdot}73} - \frac{10 \times 20^2 \times 125 \times 1{\cdot}5}{6 \times 1{\cdot}73^3}$$

$$M_{max.} = 48{,}160 \text{ lb.-feet}$$

$$\frac{48160 \times 12}{5 \times 2240} = Z = 51{\cdot}6 \qquad \textbf{Say } 9'' \times 7'' \times \textbf{0·92 (Z=51)}$$

i.e. British Standard B.S.B. 16.

8. *A wall triangular in elevation covers a girder, the apex of the triangle being over the middle of the span. Find the maximum bending moment when*

Weight of wall (w)=120 *lb. per cu. ft.; thickness of wall* (t)=18 *ins.*=1·5 *ft.; span* (L)=30 *ft.; height of triangle* (H)=16 *ft.*

$$\text{The reaction at each end} = \frac{wtLH}{4}$$

$$\left.\begin{array}{l}\text{Bending moment at}\\ \text{middle of span}\end{array}\right\} = \frac{wtLH}{4} \times \frac{L}{2} - \frac{wtLH}{4} \times \frac{L}{6}$$

$$= \frac{3wtL^2H - wtL^2H}{24} = \frac{wtL^2H}{12}$$

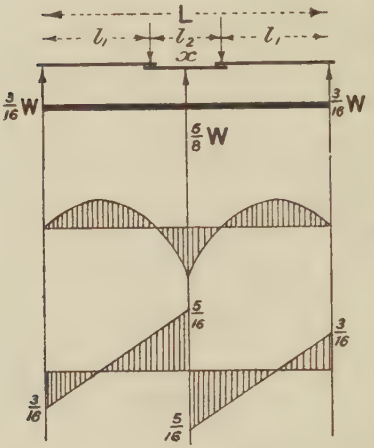
In the case mentioned above—

$$M_{max.} = \frac{120 \times 1{\cdot}5 \times 900 \times 16}{12} = \textbf{216,000 ft.-lb.}$$

If the safe stress be taken at 10,000 lb. sq. in., we have $\frac{216000 \times 12}{10000} = Z = 259$ in in. units, or, say, two R.S.J.'s $16'' \times 8'' \times 1''$.

9. *A beam of uniform section rests on three level supports.*

Each span is 12 *feet and the weight of the beam* 9,840 *lb. Find* (i) *the position of the points of contrary flexure.* (ii) *The maximum positive and negative bending moments. Sketch bending moment and shear diagrams.*



See p. 532, Vol. I.

$l_1 = \frac{3}{8}L = \frac{3}{8} \times 24 = \mathbf{9\ ft.}$

$l_2 = \frac{24}{4} = \mathbf{6\ ft.}$

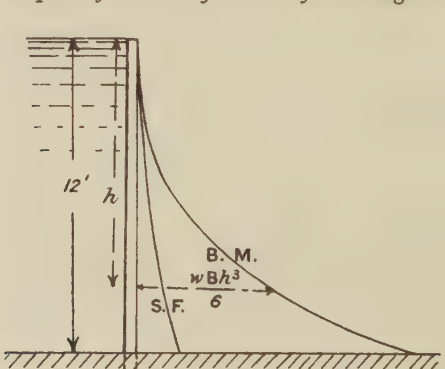
$$Mx = \tfrac{3}{16} wL \times \frac{l_2}{2} + \frac{w l_2^2}{8}$$

$$= \frac{WL}{32} = \mathbf{7{,}380\ lb.\text{-}feet.}$$

Max. bending moment on end virtual spans occurs at their middle or at $\frac{3}{16}$L from end support and amounts to—

$$\frac{w l_1^2}{8} = \frac{w}{8}(\tfrac{3}{8}L)^2 = \mathbf{4{,}140\ lb.\text{-}ft.}$$

10. *A temporary dam is formed by driving vertical piles*



which support watertight sheeting. Draw shear force and bending moment diagrams. Find the maximum B.M.

when there is water on the one side to a depth of 12 *ft.* (H). *The piles are spaced* 2′ 6″ *apart* (B).

See p. 640, Vol. I.

$$\left.\begin{array}{l}\text{Shearing force at a depth of } h \text{ feet}\\ \text{from the surface of the water}\end{array}\right\} = \frac{wBh^2}{2}$$

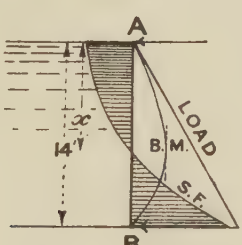
$$\text{Bending moment ditto} = \frac{wBh^2}{2} \times \frac{h}{3} = \frac{wBh^3}{6}$$

Bending moment on each pile at ground level

$$= \frac{62{\cdot}4 \times 2{\cdot}5 \times 1728}{6} = \mathbf{45{,}000\ lb.\text{-}ft.}$$

11. *A thin plate bulkhead* 14 *ft. high is strengthened by vertical beams supported top and bottom and riveted to the plating every* 2 *ft. apart. If the compartment on the one side of the bulkhead is filled with water, draw the diagram of shearing force on one of the beams, and calculate the maximum bending-moment.*

See p. 640, Vol. I.



$$\text{Total pressure} = \frac{BWH^2}{2} = \frac{2 \times 62{\cdot}4 \times 14^2}{2} = 12{,}250 \text{ lb.}$$

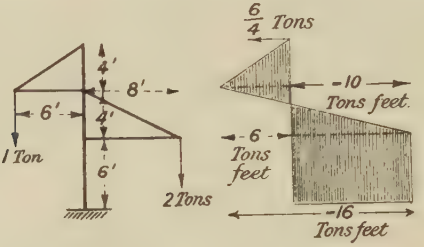
$$\text{Press. at B} = \frac{12250 \times 2}{3} = 8{,}170 \text{ lb.} \quad \text{Press. at A} = 4{,}085 \text{ lb.}$$

$$x = \frac{168}{\sqrt{3}} = 97'' \quad \text{(See p. 210, Vol. I.)}$$

$$M_{max.} = 4085 \times 97 - \tfrac{97}{168} \times \frac{146 \times 97}{2} \times \tfrac{97}{3} = 265{,}000 \text{ in.-lb.}$$

$$= \mathbf{9{\cdot}85\ ft.\text{-}tons.}$$

12. *A vertical post carries two cranes as shown; sketch and dimension the bending-moment diagram for the post.*



13. *A simple bridge is formed of telegraph poles laid side by side with all the butt ends on the one abutment. If Z be the modulus of section at the butt end and Z_x the modulus of section at a distance x ins. from the butt end, $Z_x = Z - 0{\cdot}3x$ Span = 30 ft. Diameter of poles at butt end = 12 ins.*

Find the position of the most highly stressed section when the bridge is uniformly loaded throughout its length.

Bending moment at a section distant x from butt end

$$= \frac{w}{2}(L-x)x \quad \text{(See p. 493, Vol. I.)}$$

Stress at section $f_x = \dfrac{\frac{w}{2}(Lx-x^2)}{Z-0{\cdot}3x}$

$$\frac{df_x}{dx} = \frac{(Z-0{\cdot}3x)(L-2x)-(Lx-x^2)(-0{\cdot}3)}{(Z-0{\cdot}3x)^2}$$

This is a maximum when

$$ZL-0{\cdot}3Lx-2xZ+0{\cdot}6x^2 = 0{\cdot}3x^2-0{\cdot}3Lx,\quad 0{\cdot}3x^2-2xZ = -ZL$$

$$x = \frac{Z}{0{\cdot}3} + \sqrt{\frac{Z^2}{0{\cdot}09} - \frac{ZL}{0{\cdot}3}}$$

$$Z = \frac{d^3}{10{\cdot}2} = \frac{1728}{10{\cdot}2} = 169{\cdot}5 \qquad L = 360''.$$

Inserting these values in the above expression—

$x = \mathbf{225\ ins.}$ from butt end.

14. *A tapered cantilever of circular cross-section is loaded at the extreme outer end with a load of 6 tons. The diameter at this section is 8 ins., and it increases at the rate of 0·1 in.*

per inch of length. Length of cantilever 10 *ft. Find the distance of the most highly stressed section from the outer end, also the stress at that section.*

Let the diam. at the loaded end be D and the taper t inches per in. length. Then the diam. at a distance x ins. from the loaded end is $D+xt$.

$$M_x=f_xZ_x \qquad Wx=f_x\frac{\pi}{32}(D+xt)^3 \qquad f_x=\frac{32}{\pi}Wx(D+xt)^{-3}$$

$$\frac{\pi}{32W}\times\frac{df_x}{dx}=\frac{x\,.\,d(D+xt)^{-3}}{dx}+(D+xt)^{-3}\frac{dx}{dx}$$

$$=x(-3)(D+xt)^{-4}t+(D+tx)^{-3}$$

$$=\frac{-3tx}{(D+xt)^4}+\frac{1}{(D+xt)^3}$$

$$=\frac{-3tx+D+xt}{(D+xt)^4}=0 \text{ for maximum value}$$

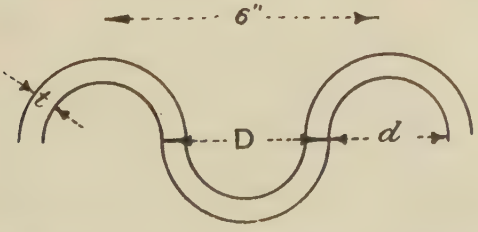
and $$D-2tx=0 \qquad x=\frac{D}{2t}.$$

The distance of the most highly stressed section from outer end of cantilever

$$x=\frac{8}{2\times0{\cdot}1}=\mathbf{40''} \qquad D_x=8+40\times0{\cdot}1=12''$$

$$Z_x=\frac{1728}{10{\cdot}2}=169{\cdot}5 \qquad f_x=\frac{6\times40}{169{\cdot}5}=\mathbf{1{\cdot}42\ tons\ sq.\ in.}$$

15. *Corrugated flooring (circular corrugations). Span,* 12 *ft. Thickness,* 0·5″. *Stress,* 6 *tons sq. in. Find the*



evenly distributed load per sq. ft. to fulfil the above-mentioned conditions.

Consider a strip 6 ins. wide—

$2d+2t=6''$ $\quad d=2{\cdot}5''$ $\quad D=3{\cdot}5''$

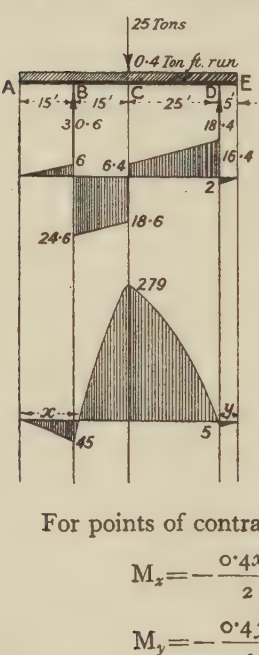
$$Z=\frac{D^4-d^4}{10{\cdot}2D}=\frac{3{\cdot}5^4-2{\cdot}5^4}{10{\cdot}2\times3{\cdot}5}=3{\cdot}1$$

$$M=\frac{Wl}{8}=\frac{W\times144}{8}=18W \qquad W=1{\cdot}03 \text{ tons}$$

Area of floor $12\times0{\cdot}5=6$ sq. ft.

Weight per sq. ft. $\frac{1{\cdot}03}{6}=0{\cdot}17$ tons=**380 lb.**

16. *A beam* 60 *feet in length rests on supports* 40 *ft. apart, the left-hand end overhangs* 15 *ft., the beam itself weighs* 0·4 *ton per ft. run and it supports a load of* 25 *tons in the middle of the beam. Find the bending moment over the supports and at the middle of the beam, also find the position of the points of contrary flexure. Draw bending moment and shear diagrams for the beam.*



Taking moments about B.

$40R_D=0{\cdot}4\times60\times15+25\times15$

$R_D=18{\cdot}4$ tons, $R_B=30{\cdot}6$ tons

$$M_B=-\frac{6\times15}{2}=-\mathbf{45} \text{ tons-ft.}$$

$M_C=-12\times15+18{\cdot}4\times25$

$=\mathbf{279}$ tons-ft.

$$M_D=-\frac{2\times5}{2}=-\mathbf{5} \text{ tons-ft.}$$

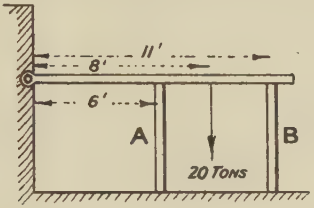
For points of contrary flexure

$$M_x=-\frac{0{\cdot}4x^2}{2}+30{\cdot}6(x-15)=0 \qquad x=\mathbf{16{\cdot}9 \text{ ft.}}$$

$$M_y=-\frac{0{\cdot}4y^2}{2}+18{\cdot}4(y-5)=0 \qquad y=\mathbf{5{\cdot}3 \text{ ft.}}$$

17. *A rigid beam is supported as shown, find the load on the props* A *and* B. *Sectional area of* A, 1·8 *sq. ins.; sectional*

area of B, 2·3 *sq. ins.; modulus of elasticity of* A, 12,000 *tons sq. in.; modulus of elasticity of* B, 10,000 *tons sq. in.*



Taking moments about hinge—

$$20\times 8 = W_A\times 6 + W_B\times 11$$

$$W_A + \tfrac{11}{6}W_B = 26{\cdot}7 \text{ tons}$$

$$x_A = \frac{W_A l}{A_A E_A} \qquad x_B = \frac{W_B l}{A_B E_B} = \tfrac{11}{6}x_A$$

$$\frac{11W_A l}{6A_A E_A} = \frac{W_B l}{A_B E_B}$$

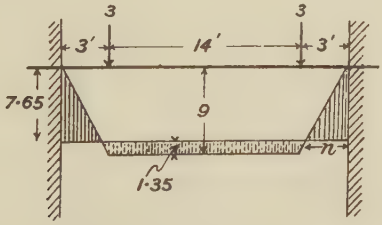
$$W_A = \frac{6W_B\times 1{\cdot}8\times 12000}{2{\cdot}3\times 10000\times 11} = 0{\cdot}51W_B$$

$$0{\cdot}51W_B + \tfrac{11}{6}W_B = 26{\cdot}7 \qquad W_B = \mathbf{11{\cdot}4 \text{ tons}}$$

$$W_A = 26{\cdot}7 - \tfrac{11}{6}\times 11{\cdot}4 = \mathbf{5{\cdot}8 \text{ tons}}$$

Load on hinge $= 20 - 5{\cdot}8 - 11{\cdot}4 = \mathbf{2{\cdot}8 \text{ tons.}}$

18. *A beam of uniform section throughout is built into a wall at each end and loaded with two equal loads of* 3 *tons*



each which are symmetrically placed about the middle of the beam. Find the bending moment close to the wall and in the middle of the beam, also find the position of the points of

contrary flexure. Length of beam, 20 feet; distance of each load from the middle of the beam, 7 feet.

Area of bending moment diagram for beam with free ends

$$3\times3\times17=153$$

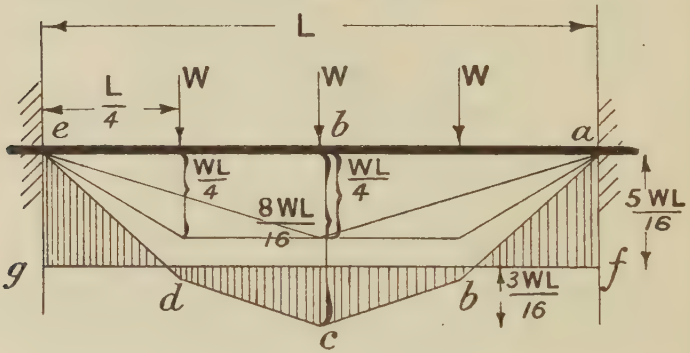
$$\text{Mean height}=\tfrac{153}{20}=7{\cdot}65$$

Bending moment at wall=**7·65 ft.-tons**

Bending moment between loads=**1·35 ft.-tons**

$$\frac{1{\cdot}35}{7{\cdot}65}=\frac{3-n}{n} \qquad n=\mathbf{2{\cdot}55\ ft.}$$

19. *A beam built in at both ends is loaded with three equal weights which divide the beam into four equal parts. Construct a bending moment diagram and find the bending moment at the middle and ends of the beam.*



Construct a bending moment diagram *a b c d e* for a free-ended beam.

Find the area of the diagram—

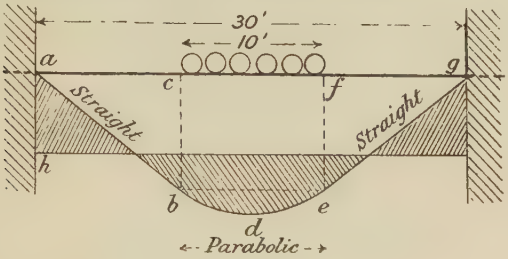
$$\text{Mean height}=\frac{\text{Area}}{\text{L}}=af \text{ or } eg.$$

Draw *fg* parallel to the unbent beam.

Then the shaded diagram shows the bending moment at every section. At *c* the bending moment= $\frac{3}{16}$WL, and at the ends $\frac{5}{16}$W.L.

20. *A built-in beam 30 ft. long between the supports is*

loaded over its central 10 *ft. with* 1·75 *tons per ft. run. Find the bending moments at the middle and ends.*



B.M. at middle when freely supported

$$= \frac{1{\cdot}75 \times 10 \times 10}{2} + \frac{1{\cdot}75 \times 10^2}{8}$$

$= 87{\cdot}5 + 21{\cdot}87 \qquad = 109{\cdot}37$ ft.-tons

Area of parabolic portion $bde = \frac{2}{3} \times 21{\cdot}87 \times 10 = 145{\cdot}8$

Area $abc + feg \qquad = 2\left(\frac{87{\cdot}5 \times 10}{2}\right) = 875{\cdot}0$

Area $cbef \qquad = 87{\cdot}5 \times 10 \qquad = 875{\cdot}0$

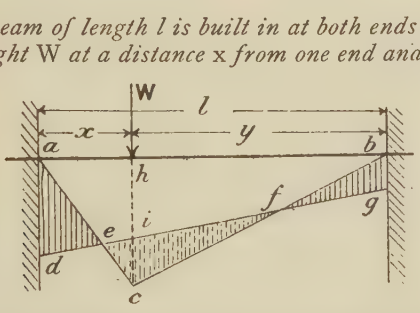
1895·8

Mean depth of diagram, viz. $ah = \frac{1895{\cdot}8}{30} = 63{\cdot}2$

Pier moment = **63·2 tons-ft.**

Bending moment at middle = 109·4 − 63·2 = **46·2 tons-ft.**

21. *A beam of length* l *is built in at both ends and loaded with a weight* W *at a distance* x *from one end and* y *from the*



other. Find expressions for the pier bending moments at each end.

Let *abc* be the bending moment diagram when the ends are free. The shaded figure represents the bending moment diagram with fixed ends. It has been shown (p. 529, Vol. I.) that the area *abc*=area *adgb*, also that the c. of g. of the two areas must lie on the same perpendicular.

Let $ad=A$ $\quad bg=B$

$R_a=\frac{Wy}{l}$. Bending moment under W with free ends$=\frac{Wxy}{l}$

$$\text{Area } abc=\frac{Wxy}{l}\times\frac{l}{2}=\frac{Wxy}{2}$$

$$\text{Area } adbg=\frac{l}{2}(A+B)=\frac{Wxy}{2}$$

The condition must also be fulfilled that the two cs. of g. lie on the same vertical.

Hence $\frac{l}{3}\left(\frac{2B+A}{B+A}\right)=\frac{l+x}{3}$ (See p. 60, Vol. I., also example **15,** Chap. III., Vol. II.)

$$\frac{2B+A}{B+A}=\frac{l+x}{l}=\frac{2x+y}{l} \qquad \frac{B}{B+A}=\frac{x}{l}$$

$$B=(B+A)\frac{x}{l}=\frac{Wx^2y}{l^2} \qquad A=\frac{Wxy^2}{l^2}.$$

22. *A beam* 20 *ft. long is built in at both ends and is loaded with* 6 *tons at a distance of* 7 *ft. from one pier. Find the pier moments and the bending moment under the load.*

(See last example.)

$$\text{Max. pier moment}=\frac{6\times7\times13^2}{400}=\mathbf{17{\cdot}7}\text{ tons-ft.}$$

$$\text{Min. pier moment}=\frac{6\times7^2\times13}{400}=\mathbf{9{\cdot}6}\text{ tons-ft.}$$

Bending moment under load with free-ended beam, viz. *hc* $=\frac{6\times7\times13}{20}=27{\cdot}3$ tons-ft.

or the sum of the pier moments$=17{\cdot}7+9{\cdot}6=27{\cdot}3$ tons-ft.

The depth $hi=9{\cdot}6+\frac{13}{20}(17{\cdot}7-9{\cdot}6)=14{\cdot}9$

The depth *ic* or the bending moment under the load

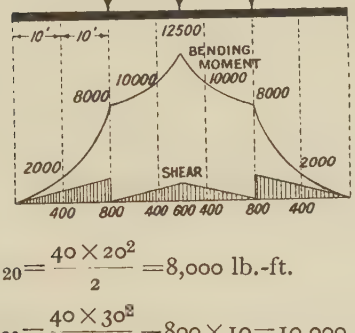
$=27{\cdot}3-14{\cdot}9=\mathbf{12{\cdot}4}$ **tons-ft.**

23. *A floating balk of timber of the same cross-section throughout supports loads as shown. Construct bending moment and shear diagrams.*

The load is uniformly distributed by the water pressure

$$=\frac{2800}{70}=40 \text{ lb. per ft. run}$$

$$M_{10}=\frac{40\times 10^2}{2}=2{,}000 \text{ lb.-ft.}$$



$$M_{20}=\frac{40\times 20^2}{2}=8{,}000 \text{ lb.-ft.}$$

$$M_{30}=\frac{40\times 30^2}{2}-800\times 10=10{,}000 \text{ lb.-ft.}$$

$$M_{35}=\frac{40\times 35^2}{2}-800\times 15=12{,}500 \text{ lb.-ft.}$$

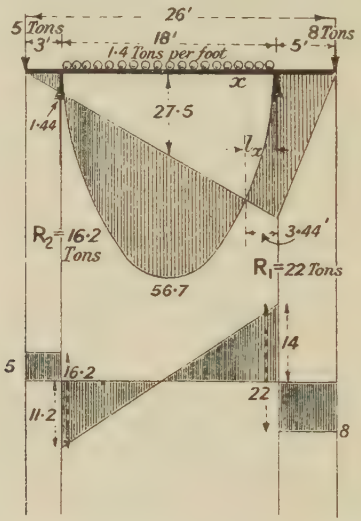
$S_{10}=40\times 10=400$ lb. $\quad S_{20}=40\times 20=800$ lb.

$S_{30}=40\times 10=400$ „ $\quad S_{35}=40\times 15=600$ „

24. *A beam rests on two walls 18 ft. apart, the right-hand end of the beam overhangs 5 ft., and the left-hand end 3 ft. Between the walls the beam supports a uniformly distributed load of 1·4 tons per foot run. At the extreme right-hand end there is a load of 8 tons, and at the extreme left-hand end a load of 5 tons. Construct bending moment and shear force diagrams and give their values at all the important sections. Find where the points of contrary flexure occur.*

Construct the parabolic bending moment diagram for the distributed load over the central span, the depth of which is $\frac{1\cdot4\times 18^2}{8}=56\cdot7$ foot-tons to the scale chosen.

The bending moment at the left-hand support is $5\times3=15$ foot-tons, and at the right support $8\times5=40$ foot-tons.



To find the points of contrary flexure we have—

$$8(5+l_x)+\frac{1\cdot4l_x^2}{2}=22l_x$$

$l_x=$ **3·44** feet and **16·56** feet from the right-hand support.

25. *A beam 38 ft. long rests on supports 26 ft. apart. The left-hand end overhangs 7 ft. Starting from the left-hand end, the loads are as follows :—4 tons at end, 2·5 tons at 8 ft., 3·2 tons at 17 ft., 5·6 tons at 30 ft., 1·9 tons at 37 ft.*

Construct shear and bending moment diagrams, and state the value of the bending moment at the important points ; also find the position of the points of contrary flexure.

$R_1\times26+1\cdot9\times4=4\times33+2\cdot5\times25+3\cdot2\times16+5\cdot6\times3$

$R_1=9\cdot8$ tons $\qquad R_2=7\cdot4$ tons

$M_y=4\times7=28$ tons-ft. $\qquad M_x=4\times1\cdot9=7\cdot6$ tons-ft.

$M_a=4\times8-9\cdot8\times1=22\cdot2$ tons-ft. (hog)

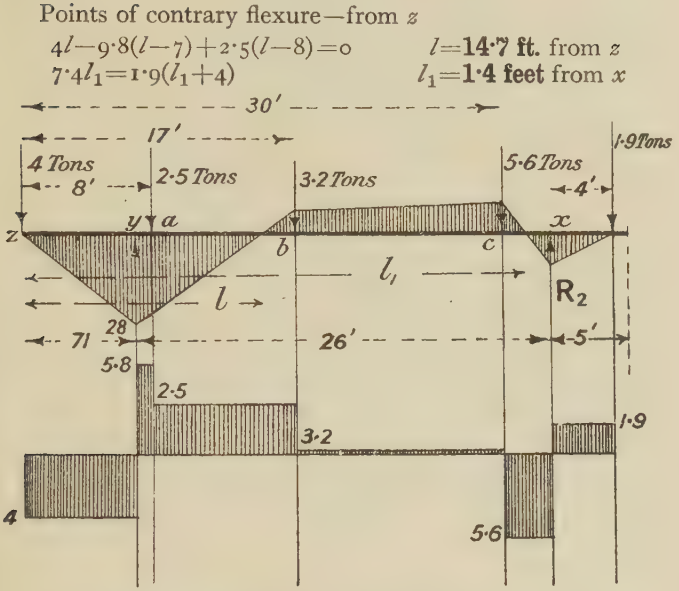
$M_c=1\cdot9\times7-7\cdot4\times3=8\cdot9$ tons-ft. (sag)

$M_b=1\cdot9\times20+5\cdot6\times13-7\cdot4\times16=7\cdot6$ tons ft. (sag)

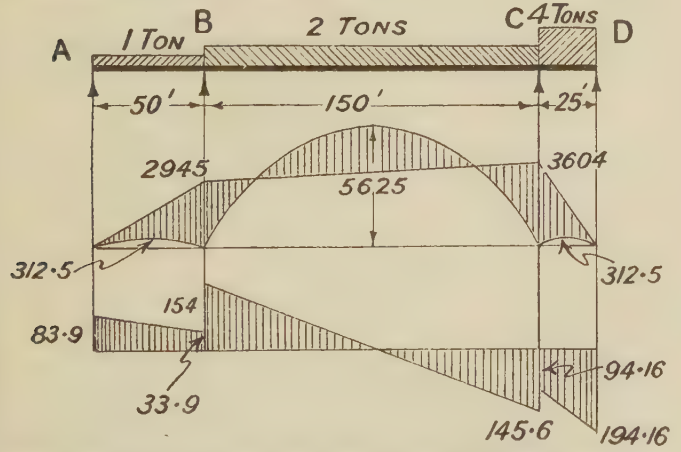
Points of contrary flexure—from z

$4l-9{\cdot}8(l-7)+2{\cdot}5(l-8)=0$ $\quad l=$ **14·7 ft.** from z

$7{\cdot}4l_1=1{\cdot}9(l_1+4)$ $\quad l_1=$ **1·4 feet** from x



26. *Construct bending moment and shear diagrams for a continuous girder of three spans* 50 *ft.*, 150 *ft.*, *and* 25 *ft.*,



when loaded with 1 *ton*, 2 *tons*, 4 *tons per foot run respectively.*

(Edinburgh University.)

Bending moments, when freely supported—

Span AB $M=\frac{1\times 50^2}{8}=312{\cdot}5$ foot-tons

BC $M=\frac{2\times 150^2}{8}=5{,}625$ foot-tons

CD $M=\frac{4\times 25^2}{8}=312{\cdot}5$ foot-tons.

Bending moments at supports B and C.

Spans AB and BC—

$M_A l_1+2M_B(l_1+l_2)+M_C l_2=\frac{1}{4}(w_1 l_1^3+w_2 l_2^3)$

$M_A\times 50+2M_B(50+150)+M_C\times 150=\frac{1}{4}(1\times 50^3+2\times 150^3)$

$M_A=0$. Then

$400M_B+150M_C=1{,}718{,}750 \qquad M_B=4297-0{\cdot}375M_C$

Spans BC and CD—

$M_B\times 150+2M_C(150+25)+M_D\times 25=\frac{1}{4}(2\times 150^3+4\times 25^3)$

$M_D=0$. Then

$6M_B+14M_C=68{,}125 \qquad M_B=11{,}354-2{\cdot}333M_C$

$M_C=$ **3,604 foot-tons** $\qquad M_B=$ **2,945 foot-tons.**

Reaction at supports

$R_A\times 50-50\times 25=M_B=2{,}945 \qquad R_A=$ **83·9 tons.**

$R_A\times 200+R_B\times 150-50\times 175-300\times 75=M_C=3{,}604$

$R_B=$ **120·49 tons.**

$R_A\times 225+R_B\times 175+R_C\times 25-50\times 200-300\times 100$
$-100\times 12{\cdot}5=M_D=0$

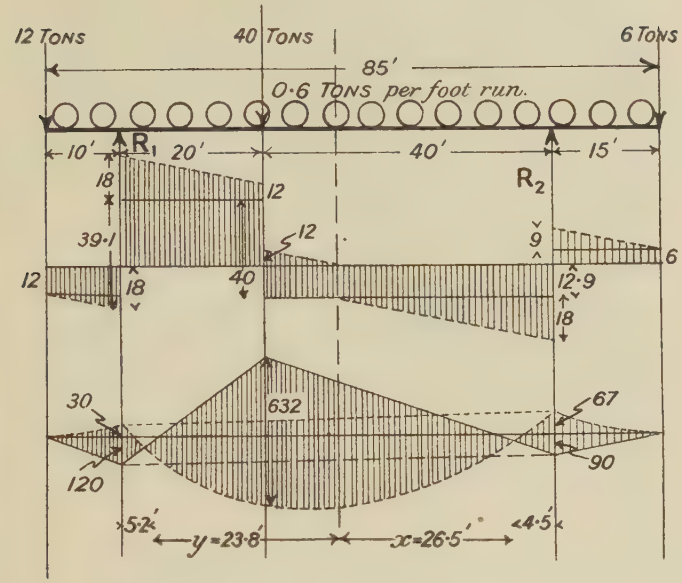
$R_C=$ **51·46 tons.**

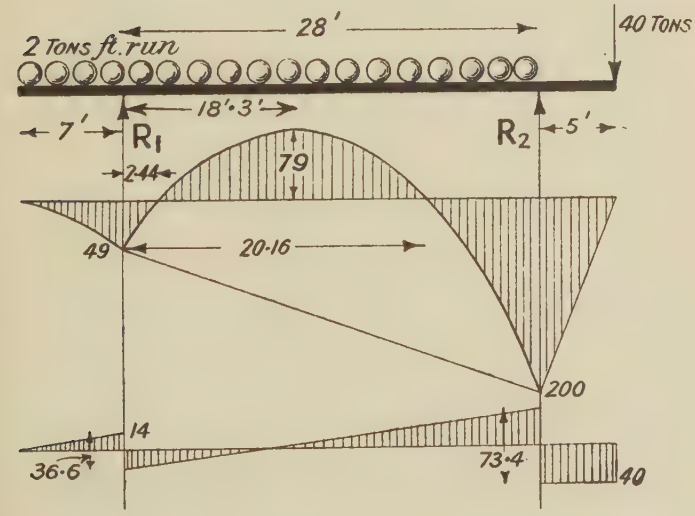
$R_D\times 25-100\times 12{\cdot}5=M_C=3{,}604$

$R_D=$ **194·16 tons.**

27. *Construct bending moment and shear diagrams for the two beams when loaded as shown in the figures.*

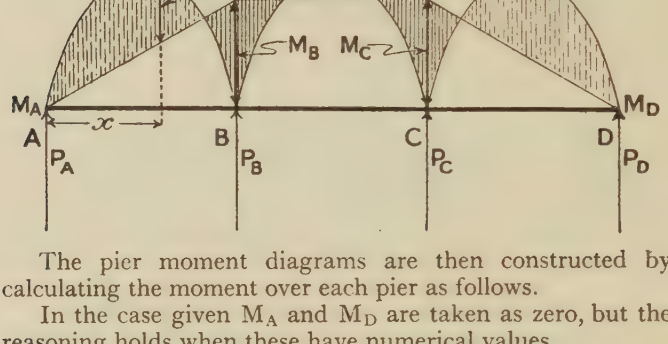
From the examples already worked out the construction should be readily followed from the diagrams.


12 TONS
40 TONS
6 TONS
85'
0·6 TONS per foot run.
10'
R1
20'
40'
15'
R2
18
12
39·1
12
9
6
12
18
40
12·9
18
30
632
67
120
90
5·2'
y=23·8'
x=26·5'
4·5'


28'
40 TONS
2 TONS ft. run
7'
18'·3'
R1
R2
5'
24·4
79
49
20·16
200
14
36·6
73·4
40

28. Clapeyron's Theorem of Three Moments

When a continuous beam rests on three or more supports at the same level, not necessarily equidistant, and the load w per foot run is evenly distributed, a convenient expression can be found for the reactions at the supports and the bending moments on the various spans. In the first place, bending moment diagrams are constructed for the distributed load for each span regarded as free ended.



The pier moment diagrams are then constructed by calculating the moment over each pier as follows.

In the case given M_A and M_D are taken as zero, but the reasoning holds when these have numerical values.

Starting from A we have for any section distant x from A

$$Mx = M_A + P_A x - \frac{wx^2}{2} \quad \ldots\ldots \text{(i)}$$

At B where $x = L_{AB}$

$$M_B = M_A + P_A L_{AB} - \frac{wL^2_{AB}}{2} \quad \ldots\ldots \text{(ii)}$$

$$P_A = \frac{M_B - M_A}{L_{AB}} + \frac{wL_{AB}}{2} \quad \ldots\ldots \text{(iii)}$$

Starting from C backwards towards B

$$-M_B = M_C + P_C L_{CB} - \frac{w}{2} L^2_{CB} \quad \ldots\ldots \text{(iv)}$$

$$P_C = \frac{M_B - M_C}{L_{CB}} + \frac{wL_{CB}}{2} \quad \ldots\ldots \text{(v)}$$

but $$\frac{M}{EI} = \frac{d^2y}{dx^2}$$ (See p. 509, *M.A.E.*, Vol. I.)

Hence for the span AB $EI\frac{d^2y}{dx^2}=M_A+P_Ax-\frac{wx^2}{2}$. (vi)

The slope $\frac{dy}{dx}$ is given by

$$EI\frac{dy}{dx}=M_Ax+P_A\frac{x^2}{2}-\frac{wx^3}{6}+C \quad . \quad . \quad . \quad \text{(vii)}$$

where C is a constant of integration.

The deflection is given by :

$$EIy=\frac{M_Ax^2}{2}+P_A\frac{x^3}{6}-\frac{wx^4}{24}+Cx \quad . \quad . \quad \text{(viii)}$$

The constant of integration is zero because the deflection y is zero at the end of the beam when x is zero. Then since y is zero when $x=L_{AB}$, we have

$$\frac{M_AL^2_{AB}}{2}+\frac{P_AL^2_{AB}}{6}-\frac{wL^4_{AB}}{24}+CL_{AB}=0$$

and $$C=-\frac{M_AL_{AB}}{2}-\frac{P_AL^2_{AB}}{6}+\frac{wL^3_{AB}}{24} \quad . \quad \text{(ix)}$$

The slope $\frac{dy}{dx}$ at B is from equation (vi)

$$EI\frac{dy}{dx}=M_AL_{AB}+\frac{P_AL^2_{AB}}{2}-\frac{wL^3_{AB}}{6}$$
$$-\frac{M_AL_{AB}}{2}-\frac{P_AL^2_{AB}}{6}+\frac{wL^3_{AB}}{24}$$

$$EI\frac{dy}{dx}=\frac{M_AL_{AB}}{2}+\frac{P_AL^2_{AB}}{3}-\frac{wL^3_{AB}}{8} \quad . \quad . \quad \text{(x)}$$

Similarly starting from C—

$$-EI\frac{dy}{dx}=\frac{M_CL_{CB}}{2}+\frac{P_CL^2_{CB}}{3}-\frac{wL^3_{CB}}{8} \quad . \quad . \quad \text{(xi)}$$

Then

$$\frac{M_AL_{AB}}{2}+\frac{P_AL^2_{AB}}{3}-\frac{wL^3_{AB}}{8}=-\frac{M_CL_{CB}}{2}-\frac{P_CL^2_{CB}}{3}+\frac{wL^3_{CB}}{8}$$

Substituting the values of P_A and P_C from (iii) and (v), we get

$$\frac{M_AL_{AB}}{2}+M_BL_{AB}+\frac{wL^3_{AB}}{8}=-\frac{M_CL_{CB}}{2}-M_BL_{CB}-\frac{wL^3_{CB}}{8}$$

or $$(M_A+2M_B)L_{AB}+(M_C+2M_B)L_{CB}+\frac{w}{4}(L^3_{AB}+L^3_{CB})=0$$

For more complex cases the reader should refer to treatises on structures such as—

Fidler's "Practical Treatise on Bridge Construction," Griffin.

Morley's "Theory of Structures," Longmans.

Warren's "Engineering Construction in Steel and Timber," Longmans.

Husband and Harby's "Structural Engineering," Longmans.

29. *A continuous girder rests on four level piers*, A, B, C, D.

Length, AB 15 *feet*, BC 20 *ft.*, CD 10 *ft.*, D *to overhanging end* 5 *ft.*

Load A *to* B, 2 *tons per ft. run*; B *to* C, 1 *ton per ft. run*; C *to end*, 3 *tons per ft. run. Find the reactions and bending moments at each pier.*

$$M_A = 0$$

$$M_A \times 15 + 2M_B \times 35 + M_C \times 20 = \tfrac{1}{4}(2 \times 15^3 + 1 \times 20^3)$$

$$70M_B + 20M_C = 3687{\cdot}5 \quad . \quad . \quad . \quad . \quad \text{(i)}$$

$$M_B \times 20 + 2M_C \times 30 + M_D \times 10 = \tfrac{1}{4}(1 \times 20^3 + 3 \times 10^3)$$

$$20M_B + 60M_C + 10M_D = 2750$$

$$M_D = \frac{3 \times 5^2}{2} = \mathbf{37{\cdot}5\ tons\text{-}feet}$$

$$20M_B + 60M_C = 2375 \qquad 20M_C = \frac{2375 - 20M_B}{3} \quad . \quad \text{(ii)}$$

From (i) $20M_C = 3687{\cdot}5 - 70M_B$ $M_B = \mathbf{45{\cdot}73\ tons\text{-}feet}$

From (ii)

$$20 \times 45{\cdot}73 + 60M_C = 2375 \qquad M_C = \mathbf{24{\cdot}34\ tons\text{-}feet} \quad \text{(iii)}$$

Reactions—take moments about B

$$R_A \times 15 - 2 \times 15 \times \tfrac{15}{2} = -45{\cdot}73 \qquad \mathbf{R_A = 11{\cdot}95\ tons}$$

at C $\quad 11{\cdot}95 \times 35 + R_B \times 20 - 30 \times 27{\cdot}5 - 20 \times 10 = 24{\cdot}34$.

$$\mathbf{R_B = 29{\cdot}12\ tons}$$

at D $\quad 11{\cdot}95 \times 45 + 29{\cdot}12 \times 30 + 10R_C - 30 \times 37{\cdot}5 - 20 \times 20$

$$- 30 \times 5 = -37{\cdot}5$$

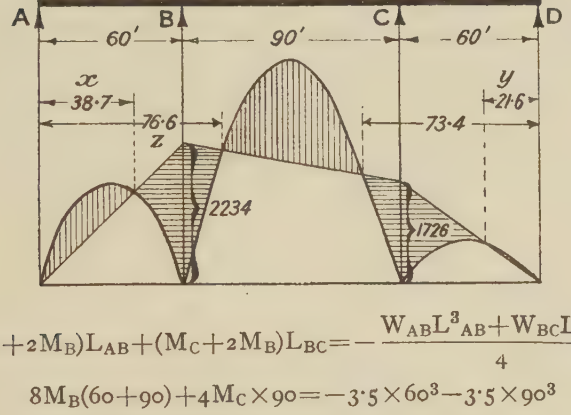
$$\mathbf{R_C = 22{\cdot}62\ tons}$$

at end $\quad 11{\cdot}95 \times 50 + 29{\cdot}12 \times 35 + 22{\cdot}62 \times 15 + 5R_D - 30 \times 42{\cdot}5$

$$- 20 \times 25 - 45 \times 7{\cdot}5 = 0$$

$$\mathbf{R_D = 31{\cdot}34\ tons.}$$

30. *A girder is continuous over three spans; the two end spans are each 60 ft. long, and the central span is 90 ft. long; the girder rests freely on its end supports, and carries a distributed load of 1½ tons per ft. run over its whole length. A moving load=2 tons per ft. run advances on to the girder, and covers one of the end spans and the central span. Determine under these conditions the reactions and bending moments at all the supports, and the points of zero bending moment.*
(Edinburgh University.)



$$(M_A+2M_B)L_{AB}+(M_C+2M_B)L_{BC}=-\frac{W_{AB}L^3_{AB}+W_{BC}L^3_{BC}}{4}$$

$$8M_B(60+90)+4M_C\times 90=-3{\cdot}5\times 60^3-3{\cdot}5\times 90^3$$

$$1200M_B+360M_C=-756{,}000-2{,}551{,}000$$

$$3{\cdot}33M_B+M_C=-9187$$

$$(M_B+2M_C)L_{BC}+(M_D+2M_C)L_{CD}=-\frac{W_{BC}L^3_{BC}+W_{CD}L^3_{CD}}{4}$$

$$360M_B+720M_C+480M_C=-2{,}551{,}000-1{\cdot}5\times 216{,}000$$

$$M_B+3{\cdot}33M_C=-7987$$

Substituting in the above equation—

$M_C=$ **−1,726 ft.-tons** $M_B=$ **−2,234 ft.-tons**

$$R_A=\frac{M_B}{L_{AB}}+\frac{W_{AB}L_{AB}}{2}=-\frac{2234}{60}+\frac{3{\cdot}5\times 60}{2}=\mathbf{67{\cdot}8\ tons}$$

$$R_D=\frac{M_C}{L_{CD}}+\frac{W_{CD}L_{CD}}{2}=-\frac{1726}{60}+\frac{1{\cdot}5\times 60}{2}=\mathbf{16{\cdot}2\ tons}$$

$$R_B = \frac{M_A - M_B}{L_{AB}} + \frac{M_C - M_B}{L_{BC}} + \frac{W_{AB}L_{AB}}{2} + \frac{W_{BC}L_{BC}}{2}$$

$$= \frac{2234}{60} + \frac{-1726+2234}{90} + \frac{3{\cdot}5\times 60}{2} + \frac{3{\cdot}5\times 90}{2} = \mathbf{305\ tons}$$

$$R_C = \frac{M_B - M_C}{L_{BC}} + \frac{M_D - M_C}{L_{CD}} + \frac{W_{BC}L_{BC}}{2} + \frac{W_{CD}L_{CD}}{2}$$

$$= -5{\cdot}6 + 28{\cdot}8 + 157{\cdot}5 + 45 = \mathbf{226\ tons.}$$

Zero bending moments.

$R_A x - \frac{3{\cdot}5x^2}{2} = 0$ $\quad 67{\cdot}8x = 1{\cdot}75x^2$ $\quad x = \mathbf{38{\cdot}7\ ft.}$ from A

$R_D y - \frac{1{\cdot}5y^2}{2} = 0$ $\quad 16{\cdot}2 = 0{\cdot}75y$ $\quad y = \mathbf{21{\cdot}6\ ft.}$ from D

$$Mz = \frac{3{\cdot}5z^2}{2} - R_A z - R_B(z-60)$$

$$= 1{\cdot}75z^2 - 67{\cdot}8z - 305(z-60)$$

When $Mz = 0$ $\quad z = \mathbf{76{\cdot}3\ ft.}$ from A or 16·6 ft. from B.

CHAPTER XIII

DEFLECTION OF BEAMS

1. *A cantilever of uniform section is built in at one end and is propped by an elastic column at the free end, the top of which just touches the section when unloaded. Find an expression for the load* P *on the prop when the cantilever supports a uniform distributed load* W. *Modulus of elasticity for cantilever* = E, *and for prop* = E_c. *Length of prop* l, *section of prop* A.

Downward deflection due to $W = \frac{WL^3}{8EI}$ (see p. 510, Vol. I.)

Upward „ „ $P = \frac{PL^3}{3EI}$

$$\text{Difference} = \left(\frac{W}{8} - \frac{P}{3}\right)\frac{L^3}{EI} = \frac{Pl}{AE_c}$$

$$P = W\left(\frac{3}{8\left(1 + \frac{3EIl}{AE_cL^3}\right)}\right)$$

2. *A cantilever consisting of a bundle of circular rods projects* 2 *feet and is loaded with* 1,000 *lb. at the free end. Deflection,* 0·1″. *Maximum stress,* 4 *tons sq. in.*

E = 30,000,000 *lb. sq. in. Find the diameter,* d, *and number,* n, *of the rods.*

$$\delta = \frac{WL^3}{3EI} \quad \text{(See p. 510, Vol. I.)}$$

$$0{\cdot}1 = \frac{1000 \times 24^3 \times 20{\cdot}4}{3 \times 30000000 \times nd^4} \qquad nd^4 = 31{\cdot}3$$

For stress conditions—

$$\frac{1000 \times 24}{4 \times 2240} = \frac{nd^3}{10{\cdot}2} \quad \text{(See p. 452, Vol. I.)}$$

$nd^3=27\cdot3 \quad 31\cdot3=27\cdot3d \quad d=\mathbf{1\cdot15''} \quad n=\frac{27\cdot3}{1\cdot15^3}=\mathbf{18\ rods.}$

3. *A cantilever* 12 *ft. long is propped level at the outer end and supports* 5 *tons* 4 *feet from that end. Find the load on the prop, assumed rigid, and the maximum bending moment.*

Bending moment at wall due to 5 tons load $=5\times8=40$ tons-ft.

Area of bending moment diagram $=\frac{40\times8}{2}=160$ tons-ft.^{2}

Distance of c. of g. of bending moment diagram from end $=\frac{2\times8}{3}+4=9\cdot33$ ft.

Momt. of b. m. diagram about free end $=160\times9\cdot33=1490$.

P=load on prop.

$\frac{PL^3}{3}=1490 \qquad P=\frac{3\times1490}{1728}=\mathbf{2\cdot59\ tons.}$

Bending moment at wall $2\cdot59\times12-40=8\cdot92$ tons-ft.

Bending moment under 5 tons load $2\cdot59\times4=\mathbf{10\cdot36\ tons\text{-}ft.}$

Alternative solution—

Deflection of free cantilever under load $=\frac{5\times8^3}{3EI}$

Slope ,, ,, beyond ,, $=\frac{5\times8^2}{2EI}$

Deflection of free cantilever at end $=\frac{5\times8^3}{3EI}+\frac{5\times8^2\times4}{2EI}$

Upward deflection at end due to P $=\frac{P\times12^3}{3EI}$

$\frac{P\times12^3}{3EI}=\frac{5\times8^3}{3EI}+\frac{5\times8^2\times4}{2EI} \qquad P=\mathbf{2\cdot59\ tons.}$

4. *A cantilever is loaded with an evenly distributed load and is propped at the free end. Find how much the prop must be set below the unloaded position to ensure that the bending moment at the wall shall be one-half as great as when there is no prop. Also find the load on the prop* P. *Length* L=80″; *stress*=5 *tons sq. in.; depth of section (symmetrical)*=8″; E=12,000 *tons sq. in.; load per foot run*=0·2 *ton.*

$$PL=\frac{WL}{4} \qquad P=\frac{W}{4}=\frac{0{\cdot}2\times 80}{4\times 12}=0{\cdot}33 \text{ ton}$$

$$Z=\frac{0{\cdot}2\times 80^2}{12\times 4\times 5}=5{\cdot}3 \qquad I=21{\cdot}2$$

$$\delta=\frac{WL^3}{8EI}-\frac{WL^3}{4\times 3EI}=\frac{WL^3}{24EI}$$

$$\delta=\frac{80\times 0{\cdot}2\times 80^3}{12\times 24\times 12000\times 21{\cdot}2}=\mathbf{0{\cdot}11}''\text{ below.}$$

5. *Find the deflection at the free end of a cantilever of uniform section when loaded with* 4 *tons at the extreme end and* 6 *tons in the middle. Length,* 14 *ft.; depth of section,* 18″; *stress,* 5 *tons sq. inch;* E, 13,000 *tons sq. inch.*

$M=4\times 168+6\times 84=1176$ inch-tons

$Z=\frac{1176}{5}=235 \qquad I=235\times 9=2115$

$$\text{Deflection due to end load}=\frac{4\times 168^3}{3\times 13000\times 2115}=0{\cdot}23''$$

$$\text{Defln. due to middle load}=\frac{\frac{6\times 84\times 84}{2}(84+\frac{2}{3}84)}{13000\times 2115}=0{\cdot}11''$$

Total deflection=**0·34″.**

6. *A cantilever is uniformly loaded throughout its length. Find where a prop must be placed in order that it may support* (i) *one-half;* (ii) *one-quarter of the load, the top of the prop being at the same height as the unbent cantilever.*

Let L=length of cantilever, x=distance from the wall, w=load per unit length. (See p. 512, Vol. I.)

$$\text{Deflection at } x=\delta_x=\frac{w}{24EI}(6L^2x^2+x^4-4Lx^3)$$

Case (i)—

$$\text{Upward deflection at prop due to prop}=\frac{\frac{wL}{2}x^3}{3EI}=\frac{wLx^3}{6EI}$$

$$\frac{wLx^3}{6EI}=\frac{w}{24EI}(6L^2x^2+x^4-4Lx^3)$$

$$\left(\frac{x}{L}\right)^2-8\left(\frac{x}{L}\right)=-6 \qquad x=\mathbf{0{\cdot}838L.}$$

Case (ii)—

Upward deflection at prop due to prop $= \dfrac{wLx^3}{12EI}$

which gives $\left(\dfrac{x}{L}\right)^2 - 6\left(\dfrac{x}{L}\right) = -6$ $\qquad x = \mathbf{1{\cdot}268L}$

i.e., the cantilever must be extended 0·268L beyond the loaded portion with the prop at the extreme end.

7. *A cantilever* 20 *ft. in length is propped level at a point* 15 *ft. from the wall. Find the load on the prop when the cantilever is covered with a load of* 0·5 *ton per ft. run.*

Let l be the length of the cantilever.

x be the distance of the prop from the wall.

(See p. 512, Vol. I.)

Downward deflection at x due to the distrib. load $\left.\vphantom{\frac{1}{1}}\right\} \delta_x = \dfrac{w}{24EI}(6l^2x^2 - 4lx^3 + x^4)$

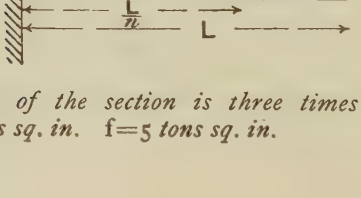
$$= \frac{w}{24EI}\left\{6l^4\left(\tfrac{3}{4}\right)^2 - 4l^4\left(\tfrac{3}{4}\right)^3 + l^4\left(\tfrac{3}{4}\right)^4\right\} = \frac{wl^4}{12EI}$$

Upward deflection due to the prop at x $\quad \delta_p = \dfrac{P \times 3^3 l^3}{3 \times 4^3 \times EI}$

Since the cantilever is level at the prop :

$$\frac{P \times 3^3 l^3}{3 \times 4^3 \times EI} = \frac{wl^4}{12EI} \qquad P = 0{\cdot}594wl = \mathbf{5{\cdot}94\ tons.}$$

8. *Find the deflection at the free end of a cantilever of rectangular section* 12 *ft. long when loaded with* 5 *tons at a point* 8 *ft. from the support.*



The depth of the section is three times the breadth. E=12,000 *tons sq. in.* f=5 *tons sq. in.*

$$\text{Area of the bending moment diagram} = \frac{WL^2}{2n^2}$$

$$\text{Distance of c. of g. from free end} = L - \frac{L}{3n}$$

$$\delta = \frac{WL^3}{6n^3EI}(3n-1) \qquad \frac{WL}{n} = 5\times 8\times 12 = f\frac{BH^2}{6} = \frac{f_9 B^3}{6}$$

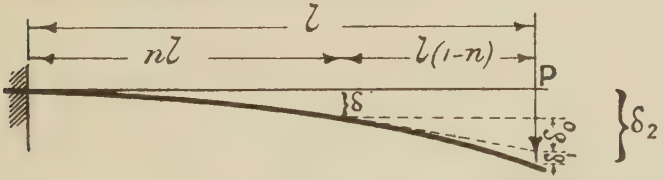
$$B = 4'' \qquad H = 12'' \qquad I = \frac{BH^3}{12} = 576 \qquad n = \tfrac{3}{2}$$

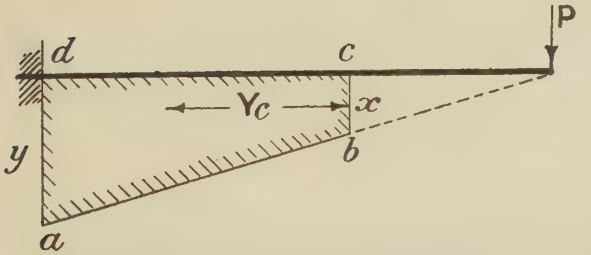
$$\delta = \frac{5\times 144^3\times 7\times 2^3}{6\times 3^3\times 12000\times 576\times 2} = \mathbf{0{\cdot}37\ in.}$$

9. *Show that the deflection at any point of a cantilever distant* nl *from the fixed end* (*where* l *is the total length*) *is given by the expression :*

$$\delta = \frac{n^2(3-n)}{2}\cdot\frac{Pl^3}{3EI}$$

the load P *being applied at the free end*





Draw a tangent to the bent cantilever at the point in question distant $l(1-n)$ from the free end.

$$\delta_1 = \frac{Pl^3}{3EI}(1-n)^3$$

From the slope of the tangent we can get δ_0, which is the area of the bending moment diagram $\times \frac{l(1-n)}{EI}$

$$\delta_0 = \frac{x+y}{2EI} \times nl \times l(1-n) = \frac{x+y}{2EI} nl^2(1-n)$$

But $x = Pl(1-n)$ and $y = Pl$, inserting these values

$$\delta_0 = \frac{Pl(1-n)+Pl}{2EI} nl^2(1-n) \qquad \delta_2 = \frac{Pl^3}{3EI} \qquad \delta = \delta_2 - \delta_1 - \delta_0$$

Substituting the above values we get $\left.\right\}$ $\quad \delta = \frac{Pl^3}{3EI} \times \frac{n^2(3-n)}{2}$.

This result can also be obtained by taking the moment of the area *abcd* (*i.e.* the portion of the bending moment diagram over the length in question).

$$\delta = \frac{x+y}{2EI} \times nl\left(\frac{x+2y}{x+y} \times \frac{nl}{3}\right) \qquad \text{(See p. 61, Vol. I.)}$$

$$= \frac{x+2y(nl)^2}{6EI} = \frac{xn^2l^2+2yn^2l^2}{6EI} = \frac{Pn^2l^3(1-n)+2Pl^3n^2}{6EI}$$

$$= \frac{Pl^3}{6EI}(n^2-n^3+2n^2) = \frac{Pl^3n^2}{6EI}(3-n) = \frac{n^2(3-n)}{2} \times \frac{Pl^3}{3EI}.$$

The first term disappears when $n=1$; the end deflection then becomes $\frac{Pl^3}{3EI}$ as found by other methods.

10. *A pole for supporting a trolley wire is* 20 *ft. high and has a horizontal arm* 12 *ft. in length. The pole is* $8\frac{1}{2}$ *inches diameter and* $\frac{1}{4}$ *inch thick. Find the vertical and horizontal movements of the end of the arm when loaded with* 180 *lb. The arm itself is sufficiently stayed to prevent any material bending. Young's Modulus of Elasticity* 12,000 *tons per square inch.*

$$\text{Z for post} = \frac{8{\cdot}5^4-8^4}{10{\cdot}2\times 8{\cdot}5} = 13 \qquad I = 13\times 4{\cdot}25 = 55{\cdot}2$$

$$\delta = \frac{ML^2}{2EI} = \frac{180\times 144\times 240^2}{2\times 12000\times 55{\cdot}2\times 2240} = \mathbf{0{\cdot}50''} \text{ horizontally.}$$

$$\text{Slope at top of pole} = \frac{ML}{EI} = \frac{180\times 240\times 144}{12000\times 2240\times 55{\cdot}2} = \frac{1}{239}$$

$$\delta \text{ vertically} = \tfrac{144}{239} = \mathbf{0{\cdot}60''}.$$

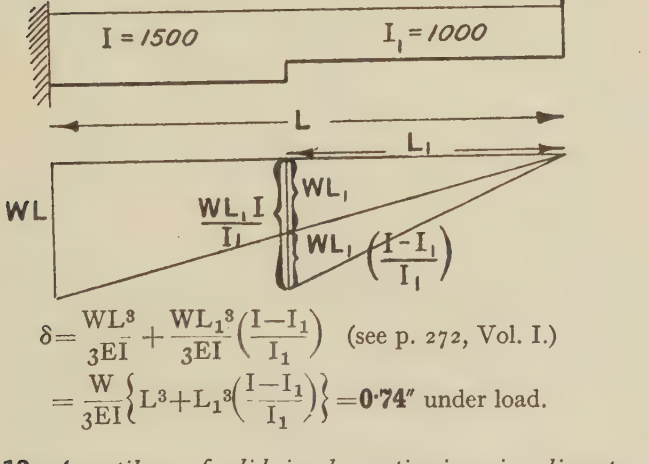
11. *A horizontal beam of uniform section is built into walls distant* l *ft. apart, one of which settles vertically through a distance* x *ft. while the other remains in its original position. Show that the reactions at the ends are* $\frac{12EIx}{l^3}$, *and the bending moments at the ends are* $\frac{6EIx}{l^2}$.

There is a virtual joint in the middle of the beam (see p. 504, Vol. I.).

$$\delta = \frac{x}{2} = \frac{Rl^3}{2^3 \times 3EI} \qquad R = \frac{12EIx}{l^3} \qquad M = \frac{Rl}{2} = \frac{6EIx}{l^2}.$$

12. *The figure represents the elevation of a loaded cantilever. Find the deflection immediately under the load* W.

W = 2 *tons.* E = 12,000 *tons per sq. in.*



$$\delta = \frac{WL^3}{3EI} + \frac{WL_1^3}{3EI}\left(\frac{I-I_1}{I_1}\right) \quad \text{(see p. 272, Vol. I.)}$$

$$= \frac{W}{3EI}\left\{L^3 + L_1^3\left(\frac{I-I_1}{I_1}\right)\right\} = \mathbf{0{\cdot}74''} \text{ under load.}$$

13. *A cantilever of solid circular section is* 12 *ins. diameter for a length of* 6 *ft. from the supporting wall; the diameter then changes abruptly to* 8 *ins. at which it remains for a length of* 7 *ft. Find the deflection when loaded with* 3 *tons on the extreme end. The weight of the cantilever itself may be neglected.*

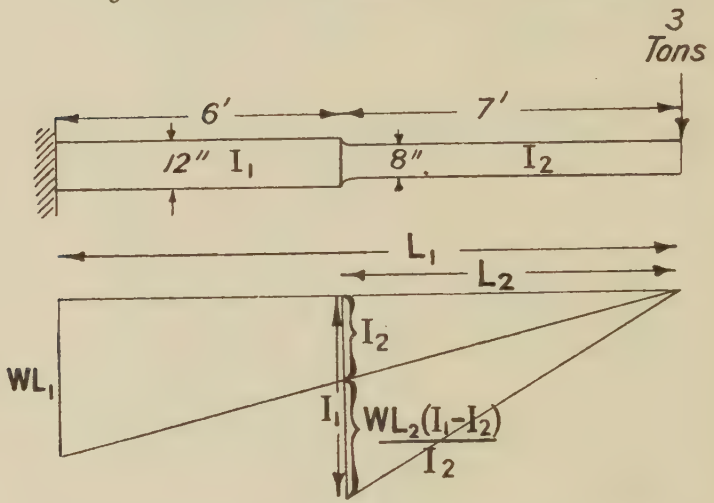
Modulus of elasticity = 12,000 *tons per sq. in.*

(See p. 272, Vol. I.)

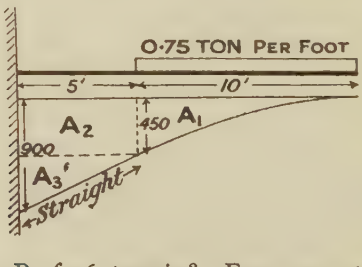
$$\delta = \left\{ \frac{WL_1 \times L_1}{2} \times \tfrac{2}{3}L_1 + \frac{WL_2(I_1 - I_2)L_2}{2I_2} \times \tfrac{2}{3}L_2 \right\} \frac{1}{EI_1}$$

$$= \frac{WL_1^3}{3EI_1} + \frac{WL_2^3}{3EI_1} \times \frac{I_1 - I_2}{I_2}$$

$$= \frac{3 \times 20{\cdot}4}{3 \times 12000 \times 12^4} \left\{ 156^3 + 84^3 \left(\frac{12^4 - 8^4}{8^4} \right) \right\} = \mathbf{0{\cdot}51 \text{ in.}}$$



14. *A cantilever* 15′ *long is loaded for* 10′ *from the free end with a load of* $\frac{3}{4}$ *ton per ft. run. The section is rect-*



angular, H=3B, *f*=6 *tons-in.*^{2} E=13,000 *tons-in.*^{2} *Find the deflection.*

Max. B.M. $=7{\cdot}5\times10=75$ tons-ft. $=900$ tons-ins.

$$M=fZ \qquad \therefore\ Z=\tfrac{900}{6}=150=\frac{9B^3}{6}$$

$B=\sqrt[3]{100}=4{\cdot}64$ ins. $\qquad H=13{\cdot}92$ ins.

$I=150\times6{\cdot}96=1046$ in.^{4}

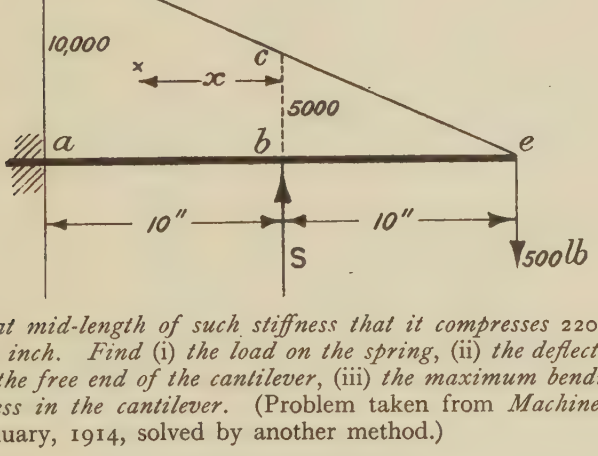
$A_1=\frac{1}{3}\times120\times450=18{,}000.$ $\qquad L_{c1}=\frac{3}{4}\times120=90''$

$A_2=60\times450=27{,}000$ $\qquad L_{c2}=30+120=150''$

$A_3=30\times450=13{,}500$ $\qquad L_{c3}=40+120=160''$

$$\delta=\frac{AL_c}{EI}=\frac{18000\times90+27000\times150+13500\times160}{13000\times1046}=\mathbf{0{\cdot}57\ in.}$$

15. *A cantilever* 6″ *wide* $\frac{3}{8}$″ *deep and* 20″ *long is loaded at its outer end with* 500 *lb. A helical spring is placed under*



it at mid-length of such stiffness that it compresses 220 *lb. per inch. Find* (i) *the load on the spring*, (ii) *the deflection at the free end of the cantilever*, (iii) *the maximum bending stress in the cantilever.* (Problem taken from *Machinery*, January, 1914, solved by another method.)

Area $abcd=75{,}000$

$$x=\tfrac{10}{3}\left(\frac{2\times2+1}{2+1}\right)=5{\cdot}56''$$ (See p. 60, *M.A.E.*, Vol. I.)

$$I=\frac{bh^3}{12}=\frac{6\times0{\cdot}375^3}{12}=0{\cdot}0264$$

Downward deflection due to 500 lb. $\qquad \delta_b=\dfrac{75000\times5{\cdot}56}{30000000\times0{\cdot}0264}=0{\cdot}527''.$

Let the load on the spring be P $= 220y$

$$\text{Upward deflection due to spring} = \frac{220y \times 10^3}{3 \times 30000000 \times 0{\cdot}0264} = 0{\cdot}0925y$$

$\therefore\ y = 0{\cdot}527 - 0{\cdot}0925y$ and $y = 0{\cdot}482''$.

$$\text{Upward deflection due to spring} = 0{\cdot}0925 \times 0{\cdot}482 = 0{\cdot}0445''$$

(i) Load on spring $= 220 \times 0{\cdot}482 =$ **106 lb.**

(ii) $\delta_e = \dfrac{500 \times 20^3}{3 \times 30000000 \times 0{\cdot}0264} - 2 \times 0{\cdot}0445 =$ **1·59″.**

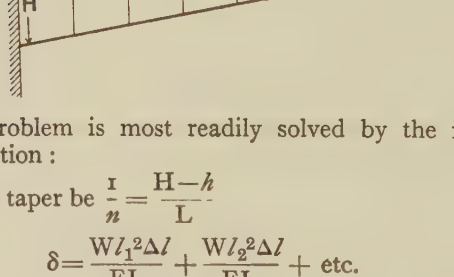
(iii) $M_a = 500 \times 20 - 220 \times 0{\cdot}482 \times 10 = 8940$ lb.-inches.

$$Z = \frac{6 \times 0{\cdot}375^2}{6} = 0{\cdot}14$$

$$f = \frac{8940}{0{\cdot}14} =$$ **64,000 lb.** per sq. inch.

Provided the elastic limit is not exceeded.

16. *A cantilever of rectangular section is* 3 *ins. deep at the free end and* 9 *inches at the fixed end. The width* b *of the section throughout is* 3 *ins. It supports a weight of* 1·1 *tons at the free end. Find the defln. Length of cantilever* (L), 60″ ; *modulus of elasticity* (E), 700 *tons sq. in.*



This problem is most readily solved by the following approximation :

Let the taper be $\dfrac{1}{n} = \dfrac{H-h}{L}$

$$\delta = \frac{W l_1^2 \Delta l}{EI_1} + \frac{W l_2^2 \Delta l}{EI_2} + \text{etc.}$$

$$I_1 = \frac{b}{12}\left(h + \frac{l_1}{n}\right)^3 \qquad I_2 = \frac{b}{12}\left(h + \frac{l_2}{n}\right)^3$$

Hence $$\delta = \frac{W\Delta l}{E\frac{b}{12}} \cdot \left(\frac{l_1^2}{\left(h+\frac{l_1}{n}\right)^3} + \frac{l_2^2}{\left(h+\frac{l_2}{n}\right)^3} + \text{etc.} \right)$$

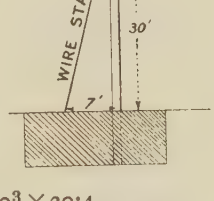
In this case $\Delta l = 10''$ $n = 10$

$$\delta = \frac{1{\cdot}1 \times 10}{700 \times 0{\cdot}25}\left(\frac{25}{43} + \frac{225}{91{\cdot}3} + \frac{625}{166} + \frac{1225}{275} + \frac{2025}{421} + \frac{3025}{614}\right)$$

$$= \mathbf{1{\cdot}32 \text{ inches.}}$$

By integration $\delta = 1{\cdot}315$ ins. Another method of dealing with tapered beams is given in Vol. I., p. 272.

17. *The figure represents a wooden mast of uniform diameter built into a concrete bed. It is subjected to a horizontal pull* P *at a distance of* 30 *ft. from the ground. The length of the wire stay is such that it just becomes taut when the bending stress in the mast is* 1,000 *lb. per sq. in. Show that when the mast is unloaded the slack in the stay is* 0·8 *inch (approximately). Diameter of mast,* 12 *ins.; Young's modulus for the mast,* 2,000,000 *lb. per sq. in.*



Length of stay

$= \sqrt{7^2+30^2} = 30{\cdot}8$ ft. $\quad M = fz$

$$P \times 360 = \frac{1000 \times 12^3}{10{\cdot}2} = 472 \text{ lb.}$$

Deflection due to $P = \frac{PL^3}{3EI} = \frac{472 \times 360^3 \times 20{\cdot}4}{3 \times 2000000 \times 12^4} = 3{\cdot}6''$

Component of deflection in direction of stay $= 3{\cdot}6 \times \frac{7}{30{\cdot}8} = \mathbf{0{\cdot}82''}$.

18. *A beam of* 20 *feet span is centrally loaded. Find the deflection at the middle when the ends of the beam slope* 1°.

Slope at ends

$$= \frac{\text{Area of half the bending moment diagram}}{EI} = \frac{Wl^2}{16EI}$$

Slope corresponding to $1° = \frac{\pi}{180}$

$$\delta = \text{slope} \times \frac{l}{3} = \frac{\pi l}{3 \times 180} = \frac{3{\cdot}14 \times 20 \times 12}{3 \times 180} = \mathbf{1{\cdot}4''}.$$

19. *Find the relative strength and deflection of a solid* 2*-inch square beam and one of the same size consisting of a*

bundle of $\frac{1}{2}$-inch square rods, all other conditions being the same. Friction between the rods being neglected.

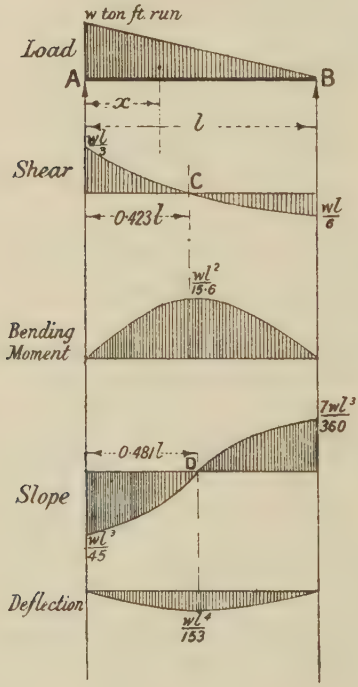
The strength varies as the modulus of the section:

$$\frac{\text{Solid}}{\text{Bundle}} = \frac{\frac{S^3}{6}}{\frac{16Sr^3}{6}} = \frac{2^3}{16 \times 0{\cdot}5^3} = 4.$$

The deflection varies inversely as the moment of inertia of the section:

$$\frac{\text{Bundle}}{\text{Solid}} = \frac{\frac{16Sr^4}{12}}{\frac{S^4}{12}} = \frac{16 \times 0{\cdot}5^4}{2^4} = 16.$$

Thus the solid beam is 4 times as strong and 16 times as stiff as the beam made up of a bundle of rods.



20. *A free-ended beam supports a wall of triangular elevation, the base coincides with the beam and the vertical side is over one of the supports and the apex over the other. Construct load, shear, bending moment, slope and deflection diagrams.*

21. *Three beams, each 20 ft. long placed side by side and freely supported at each end, support jointly a central load* W *tons. The two outside beams are 9″ deep and 3″ broad; the centre one is 3″ deep and 9″ broad. The stress in any beam must not exceed 2 tons per sq. inch. Find* W *and the load supported by each beam.*

$$I_1 \text{ for middle beam} = \frac{9 \times 27}{12} = 20{\cdot}25$$

$$I_2 \text{ for outside beams} = \frac{2 \times 3 \times 729}{12} = 364{\cdot}5$$

W_1=load on middle beam, W_2=load on both outside beams.

Since all deflect same amount $\frac{W_1}{W_2} = \frac{I_1}{I_2} = \frac{20{\cdot}25}{364{\cdot}5} = \frac{1}{18}$

$$W_1 = \frac{W}{19} \qquad W_2 = \tfrac{18}{19}W \qquad Z_1 = \frac{9 \times 3^2}{6} = 13{\cdot}5$$

$$Z_2 = \frac{2 \times 3 \times 9^2}{6} = 81 \qquad \frac{W_2 l}{4} = f z_2 \qquad W_2 = \frac{2 \times 81 \times 4}{240} = \mathbf{2{\cdot}7 \text{ tons}}$$

$$W = \tfrac{19}{18}W_2 = \tfrac{19}{18} \times 2{\cdot}7 = 2{\cdot}85 \text{ tons} \qquad W_1 = \frac{W_2}{18} = \frac{2{\cdot}7}{18} = \mathbf{0{\cdot}15 \text{ ton}}$$

Each outside beam supports $\frac{2{\cdot}7}{2} = \mathbf{1{\cdot}35 \text{ tons.}}$

22. *A girder 300 ft. long rests on three piers equally spaced and is loaded with 1·5 tons per ft. run. Find* (i) *the load on each pier;* (ii) *the bending moment and the shear at the central pier;* (iii) *the position of the points of zero bending moment.*

(See p. 532, Vol. I.)

$$R_1 = R_3 = \tfrac{3}{16}wL = \frac{3 \times 1{\cdot}5 \times 300}{16} = \mathbf{84{\cdot}4 \text{ tons}}$$

$$R_2 = \tfrac{5}{8}wL = \frac{5 \times 1{\cdot}5 \times 300}{8} = \mathbf{281{\cdot}2 \text{ tons.}}$$

Bending moment at central pier $= 84{\cdot}4 \times 150 - \frac{1{\cdot}5 \times 150^2}{2} = \mathbf{-4220 \text{ tons-ft.}}$

Shearing force $= \tfrac{1}{2} \times 281{\cdot}2 = \mathbf{140{\cdot}6 \text{ tons}}$

Position of points of zero bending moment $84{\cdot}4 \times x = \frac{1{\cdot}5x^2}{2}$

$x = \mathbf{112{\cdot}5 \text{ ft.}}$ from the end piers.

23. *A shaft runs in three short bearings. Find the maximum bending moment when* (i) *the bearings are all in line ;* (ii) *the central bearing is set low in order to make the load the same on all the bearings. Find the distribution of load on the bearings required to make the bending moment a minimum, also the amount the central bearing must be set low to secure this condition.*

(i) See p. 532, Vol. I., where it is shown that the maximum bending moment occurs at the central bearing, and is $-\frac{w\mathrm{L}^2}{\mathbf{32}}$, where w is the weight of shaft per inch run and L is the whole length of the shaft.

(ii) The central bearing supports one-third of the weight of the shaft. Then :

$$\left.\begin{array}{l}\text{Upward bending moment due to}\\ \text{load on central bearing}\end{array}\right\} = \frac{w\mathrm{L}^2}{3\times 4} = \frac{w\mathrm{L}^2}{12}$$

$$\text{Bending moment at middle if there were no bearing} = \frac{w\mathrm{L}^2}{8}$$

$$\left.\begin{array}{l}\text{Resulting bending moment}\\ \text{at central bearing}\end{array}\right\} = \frac{w\mathrm{L}^2}{8} - \frac{w\mathrm{L}^2}{12} = \frac{w\mathbf{L}^2}{\mathbf{24}}$$

The amount the central bearing must be set low to ensure an equal load on each is :

$$\frac{5\mathrm{W}l^3}{384\mathrm{EI}} - \frac{\mathrm{W}l^3}{3\times 48\mathrm{EI}} = \frac{\mathrm{W}l^3}{164\mathrm{EI}}$$

In this case there is no reversal of bending moment as in case (i), and the maximum is not at the central bearing.

The bending moment at any section distant l from the end bearing is $\mathrm{M} = \frac{w}{3}\mathrm{L}l - \frac{wl^2}{2}$ $\qquad \frac{d\mathrm{M}}{dl} = \frac{w\mathrm{L}}{3} - \frac{2wl}{2}$

which gives a maximum when $\frac{\mathrm{L}}{3} = l$.

The bending moment is then $\mathrm{M}_{\max.} = \frac{w}{3}\cdot\frac{\mathrm{L}^2}{3} - \frac{w\mathrm{L}^2}{2\times 9} = \frac{w\mathrm{L}^2}{18}$

Thus—

$$\frac{\text{B.M. when the load is the same on all bearings}}{\text{B.M. when the bearings are in line}} = \frac{32}{18} = 1{\cdot}78.$$

It is often urged by designers that the central bearing should be set low in order to equalise the load on each, but

by doing so the bending moment on the shaft is 78 per cent. greater than when the bearings are in line. The best result is obtained when the bending moment over the middle bearing is equal to the bending moment between the end and middle bearings. Let the span between $x.x$, Fig. 525, Vol. I., be l_c.

The reaction at the end bearings $= \frac{wl_1}{2}$

The bending moment at the central bearing

$$= M_c = \frac{wl_1}{2} \times \frac{l_c}{2} + \frac{wl_c}{2} \times \frac{l_c}{4} \qquad M_c = \frac{w}{4}\left(l_1 l_c + \frac{l_c^2}{2}\right)$$

The bending moment in the middle of the l_1 span :

$$M_1 = \frac{wl_1^2}{8} \qquad \text{make } M_1 = M_c$$

Then $\frac{l_1^2}{8} = \frac{2l_1 l_c + l_c^2}{8}$ and $l_1^2 - 2l_1 l_c + l_c^2 = 2l_c^2$

$$l_1 - l_c = \sqrt{2}l_c \qquad l_1 = l_c(\sqrt{2}+1) \quad \textbf{but} \quad l_c = L - 2l_1$$

Hence $l_1 = (\sqrt{2}-1)L = 0{\cdot}414L \qquad l_c = 0{\cdot}172L$

The load on the central bearing is then :

$$0{\cdot}172wL + 0{\cdot}414wL = \mathbf{0{\cdot}586}wL$$

or, say, 58 per cent. of the whole weight of the shaft, and 21 per cent. on each end bearing.

The amount the central bearing must be set low to get the minimum bending moment is :

$$\frac{wL^4}{384EI}(5 - 8 \times 0{\cdot}586) = \frac{wL^4}{1230EI}$$

The bending moment is $\frac{w \times 0{\cdot}414^2 L^2}{8} = \frac{wL^2}{46{\cdot}7}$

Hence $\frac{\text{BM. under best conditions}}{\text{B.M. when bearings are in line}} = \frac{32}{46{\cdot}7} = 0{\cdot}685$

23a. *Find the bending stress in an unloaded shaft running in three bearings. Length overall,* 12 *feet ; diameter,* 10 *inches ; weight per cubic inch,* 0·28 *lb.* (i) *Bearings in line ;* (ii) *load equal on all bearings ;* (iii) *central bearing set low to give least bending moment.*

Weight per inch run $= 78{\cdot}54 \times 0{\cdot}28 = 22$ lb.

(i) Bending moment $= \frac{22 \times 144^2}{32} = 14{,}300$ inch-lb.

Stress $f = \frac{M}{Z} = \frac{14300 \times 10{\cdot}2}{10^3} =$ **145 lb.** sq. inch.

(ii) Stress $= 145 \times 1{\cdot}78 =$ **258 lb.** sq. inch.

(iii) Stress $= 145 \times 0{\cdot}685 =$ **100 lb.** sq. inch.

In this case the central bearing would be set low by :

$$\frac{22 \times 144^4 \times 20{\cdot}4}{1230 \times 30{,}000{,}000 \times 10^4} = \frac{1}{1910}''.$$

It will be seen from the above that the stress due to this cause is by no means serious, and under ordinary working conditions it is out of the question to attempt to work to $\frac{1}{2000}$ inch in setting up bearings.

If bearings get seriously out of line through wear or otherwise the stress may become great enough to fracture the shaft, especially as the stress is reversed at every revolution.

To keep the wear equal on all bearings the central bearing should be made longer than the end bearings in the ratio $\frac{58}{22} = 2{\cdot}6$, or for the first case considered $\frac{10}{3} = 3{\cdot}3$, or in general about 3.

24. *A shaft* 8 *ins. diam. and* 20 *ft. long is supported on three bearings and carries two wheels, each weighing* 2 *tons, which are symmetrically placed about the middle of the shaft and* 6 *ft. apart. Find by how much the central bearing must be set low in order that each bearing may be equally loaded.* E = 12,000 *tons sq. in.*

Weight of shaft $= \frac{0{\cdot}785 \times 8^2 \times 0{\cdot}28 \times 20 \times 12}{2240} = 1{\cdot}51$ tons.

Deflection due to weight of shaft when resting freely on end bearings $= \frac{5Wl^3}{384EI} = 0{\cdot}112''$

Bending moment in middle of shaft due to wheels $= 2 \times 84 = 168$ inch-tons.

Moment of bending moment diagram for wheels about one end bearing $= 168(36 \times 102 + \frac{84}{2} \times \frac{2}{3} \times 84) = 1{,}012{,}000$.

Central deflection due to wheels $= \frac{1012000 \times 20{\cdot}4}{12000 \times 8^4} = 0{\cdot}42''$.

Deflection due to weight of shaft and wheels when resting freely on end bearings $= 0{\cdot}112 + 0{\cdot}42 = 0{\cdot}532''$.

Load on central bearing $= \frac{4+1\cdot51}{3} = 1\cdot84$ tons.

Upward deflection due to load on central bearing :

$$= \frac{1\cdot84 \times 240^3 \times 20\cdot4}{48 \times 12000 \times 8^4} = 0\cdot22''$$

Thus the central bearing must be set low by :

$$0\cdot532 - 0\cdot22 = \mathbf{0\cdot312''}.$$

25. *A girder, assumed to be rigid, is supported horizontally on three piers, A, B, C, equally spaced and which rest on sloping ground. The piers yield 0·1" for the following loads : A, 30 tons ; B, 40 tons ; C, 60 tons. Find the amount each pier yields when a load of 120 tons is placed at B. Also find where the load should be placed in order that the girder may remain horizontal. Span of each, 20 ft.*

Let x=the compression of the piers.

$$W_A = 300x_A, \qquad W_B = 400x_B, \qquad W_C = 600x_C$$

$$300x_A + 400x_B + 600x_C = 120$$

$$x_B = \frac{x_A + x_C}{2}, \; x_A = 2x_C \quad \text{(by moments)} \qquad x_B = \frac{3x_A}{4}$$

Substitute in the above equation :

$$300x_A + 400 \times \frac{3x_A}{4} + \frac{600x_A}{2} = 120$$

$$x_A = \mathbf{0\cdot13''} \qquad x_B = \mathbf{0\cdot10''} \qquad x_C = \mathbf{0\cdot066''}.$$

When the girder remains horizontal :

$$x_A = x_B = x_C$$

Ratios $W_A = 30$ $W_B = 40$ $W_C = 60$ Total $= 130$.

$$W_A = \frac{30 \times 120}{130} = 27\cdot7 \qquad W_B = 36\cdot9 \qquad W_C = 55\cdot4 \text{ tons.}$$

Let x=distance of load from B_1 between B and C :

$$27\cdot7(20+x) + 36\cdot9x = 55\cdot4(20-x)$$

$x = \mathbf{4\cdot6\ feet}$ from B between B and C.

26. *A water main is supported by two longitudinal beams over a 30-ft. span. Find* (i) *the weight supported by each beam ;* (ii) *the stress in the beams and in the main. Depth of main, 3 ft. ; depth of beams, 2 ft. ; moment of inertia of main* (I_m), *9000 in. units ; moment of inertia of each beam*

(I), 6000 *in. units ;* E_m *for main,* 12,000 *tons sq. in. ;* E *for beams,* 8000 *tons sq. in. ; weight of main and water per ft. run,* 600 *lb. ; weight of each beam per ft. run,* 240 *lb.*

The deflection of the main is the same as that of the beams :

$$\delta_m = \frac{5 \times W_m \times 360^3}{384 \times 1200 \times 9000} = \frac{5 \times W \times 360^3}{384 \times 8000 \times 2 \times 6000}$$

$W_m = 1{\cdot}125\,W$ where W is the weight borne by the two beams and W_m the weight taken by the main :

$W_m + W = 30(600+480) = 32{,}400$ lb.

$2{\cdot}125W = 32400$ $W = 15{,}200$ lb. or **7600 lb.** on each beam.

$W_m = 17{,}100$ lb.

$$\text{Stress in main} = \frac{17100 \times 360 \times 18}{8 \times 9000} = \mathbf{1540\ lb.}\ \text{sq. in.}$$

$$\text{Stress in beams} = \frac{15200 \times 360 \times 12}{8 \times 12000} = \mathbf{680\ lb.}\ \text{sq. in.}$$

27. *An initially straight flat steel bar* 1″ *wide in the middle and tapering to nothing at the ends,* 0·25″ *thick, is bent to a semicircle, the ends are held in position by a wire attached to them. Find the tension in the wire* (t). *Bending stress in bar* (f), 8 *tons sq. in.* E = 12,500 *tons sq. in.*

$$M = \frac{EI}{\rho} = \frac{fI}{y} \qquad \text{(See p. 506, Vol. I.)}$$

$$\rho = \frac{12500 \times 0{\cdot}125}{8} = 200'' \qquad M = t\rho = fz$$

$$t = \frac{8 \times 0{\cdot}25^2 \times 2240}{6 \times 200} = \mathbf{0{\cdot}93\ lb.}$$

28. *The following results were obtained from a bending test of a beam of rectangular section* 4″ *deep,* 2″ *wide, on a span of* 60″. *Find* E *for the material.*

Load in middle, lb.	1000	2000	3000	4000	5000	6000
Deflection, inches	0·023	0·037	0·052	0·066	0·081	0·095

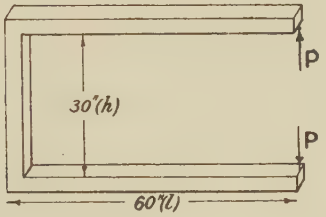
Find the mean deflection by differences thus :

0·095	0·081	0·066
0·052	0·037	0·023
0·043	0·044	0·043

Mean 0·0433″ for 3000 lb.

$$E=\frac{WL^3}{48\delta I}=\frac{3000\times 60^3\times 12}{48\times 0{\cdot}0433\times 2\times 4^3}=\textbf{29,400,000 lb. sq. in.}$$

29. *Find the load* P *required to spring open the frame shown in the figure by* 2 *ins.* E = 12,000 *tons sq. in. Section* 3·5″ *square.*



$$\text{Deflection of horizontal arm} = \frac{Pl^3}{3EI}$$

$$\text{Bending moment on vertical} = Pl$$

$$\text{Area of bending moment diagram} = Plh$$

$$\text{Slope of ends of vertical} = \frac{Plh}{2EI}$$

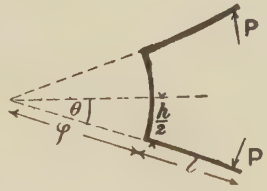
$$\left.\begin{matrix}\text{Deflection at ends of horizontal}\\ \text{arm due to bending of vertical}\end{matrix}\right\} = \frac{Pl^2h}{2EI}$$

$$\text{Total deflection} = \frac{Pl^2}{EI}\left(\frac{l}{3}+\frac{h}{2}\right)=1 \text{ inch}$$

$$P=\frac{12000\times 3{\cdot}5^4}{12\times 60^2(20+15)}=\textbf{1·19 tons.}$$

Alternative solution for deflection due to bending of vertical.

Deflection of each tip of horizontal arm due to bending of vertical $=l\theta$ $\qquad \frac{h}{2}=\rho\theta.$



But $\rho=\frac{EI}{M}$

$$\therefore \delta=l\theta=\frac{lh}{2\rho}=\frac{lhM}{2EI} \quad \delta=\frac{Pl^2h}{2EI}.$$

30. *A R.S.J.* 30 *ft. long rests symmetrically across the top of a wall* 16 *ft. wide. A weight of* 9 *tons is placed at*

each extreme end; neglecting the weight of the beam, find (1) *the stress in the material;* (2) *the radius of curvature of that portion of the beam which is over the wall;* (3) *the greatest distance between the underside of the beam and the wall.*

R.S.J. B=8 *ins.* H=14 *ins.* t=1·2 *ins.*

E=12,000 *tons per sq. inch.*

Approx. $Z = 8 \times 14 \times 1{\cdot}2 = 134{\cdot}4$. $I = 940{\cdot}8$.

(i) $f = \dfrac{9 \times 7 \times 12}{134{\cdot}4} =$ **5·6 tons** sq. in.

(ii) $\rho = \dfrac{EI}{M} = \dfrac{12000 \times 940{\cdot}8}{9 \times 84} =$ **15,000 ins.**

(iii) $\delta = \dfrac{Ml^2}{8EI} = \dfrac{9 \times 84 \times 192 \times 192}{8 \times 12000 \times 940{\cdot}8} =$ **0·31 in.**

31. *A beam of uniform section rests symmetrically on two piers. It is centrally loaded, and the deflection beneath the load is 2 ins. Find the slope θ of the beam at the piers and the height of the free ends above the level of the piers. Length of beam, 36 ft. Distance apart of piers, 26 ft. Neglect the weight of the beam.*

$$\delta_c = \frac{Wl^3}{48EI} = \frac{1}{6}$$

Bending moment under load $= \dfrac{W \times 26}{4}$

$EI\theta = \dfrac{13W}{2} \times \dfrac{13}{2} = \dfrac{169W}{4}$ at the pier. (See p. 508, Vol. I.)

$EI\delta_e = \dfrac{169W}{4} \times 5$ at the free end.

$EI\delta_c = \dfrac{2}{3} \times \dfrac{169W}{4} \times 13 = \dfrac{EI}{6}$ at the centre.

$\dfrac{W}{EI} = \dfrac{1}{6} \times \dfrac{48}{26^3} = \frac{1}{2200}$ ft.-tons.

$\delta_e = \dfrac{W}{EI} \times \dfrac{5 \times 169}{4} = 0{\cdot}096' =$ **1·15 ins.** at the free end.

$\theta = \dfrac{W}{EI} \times \dfrac{169}{4} = \dfrac{42{\cdot}25}{2200} =$ **0·0192 radian** at the pier.

32. *Find the maximum bending moment and central deflection of a beam loaded thus :* A, 2 *tons at* 3 *ft. from left abutment ;* B, 1·5 *tons at* 5 *ft. ;* C, 3 *tons at* 7 *ft. ;* D, 4 *tons at* 12 *ft. Length of beam,* 20 *feet. R.S.J. section* $H=2b=12t$. *Stress* 6 *tons sq. inch.* $E=12{,}000$ *tons per sq. inch.*

Reactions : $R_2\times 20=2\times 3+1{\cdot}5\times 5+3\times 7+4\times 12$

$R_2=4{\cdot}125$ tons $\qquad R_1=6{\cdot}375$ tons

Bending moment :

$M_A=6{\cdot}375\times 3=19{\cdot}12$ tons-feet

$M_B=6{\cdot}375\times 5-2\times 2=27{\cdot}87$ tons-feet

$M_C=4{\cdot}125\times 13-4\times 5=33{\cdot}5$ tons-feet (maximum)

$M_D=4{\cdot}125\times 8=33$ tons-feet

$$Z=\frac{33{\cdot}5\times 12}{6}=67=btH=\frac{H^3}{24}$$

$H=11{\cdot}7''$, say $12''$, $\quad b=6''$, $\quad t=1''$.

Deflection (see p. 523, Vol. I.) :

Load.	Position.	K	W	KW
A	0·15	0·0090	2	0·0180
B	0·25	0·0142	1·5	0·0213
C	0·35	0·0180	3	0·0540
D	0·60	0·0196	4	0·0784
				0·1717

$$\delta=\frac{0{\cdot}1717\times 240^3}{12000\times 67\times 6}=\mathbf{0{\cdot}49''}.$$

As a rough check. If the load were evenly distributed $\delta=0{\cdot}39''$, and if centrally placed $\delta=0{\cdot}63''$; with the distribution given, it should be between the two, which it is.

33. *A beam* 40 *ft. long and freely supported at each end is loaded as follows :*

Distance from left-hand end (ft.)	4	12	16	28	32
Load in tons	2	3	1	6	4

Find the deflection at a point 24 *ft. from the same end.*

E . . . . . 12,000 *tons per square in.*

I . . . . . 2,560 *in.*^{4} *units.*

(See p. 523, Vol. I.)

Position of load.	K	KW
0·1	0·0055	0·0110
0·3	0·0150	0·0450
0·4	0·0180	0·0180
0·7	0·0165	0·0990
0·8	0·0120	0·0480
		0·2210

$$\delta = \frac{0{\cdot}221 \times 480^3}{12000 \times 2560} = \mathbf{0{\cdot}8''}.$$

Rough check. Assume that the load is evenly distributed. Then :

$$\delta = \frac{5Wl^3}{384EI} = \frac{5 \times 16 \times 480^3}{384 \times 12000 \times 2560} \qquad \delta = 0{\cdot}75''.$$

34. *A 3″ square bar bent to form an equilateral triangle ABC with the base BC horizontal but not joined at the apex. Length of sides 5 ft. Find the horizontal force* P *required to open the apex by 6 ins.* E = 12,000 *tons sq. in.*

Take a point on AB distant x from A.

The work done by P in bending AB is

$$\frac{1}{2EI}\int M_x^2 dx = \frac{P^2 \cos^2 30}{2EI}\int_0^{60} x^2 dx$$

$$= \frac{3P^2}{8EI} \times \frac{60^3}{3} = \frac{27000P^2}{EI}$$

Work done by P in bending BC :

Bending moment $= P \times \overline{AB} \cos 30 = \frac{\sqrt{3}}{2} P \times \overline{AB}$

$$\text{Work done} = \frac{P^2 \times \overline{AB}^2 \times 3 \times \overline{BC}}{2EI \times 4} = \frac{810000P^2}{EI}$$

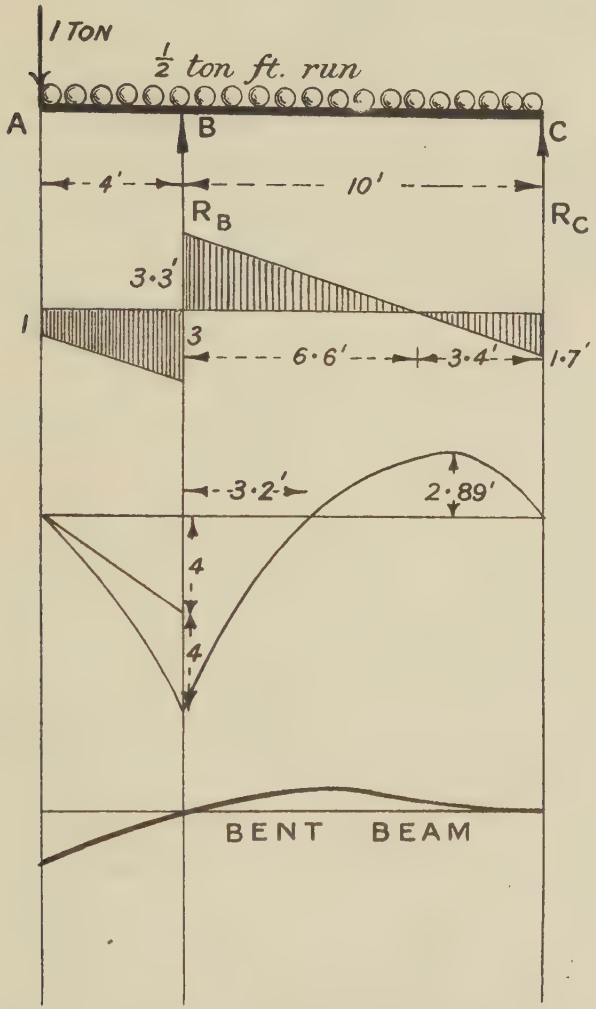
Total work done in bending the members of the triangle, neglecting the direct strains on the members, which are very small in comparison with the bending.

$$\text{Work} = (2 \times 27000 + 810000)\frac{P^2}{EI} = \frac{864000P^2}{EI}$$

We also have the work done by $P = P \times 6$:

$$\therefore\ 6P = \frac{864000P^2}{EI} \qquad P = \frac{6 \times 12000 \times 3^4}{864000 \times 12} = \mathbf{0{\cdot}562\ ton.}$$

35. *A beam* ABC 14 *ft. long is supported at* B *and* C,



B *being* 4 *ft. from* A. *It is loaded with* 0·5 *ton per ft. run over the whole beam and* 1 *ton at the overhanging end* A.

Construct diagrams of shearing force and bending moment, also the form of the bent beam.

Taking moments about B $1\times4=R_c\times10$

$R_c=0{\cdot}4$ ton due to load at A.

Also $0{\cdot}5\times4\times2=R_c'\times10$

$R_c'=0{\cdot}4$ ton due to distributed load on AB.

$\frac{0{\cdot}5\times10}{2}=R_c''=2{\cdot}5$ tons due to distributed load on BC.

$R_c=2{\cdot}5-0{\cdot}4-0{\cdot}4=1{\cdot}7$ tons, which is the shear at C.

Reaction at $B=1+\frac{1{\cdot}4}{2}-1{\cdot}7=6{\cdot}3$ tons.

The bending moment is zero at the section where area of shear diagram starting from B is equal to the overhanging shear diagram, or $\frac{3+1}{2}\times4=\frac{\left(3{\cdot}3+3{\cdot}3-\frac{y}{2}\right)y}{2}$ $y=3{\cdot}2$ feet.

The maximum bending moment occurs at the section where the shear is zero.

From similar triangles we have: $\frac{3{\cdot}3}{1{\cdot}7}=\frac{x}{z}$ $3{\cdot}3z=1{\cdot}7x$

Also $x+z=10$ $z=10-x$

Then $33-3{\cdot}3x=1{\cdot}7x$ $x=6{\cdot}6$ ft. $z=3{\cdot}4$ ft.

The bending moment is equal to the area of the shear diagram, viz. $\frac{1{\cdot}7\times3{\cdot}4}{2}$ 2·89 tons-feet.

36. *The planks used for timbering the sides of an earthwork trench are bent* 5 *ins. by the lateral pressure of the earthwork. The planks are* 3″ *thick and* 8 *ft. long, their ends being freely supported.*

Find (i) *the pressure per sq. foot exerted by the earthwork.* (ii) *The stress in the planks.* E=900,000 *lb. sq. inch.*

$$W=\frac{384EI\delta}{5l^3}=10{,}550 \text{ lb. for 1 ft. wide}$$

(i) Press. per sq. foot $=\frac{10550}{8}=$ **1320 lb.**

(ii) $f=\frac{Wl}{8Z}=\frac{10550\times96\times6}{8\times12\times9}=$ **7030 lb.** sq. in.

The above is an actual case; the planks were probably near their breaking point.

37. *In a room, rectangular in plan, 20 ft. by 15 ft., the ends of a rolled joist rest on the middle of the two opposite walls; another joist of the same cross section rests on the middle of the other two walls and crosses the first-mentioned joist at right angles. When both are unloaded the bottom flange of the longer joist just touches the top flange of the shorter. A load of 4 tons is placed on the middle of the longer joist. Find:* (1) *The load supported by each joist;* (2) *a suitable section for the joists;* (3) *the central deflection. Stress in flanges, 5 tons per sq. in. Modulus of elasticity, 11,000 tons per sq. in. You may neglect the web in calculating the moment of inertia of the section.* (Edinburgh University.)

The deflection of the two joists is the same. Hence:

$$\frac{W_1 l_1^3}{48EI} = \frac{W_2 l_2^3}{48EI} \quad \text{and} \quad W_1 \times 20^3 = W_2 \times 15^3$$

$$W_1 = \tfrac{27}{64} W_2 \quad \text{also} \quad W_1 + W_2 = 4 \text{ tons}$$

(i) $\frac{27}{64} W_2 + W_2 = 4 \qquad W_2 = 2{\cdot}8$ tons $\qquad W_1 = \mathbf{1{\cdot}2}$ **tons** nrly.

(ii) $Z = \dfrac{2{\cdot}8 \times 180}{4 \times 5} = 25{\cdot}2$ or say $\mathbf{12'' \times 4'' \times \frac{9}{16}''}$

(iii) $\delta = \dfrac{2{\cdot}8 \times 180^3}{48 \times 11000 \times 27 \times 6} = \mathbf{0{\cdot}19''}$.

38. *It is required to lift a load of 2 tons from the middle of a beam 28 ft. clear span, but it is not strong enough. Another beam of the same section is available but it is too short to be used side by side with the first mentioned beam, it is therefore placed across the middle of it and at right angles where it rests on supports 20 ft. apart. When unloaded the two beams just touch one another. The load is slung from the uppermost beam. Find the load supported by each beam and the maximum bending moments to which they are respectively subjected.*

The deflection of each beam is the same:

$$\frac{W_1 \times 28^3}{48EI} = \frac{W_2 \times 20^3}{48EI} \qquad W_1 = \frac{W_2}{2{\cdot}74} = W - W_2 = 2 - W_2$$

$W_2 = \mathbf{1{\cdot}46}$ **tons** $\qquad W_1 = \mathbf{0{\cdot}54}$ **ton**

Bending moment on 20 ft. span $= \dfrac{1{\cdot}46 \times 20}{4} = \mathbf{7{\cdot}3}$ tons-ft.

" " 28 " $= \dfrac{0{\cdot}54 \times 28}{4} = \mathbf{3{\cdot}78}$ "

If a single beam had been used on the 28-ft. span the bending moment in the middle would have been $\frac{2\times 28}{4}=14$ tons-ft., or 10 tons-ft. on the shorter span.

39. *A carriage spring, 32 ins. long, is built up of twelve* $2\frac{1}{2}$ *in.* $\times\frac{1}{4}$ *in. steel plates. Find the greatest central load it can support without the stress in any plate exceeding* 10 *tons per sq. in. What is the deflection under this load?*

$$E=30{,}000{,}000 \text{ lb. per sq. in.}$$

$2P$=central load—

$$P=\frac{2nfbt^2}{6L}=\frac{2\times 12\times 10\times 2\cdot 5\times(\frac{1}{4})^2}{6\times 32\times 2240}=\mathbf{437 \text{ lb.}}$$

$$\delta=\frac{3PL^3}{4nEbt^3}=\frac{3\times 437\times 32^3\times 4^3}{4\times 12\times 30{,}000{,}000\times 2\cdot 5}=\mathbf{0\cdot 76''.}$$

As pointed out on p. 519, Vol. I., the deflection of a plate spring is not the same when the load is being increased as it is when it is being decreased. In order to minimise this trouble strips of zinc and other metals are sometimes placed between the leaves with the object of reducing the friction with decidedly beneficial results. See a paper by H. S. Rowell, "Experiments on Laminated Springs," Institution of Automobile Engineers, April, 1926.

40. *Find the least radius to which any section of depth* d *can be bent without exceeding the elastic limit.*

$$\frac{E}{R}=\frac{f_s}{y}=\frac{2f_s}{d}$$

Hence if f is the elastic limit the least radius is R, where $R=\frac{Ed}{2f}$.

If $E=30{,}000{,}000$, $f=30{,}000$, $d=\frac{1}{2}$ inch, then the radius $R=250$ ins.

The least radius to which wire can be coiled without permanent set.

For a steel wire 0·05 in. in diameter, with an elastic limit of 80 tons per sq. inch, this radius is

$$R=\frac{13500\times 0\cdot 025}{80}=4\cdot 22 \text{ ins.}$$

For an annealed wire of similar material, with an elastic limit of 15 tons per sq. inch

$$R=\frac{13500\times 0{\cdot}025}{15}=22{\cdot}5 \text{ ins.}$$

41. *A rolled steel joist crosses a span of* 15 *ft. and is built in at the ends. At a distance of* 6 *ft. from one end an arm is riveted perpendicular to the joist, and extends* 5 *ft. horizontally. Determine the reactions, bending moments, and couples produced at the ends of the joist when a load of* 10 *cwts. is suspended from this arm.*

Let R_a and M_a be the reaction and bending moment at the end nearer the arm.

If the deflection y be measured downwards at any point across the span:

$$+EI\frac{d^2y}{dx^2}=M_a-R_ax+W\{x-6\}$$

where the brackets { } signify that the term containing them is to be neglected if the quantity inside the brackets is negative at the point considered.

Integrating: $EI\frac{dy}{dx}=M_ax-\frac{R_a}{2}x^2+\frac{W}{2}\{x-6\}^2$

Where $0=M_a\cdot 15-\frac{R_a}{2}\cdot 15^2+\frac{w}{2}\cdot 9^2$ . . . . . (i)

Integrating again: $EIy=\frac{M_a}{2}x^2-\frac{R_a}{6}x^3+\frac{W}{6}\{x-6\}^3$

where $0=\frac{M_a}{2}\cdot 15^2-\frac{R_a}{6}\cdot 15^3+\frac{W}{6}\cdot 9^3$ . . (ii)

Solving (i) and (ii)

$R_a=\frac{81}{125}W=$ **725 lb.** $M_a=\frac{53}{25}W=$ **2420 lb.-ft.**

Hence $R_b=\frac{44}{125}W=$ **395 lb.** $M_b=\frac{36}{25}W=$ **1610 lb.-ft.**

Since the ends are held rigid, the total angle of twist will be the same for the two parts of the span.

Hence the twist per unit length will be inversely proportional to the length of the parts, and the couples at the ends will be in the same ratio:

Hence $\frac{C_a}{C_b}=\frac{2}{6}$ Also $C_a+C_b=W\times 5$ lb.-ft.

Hence $C_a=3W=$ **3360 lb.-ft.** $C_b=2W=$ **2240 lb-ft.**

42. *Deflection of shaft of varied diameter supported in three bearings.*

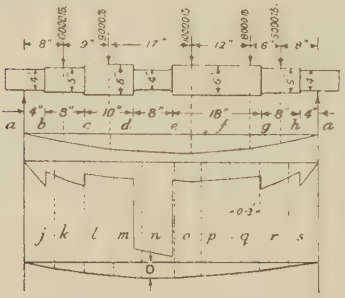


FIG. 1.

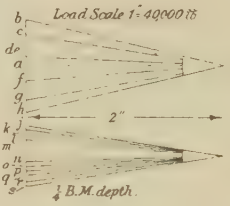


FIG. 2.

FIG. 3.

FIG. 5.

FIG. 4.

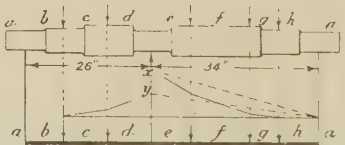


FIG. 6.

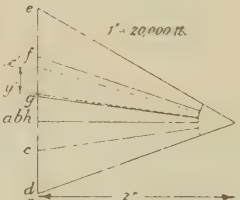


FIG. 7.

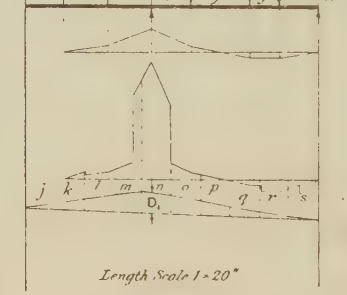


FIG. 8.

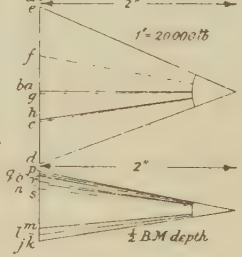
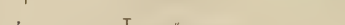


FIG. 9.

FIG. 10.

FIG. 11.

FIG. 12.



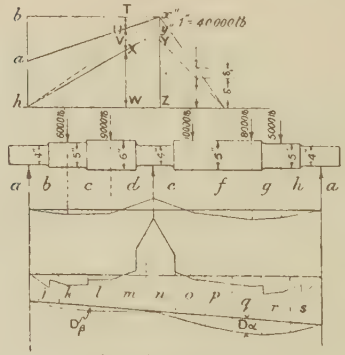


FIG. 13

FIG. 14.

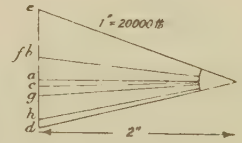


FIG. 15.

FIG. 16.

FIG. 17.

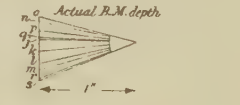


FIG. 18.

This diagram has been reduced to half-size.

(1) Construct the bending moment diagram when supported at the two end bearings only (Figs. 1 and 2; see p. 498, Vol. I.).

(2) Enlarge the depth of the bending moment diagram inversely as the diameter [4] (Fig. 3, see p. 272, Vol. I.).

Diameter of Shaft.	D^4	Ratio of enlargement.
6″	1296	1
5″	625	$\frac{1296}{625}=2{\cdot}07$
4″	256	$\frac{1296}{256}=5{\cdot}06$

(3) From the diagram thus altered construct the deflection diagram (Figs. 4 and 5, see p. 522, Vol. I.)

$$\text{Actual deflection } \delta=\frac{D\times m\times n^3\times x\times \overline{OH}\times \overline{O_1H_1}\times 4}{EI}$$

In this case 1″ on the load scale of the first bending moment diagram $m=40{,}000$ lb. Also 1″ on the length of the diagram $n=20''$. The distance between the ordinates $x=0{\cdot}3''$. The polar distance $\overline{OH}=2''$ and $\overline{O_1H_1}=2''$. The 4 is inserted because the depth of the bending moment diagram has been reduced for convenience of drawing to $\frac{1}{4}$th in the second vector polygon. I is the moment of inertia for the largest diameter of the shaft, viz. 6″. In this case

$$I=\frac{\pi D^4}{64}=63{\cdot}6 \text{ inch}^4$$

The maximum depth D of the b.m. diagram $=0{\cdot}16''$.

Taking $E=30{,}000{,}000$ lb. sq. inch.

Hence the corresponding deflection is :

$$\delta=\frac{0{\cdot}16\times 40000\times 20^3\times 0{\cdot}3\times 2\times 2\times 4}{30{,}000{,}000\times 63{\cdot}6}=0{\cdot}129''.$$

(4) Construct a bending moment diagram for the shaft when supported on the central bearing only (Figs. 6, 7). The moments about this bearing are not equal, hence the shaft is unbalanced, and consequently the two portions of the bending moment diagrams when started from the ends do not meet over the support, the unbalanced moment is given by *xy*. From the vector polygon (Fig. 7) (dotted lines) we see that an upward force *x'y'* is required at the right-hand bearing to balance the shaft. With this upward force *x'y'*, shown in Fig. 9 as *ha*, construct the bending moment diagram (Fig. 8).

(5) Correct this bending moment diagram for shaft diameters as explained in paragraph (2), thus obtaining Fig. 10.

(6) Construct the deflection diagram to correspond (Figs. 11 and 12).

In this case the maximum depth $D_1 = 0{\cdot}21''$ and the load scale m is 20,000 lb.

The deflection :

$$\delta_1 = \frac{0{\cdot}21 \times 20000 \times 20^3 \times 0{\cdot}3 \times 2 \times 2 \times 2}{30{,}000{,}000 \times 63{\cdot}6} = 0{\cdot}042''.$$

The last 2 appears in the numerator because the depth of the bending moment diagram has been reduced to $\frac{1}{2}$ for convenience in drawing.

(7) In Fig. 13 (see p. 534, Vol. I.) construct a rectangle of any convenient length and whose height hb represents the total load on the shaft. Divide it so that ab represents the load on the left-hand bearing and ah the load on the right-hand bearing. These values are taken from Fig. 2. Divide the height $x''Z$ at the other end of the rectangle at Y so that $x''Y$ and YZ are proportional respectively to the actual deflections of the shaft δ_1 and δ. Mark off $x''y''$ to represent $x'y'$ in Fig. 7 to the same scale as the loads ab and ah at the left-hand end of the rectangle. Join $y''h$, also $x''h$. From Y draw a horizontal to cut $x''h$ in V, through V draw a vertical TW.

Then, as explained on p. 534, Vol. I.,

TU represents the load on the left-hand bearing.
UX represents the load on the right-hand bearing.
XW represents the load on the central bearing.

(8) Construct the bending moment diagram (Figs. 14, 15), also the diagram corrected for variation in diameter (Fig. 16), and the final deflection diagram (Figs. 17, 18).

Then the deflection at any point a is :

$$\delta_a = \frac{D_a \times m \times n^3 \times x \times \overline{OH} \times \overline{O'H'}}{EI}$$

$$\delta_a = \frac{D_a \times 20000 \times 20^3 \times 0{\cdot}3 \times 2 \times 1}{30{,}000{,}000 \times 63{\cdot}6} = 0{\cdot}05 D_a \qquad D_a = 0{\cdot}15''$$

Hence $\delta_a = 0{\cdot}05 \times 0{\cdot}15 = 0{\cdot}0075''$

$D_\beta = 0{\cdot}04''$ and $\delta_\beta = 0{\cdot}002''$.

CHAPTER XIV

BENDING AND DIRECT STRESS

1. *A rigid beam is suspended by two bars* 0·5″ *diam. as shown in Fig.* 530, *Vol. I. The one is of copper and the other of steel. They are* 6″ *apart centre to centre. Find the load on each and the position of* W *to ensure that each bar extends an equal amount.*

W=3 *tons.* E_c=6,000 *tons sq. inch.* E_s=13,000 *tons sq. inch.*

Symbols bearing the suffix *c* refer to the copper and *s* to the steel bar.

Since the extension is the same on each we have—

$$x_c=\frac{W_c l}{AE_c}=\frac{W_s l}{AE_s} \qquad \text{hence} \qquad W_c=W_s\times\tfrac{6000}{13000}$$

$$W_c+W_s=3 \text{ tons} \qquad \text{hence} \qquad W_s\times\tfrac{6}{13}+W_s=3$$

$$W_s=\mathbf{2{\cdot}05 \text{ tons}}, \qquad \frac{2{\cdot}05}{0{\cdot}785\times 0{\cdot}5^2}=10{\cdot}5 \text{ tons sq. in.}$$

$$W_c=3-2{\cdot}05=\mathbf{0{\cdot}95 \text{ tons}} \quad =4{\cdot}85 \text{ tons sq. in.}$$

$$x_s=\frac{10{\cdot}5}{13000}=0{\cdot}0081'' \qquad x_c=\frac{4{\cdot}85}{6000}=0{\cdot}0081''.$$

Let μ be the distance of W from the copper bar, then

$$3\mu=2{\cdot}05\times 6 \qquad \mu=\mathbf{4{\cdot}1}''.$$

2. *A tie bar* 8 *ins. wide and* 1 *in. thick is loaded axially with* 50 *tons. Find the maximum stress in the material if a hole* 1·5″ *diam. is drilled through the bar with its centre at a distance of* 2″ *from the axis of the bar.*

Area of section through hole =6·5 sq. ins.

$$\text{Direct stress}=\frac{50}{6{\cdot}5}=7{\cdot}7 \text{ tons per sq. in.}$$

Taking moments about a short edge of the section remote from hole. Distance of c. of g. from edge

$$= \frac{\frac{5{\cdot}25^2}{2} + 1{\cdot}25 \times 7{\cdot}375}{6{\cdot}5} = 3{\cdot}54''.$$

Moment of inertia of section

$$I = \frac{4{\cdot}46^3 + 1{\cdot}71^3 - 3{\cdot}21^3 + 3{\cdot}54^3}{3} = 35 \text{ inch}^4 \text{ units}$$

$$Z_1 = \frac{35}{3{\cdot}54} = 9{\cdot}85 \text{ inch}^3 \text{ units} \qquad Z_2 = \frac{35}{4{\cdot}46} = 7{\cdot}85 \text{ inch}^3 \text{ units}.$$

Assuming a linear distribution of stress across the section the bending stress $f_1 = \frac{50 \times 0{\cdot}46}{9{\cdot}85} = 2{\cdot}34$ tons per sq. in. on the edge remote from hole (compression).

Total stress $= 7{\cdot}7 - 2{\cdot}34 = 5{\cdot}36$ tons sq. in. (tension)

$$f_2 = \frac{50 \times 0{\cdot}46}{7{\cdot}85} = 2{\cdot}93 \text{ tons per sq. inch}$$

on edge near hole (tension).

Total stress $= 7{\cdot}7 + 2{\cdot}93 =$ **10·63 tons per sq. in.** (tension).

Uniform stress, no hole, 6·25 tons sq. inch.

Professor Coker and others have shown that the intensity of stress round a hole in a tie bar is materially greater than the above treatment gives. On loading the bar the material probably passes the elastic limit in the neighbourhood of the hole and thereby equalises the stress, with the result that after loading the treatment we give above is approximately correct; but there is no definite experimental evidence upon this point.

3. *A tie bar 3 ins. diam. has a flaw or cavity 1·3 ins. diam. the centre of which is 1 in. from the centre of the bar. Find the elastic strength of the bar as compared with that of a solid bar.*

Moment of inertia of section about the N.A. (see p. 90, Vol. I.)

$$= \frac{\pi}{64}\left(D_e^4 - D_i^4 - \frac{16 D_e^2 D_i^2 x^2}{D_e^2 - D_i^2}\right)$$

$$= \frac{3{\cdot}14}{64}\left(81 - 2{\cdot}9 - \frac{16 \times 9 \times 1{\cdot}69 \times 1}{9 - 1{\cdot}69}\right) = 2{\cdot}2$$

$$\text{Distance of c. of g. from edge} = \frac{9\times1\cdot5-1\cdot69\times2\cdot5}{7\cdot31}$$

$$=1\cdot27'' \quad \text{or } 0\cdot23'' \text{ from centre of bar}$$

$$Z_1=\frac{2\cdot2}{1\cdot27}=1\cdot73 \qquad Z_2=\frac{2\cdot2}{1\cdot73}=1\cdot27$$

$$\text{Sectional area} = 7\cdot07-1\cdot33=5\cdot74 \text{ sq. ins.}$$

$$\text{Direct stress} = \frac{W}{5\cdot74}=0\cdot174W$$

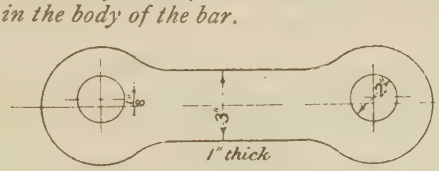
$$\text{Bending stress} = \frac{W\times0\cdot23}{1\cdot27}=0\cdot181W \text{ (tension)}$$

$$\text{Total stress} = (0\cdot174+0\cdot181)W=0\cdot355W$$

$$\text{Relative strength} = \frac{\text{Bar with hole}}{\text{Solid bar}}=\frac{0\cdot141}{0\cdot355}=\mathbf{0\cdot4}.$$

There is a well-known instance of the collapse of a big roof through a hidden flaw of this kind in one of the tie bars.

4. *The tie-bar shown was intended to be subjected to a tensile stress of* 5 *tons per sq. in., but the holes were drilled* $\frac{1}{8}$*th of an in. out of truth; calculate the actual stress on the material in the body of the bar.*



Uniform tension on section = 5 tons sq. in.

Load on bar = $5\times3=15$ tons

Bending moment = $15\times\frac{1}{8}$ inch-tons

$$\text{Bending stress} = \frac{15}{8}\div\frac{1\times3^2}{6}=1\cdot25 \text{ tons sq. in.}$$

Max. stress = **6·25** tons sq. in. tension

Min. stress = **3·75** tons sq. in. ,,

5. *A flat tie bar of rectangular section is* 0·75 *in. thick and* 12 *ins. wide, and is subjected to a tension of* 5 *tons per sq. in. evenly distributed over the section. Find the maximum tensile stress in the bar when it is partially sawn through on the narrow edge to a depth of* 2 *ins.*

Load $12 \times 0{\cdot}75 \times 5 = 45$ tons.

$$Z \text{ at sawn section} = \frac{0{\cdot}75 \times 10^2}{6} = 12{\cdot}5$$

Distance of line of loading from neutral axis $= 1''$

$$\text{Bending stress } f = \frac{45 \times 1}{12{\cdot}5} = 3{\cdot}6 \text{ tons sq. in.}$$

$$\text{Direct tension} = \frac{45}{10 \times 0{\cdot}75} = 6{\cdot}0 \text{ tons sq. in.}$$

Resultant maximum stress = **9·6** tons sq. in.

6. *A tie bar* 4 *ins. diam. is cranked in order to clear a projecting object. The distance between the centre line of the straight part and the centre of the cranked part is* 10·8 *ins. Find the diam. of the cranked portion to fulfil the condition that the maximum intensity of stress in the cranked portion shall be equal to that in the straight part of the tie bar.*

$$\text{Stress in straight portion} = \frac{P}{4\pi}$$

$$\text{Direct stress in cranked portion} = \frac{4P}{\pi D^2}$$

$$\text{Bending stress} \quad ,, \quad ,, \quad = \frac{P \times 10{\cdot}8 \times 32}{\pi D^3}$$

$$\frac{P}{4\pi} = \frac{4P}{\pi D^2} + \frac{P \times 10{\cdot}8 \times 32}{\pi D^3} \qquad D^3 = 16D + 1382$$

Solving by trial and error. $D = \mathbf{11{\cdot}6}''$.

7. *A beam* AB *is supported and loaded as shown in the*

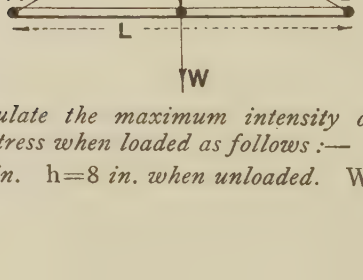


figure. Calculate the maximum intensity of tensile and compressive stress when loaded as follows :—

L = 40 *in.* h = 8 *in. when unloaded.* W = 1 *ton.*

Sectional area of beam 1·25 *sq. in. Modulus of section* 0·98 *in. units. Maximum deflection* 0·82 *in.*

Bending stress regarded as a simple beam $= \frac{WL}{4Z}$

Compressive stress due to sloping the bars $= \frac{WL}{4hA}$

Bending stress due to end thrust on the deflected beam $= \frac{WL\delta}{4hZ}$

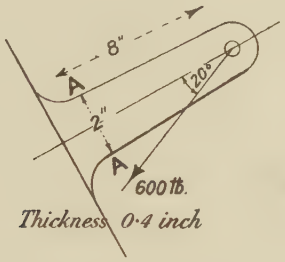
$$\text{Max. compressive stress} = \frac{WL}{4}\left(\frac{1}{Z} + \frac{1}{hA} + \frac{\delta}{hZ}\right)$$

$$= \frac{1 \times 40}{4}\left(\frac{1}{0{\cdot}98} + \frac{1}{8 \times 1{\cdot}25} + \frac{0{\cdot}82}{0{\cdot}98 \times 8}\right)$$

$$= \mathbf{12{\cdot}2} \text{ tons sq. in}$$

$$\text{Tensile stress} = \mathbf{0{\cdot}2} \text{ ton sq. in.}$$

8. *Find the stress at the upper and lower edges of the section* A *when the arm is loaded as shown.*



Thickness 0·4 inch

Force along axis = 600 cos 20° = 564 lb.

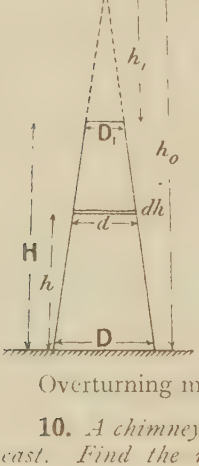
Area of section = 0·8 sq. in. Direct stress $= \frac{564}{0{\cdot}8} = 705$ lb. sq. in.

$$\text{Bending stress} = \frac{600 \sin 20° \times 8 \times 6}{0{\cdot}4 \times 2^2} = 6{,}156 \text{ lb. sq. in.}$$

Max. compressive stress on bottom edge = **6,861** lb. sq. in.
„ tensile „ on top edge = **5,451** „ „

9. *A chimney* 200 *ft. high of circular section is* 14 *ft. external diameter at the bottom and* 10 *ft. at the top. Find*

(b) overturning moment when the wind pressure is 20 lb. per sq. ft. of the projected area of chimney.



$$\frac{10}{14} = \frac{h_1}{h_o} = \frac{h_o - 200}{h_o}$$

$h_o = 700$ $\qquad d = \frac{D}{h_o}(h_o - h)$

Moment of wind press. on strip about base $= 20d \times h \times dh$

Moment on chimney

$$= \frac{20D}{h_o}\int_o^H (h_o - h)h\,dh$$

$$= 0{\cdot}4\left(\frac{h_o H^2}{2} - \frac{H^3}{3}\right)$$

$$= 0{\cdot}4\left(\frac{700 \times 200^2}{2} - \frac{200^3}{3}\right)$$

Overturning moment = **4,533,000 lb.-ft.**

10. *A chimney 150′ high is 3 ft. out of plumb towards the east. Find the maximum compressive stress at the base when there is a south wind of 10 lb. per sq. ft. External diam. 20 ft., internal 10 ft. Weight of brickwork 130 lb. c. ft.*

Weight of chimney $= \frac{0{\cdot}785(20^2 - 10^2)150 \times 130}{2240} = 2050$ tons

Direct compressive stress $= \frac{150 \times 130}{2240} = 8{\cdot}7$ tons sq. ft.

Bend. mom. due to overhang $= 2{,}050 \times 1{\cdot}5 = 3{,}080$ tons-ft.

Area exposed to wind $= 20 \times 150 = 3{,}000$ sq. ft.

Bending moment due to wind $= \frac{3000 \times 10 \times 75}{2240} = 1{,}000$ tons-ft.

Resultant bending moment $= \sqrt{1000^2 + 3080^2} = 3{,}240$ tons-ft.

$\tan\theta = \frac{1000}{3080}$ $\qquad \theta = 18°$ from east

$Z = \frac{D^4 - d^4}{10{\cdot}2D} = \frac{20^4 - 10^4}{204} = 735$ feet^3 units.

Stress due to bending $= \frac{3240}{735} = 4{\cdot}4$ tons/sq. foot.

Max. stress $= 8{\cdot}7 + 4{\cdot}4 = \mathbf{13{\cdot}1}$ tons sq. ft.

Min. stress $= 8{\cdot}7 - 4{\cdot}4 = 4{\cdot}3$,, ,,

11. *Find the maximum and minimum compression stresses at the base of a circular chimney, which may be*

assumed to have no taper. The intensity of wind pressure varies as the square root of the height above the ground.

Wind pressure at 180′ *above ground on the projected area of the chimney* $\Big\}$ P = 18 *lb. sq. ft.*

External diam. of chimney, 15′; *internal diam. of chimney*, 7′; *wt. of brickwork per cubic ft.*, 120 *lb.*; *height of chimney* (H), 200 *feet.*

p = intensity of wind pressure at a height h above the ground.

$$\frac{P}{p} = \frac{H^{\frac{1}{2}}}{h^{\frac{1}{2}}} \qquad p = \frac{P}{H^{\frac{1}{2}}} h^{\frac{1}{2}} = 1{\cdot}34 h^{\frac{1}{2}}$$

Moment of wind acting on strip of area D . dh $\Big\}$ $= Dph.dh = 1{\cdot}34 D h^{\frac{3}{2}} dh$

$$M = 1{\cdot}34 D \int_0^{200} h^{\frac{3}{2}} dh = 1{\cdot}34 \times 15 \times \tfrac{2}{5} \times 200^{\frac{5}{2}} = 4{,}500{,}000 \text{ lb.-ft.}$$

$$Z = \frac{15^4 - 9^4}{10{\cdot}2 \times 15} = 288$$

Bending stress $= \frac{4500000}{288} = 15{,}600$ lb. sq. ft.

Comp. stress $= 200 \times 120 = 24{,}000$ lb. sq. ft.

Max. comp. stress $= 39{,}600$ lb. sq. ft. $=$ **17·7 tons sq. ft.**

Min. comp. stress $= 8{,}400$ lb. sq. ft. $=$ **3·75 tons sq. ft.**

Check on bending moment by taking the parabolic load diagram—

$p_{200} = 1{\cdot}34 \times 14{\cdot}14 = 18{\cdot}9$ area $= \frac{2}{3} \times 18{\cdot}9 \times 200 = 2{,}520$

$$M = \frac{15 \times 2520 \times 3 \times 200}{5} = \textbf{4,540,000 lb.-ft.}$$

12. *A hollow circular section column has a bracket on one side which supports a load of* W *tons at a distance of* 1 *ft. from the central axis of the column also a central load of* 3W *tons. Find* W *to satisfy the condition that the compressive stress may not exceed* 10 *tons per sq. in. or the tensile stress* 1 *ton per sq. in. State whether the column fails in tension or compression. External diam.*, 8 *ins.*; *internal*, 6 *ins.*

Area of section $= 22$ sq. ins. $\qquad Z = \frac{8^4 - 6^4}{10{\cdot}2 \times 8} = 34{\cdot}2$

Compressive stress $\frac{4W}{22} + \frac{12W}{34{\cdot}2} = 10 \qquad W = 18{\cdot}8$ tons

Tensile stress $\frac{4W}{22} - \frac{12W}{34{\cdot}2} = 1 \qquad W = 6$ tons

Therefore the specified stress is reached when W = **6 tons,** and failure is likely to occur in tension.

13. *A crane of the type shown in Fig.* 135 (*Vol. I.*) *lifts a load of* 5 *tons at a radius of* 14 *ft. If the post be of hollow circular section* (D = 3*d*), *find the diam. when f* = 2 *tons sq. in.*

NOTE.—The height *h* of the post does not affect the question.

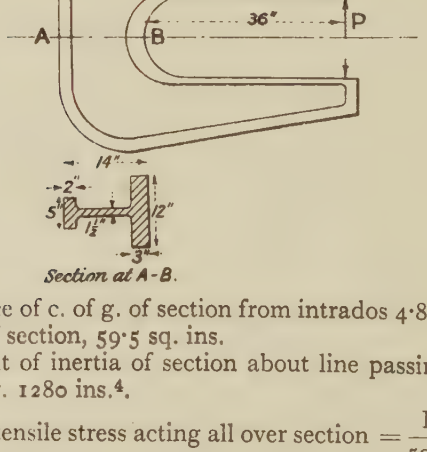
$$\text{Direct comp. stress} = \frac{5}{0{\cdot}785\left(D^2 - \frac{D^2}{9}\right)} = \frac{7{\cdot}17}{D^2}$$

$$\text{Bending stress} = \frac{5 \times 14 \times 12}{D^4 - \left(\frac{D}{3}\right)^4} = \frac{8670}{D^3}$$
$$10{\cdot}2D$$

$$\text{Then } \frac{8670}{D^3} + \frac{7{\cdot}17}{D^2} = 2$$

By trial and error $D = \mathbf{16{\cdot}4''}$ $\qquad d = \mathbf{5{\cdot}5''}$.

14. *Find the load* P *required to produce a stress of* 2 *tons sq. in. in the frame as shown.*



Section at A-B.

Distance of c. of g. of section from intrados 4·8″

Area of section, 59·5 sq. ins.

Moment of inertia of section about line passing through the c. of g. 1280 ins.⁴.

$$\text{Direct tensile stress acting all over section} = \frac{P}{59{\cdot}5}$$

Bending stress at intrados $= \frac{(36+48)P \times 4{\cdot}8}{1280}$

Tensile stress on intrados

$= \frac{P \times 40{\cdot}8 \times 4{\cdot}8}{1283} + \frac{P}{59{\cdot}5} = 2$ tons sq. in. $\quad P = 11{\cdot}8$ tons

Compressive stress on the extrados

$$= \frac{11{\cdot}8 \times 40{\cdot}8 \times 9{\cdot}2}{1283} - \frac{11{\cdot}8}{59{\cdot}5} = \mathbf{3{\cdot}26 \text{ tons sq. inch.}}$$

In the above solution the effect of the curvature of the frame has been neglected (see page 544, Vol. I.)

15. *Find the stresses in the main column of a* 100*-ton testing machine* (*see Chap. III., Q.* 3). *Eccentricity of loading,* 17·49″; *area of section,* 127 *sq. ins.*; $Z_c = 830$; $Z_t = 770$.

Direct comp. stress $= \frac{100}{127} = 0{\cdot}79$ ton sq. in.

Ben. mom. $= 1{,}749$ inch-tons

Comp. bending stress $= \frac{1749}{830} = 2{\cdot}11$ ton sq. in.

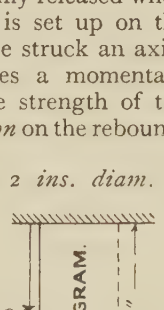
Ten. „ „ $= \frac{1749}{770} = 2{\cdot}27$ „ „

Max. comp. stress $= 2{\cdot}11 + 0{\cdot}79 = \mathbf{2{\cdot}90}$ tons sq. in.

Tensile stress $= 2{\cdot}27 - 0{\cdot}79 = \mathbf{1{\cdot}48}$ „ „

It should be noted that the load is suddenly released when the specimen breaks, and a tensile stress is set up on the rebound. If a vertical cast-iron column be struck an axial blow by a falling weight which produces a momentary compressive stress greater than the tensile strength of the material, the column may be broken in *tension* on the rebound.

16. *A vertical rod* 30 *ins. long and* 2 *ins. diam. is held rigidly at its upper end, and a rigid horizontal arm is secured to the lower end. A wt. of* 2,000 *lb. is applied to the end of the arm at a distance of* 6 *inches from the axis of the rod. Calculate* (*approximately*) *the horizontal movement of the lower end of the rod.* E = 30,000,000 *lb. sq. in.*



Bending moment $M = 2000 \times 6$ inch-lb.

Area of bending moment diagram $A = 2000 \times 6 \times 30$

Distance of c. of g. of bending moment diagram from free end of rod = 15 inches.

$$\delta = \frac{2000 \times 6 \times 30 \times 15 \times 64}{30{,}000{,}000 \times 3{\cdot}14 \times 16} = \mathbf{0{\cdot}23 \text{ in.}}$$

17. *A train weighing* 200 *tons when running over a viaduct at a speed of* 60 *m.p.h. is pulled up by continuous brakes in* 300 *yards. The rail level is* 70 *ft. above the foot of the viaduct piers. Take the weight of the girders, etc., as* 120 *tons on each pier, and the weight of the masonry as* 140 *lb. per cu. ft. The piers are of rectangular section,* 30 *ft. wide and* 15 *ft. thick at the foot, and have a batter of* 1 *in. per ft. on all sides. Find the maximum intensity of compressive stress on the edge of the footing.*

$$\text{Horizontal force} = \frac{WV^2}{2gd} = \frac{200 \times 88^2}{2 \times 32{\cdot}2 \times 300 \times 3} = 26{\cdot}7 \text{ tons}$$

Moment of force about foot of piers $= 26{\cdot}7 \times 70 = 1{,}869$ tons-ft.

Volume of pier

$$= \int_0^{70} \left(15 - \frac{2h}{12}\right)\left(30 - \frac{2h}{12}\right) dh = \int_0^{70} \left(450 - \frac{45 \times 2h}{12} + \frac{4h^2}{144}\right) dh$$

$$= 70\left(450 - \frac{45 \times 70}{12} + \frac{4 \times 4900}{144 \times 3}\right) = 16{,}300 \text{ cubic feet}$$

or the same result can be obtained by Simpson's rule (p. 42, Vol. I.).

$$\text{Weight of pier} = \frac{16300 \times 140}{2240} = 1{,}020 \text{ tons}$$

Compressive stress on section, including half weight of train

$$= \frac{1020 + 100 + 120}{30 \times 15} = 2{\cdot}75 \text{ tons sq. ft.}$$

$$\text{Bending stress} = \frac{1869 \times 6}{30 \times 15^2} = 1{\cdot}66 \text{ tons sq. ft.}$$

Maximum compressive stress $= \mathbf{4{\cdot}41 \text{ tons sq. ft.}}$

18. *A solid square section masonry structure, such as Cleopatra's needle, has been designed so that it will overturn with a wind pressure* p *per unit of side area, but after the drawings had been made it was decided to increase the structure to* n *times its original dimensions by altering the scale of the drawing. What wind pressure will then be*

required to overturn it? and what should be the relative areas of the bases in order that both may be equally stable with the same wind pressure?

Let the height be h, the breadth of the sides b, and the height above the base at which the resultant wind pressure acts xh. The weight per cubic unit of the masonry w.

$$\text{Overturning moment} = pbhxh$$

$$\text{Righting moment} = \frac{wb^2hb}{2}$$

$$p = \frac{wb^3h}{2bh^2x} = \frac{wb^2}{2xh} \quad \text{(original)}$$

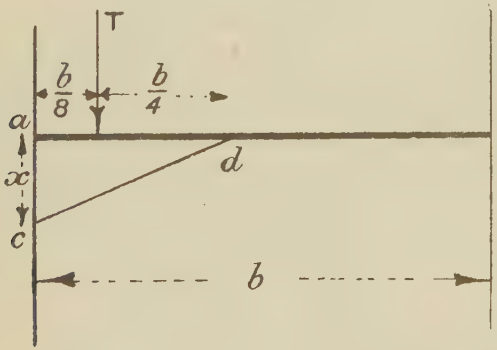
When the scale has been increased n times

$$p' = \frac{wn^2b^2}{2xnh} = \frac{wnb^2}{2xh} \quad \text{(increased)}$$

or it requires n times the wind pressure to overturn it, or its stability is increased n times.

For the same wind pressure the area of the base must be the same in both cases, in the case of Cleopatra's needle the base is smaller than the sections immediately above it.

19. *The line of thrust in a masonry structure passes through the section at a point distant* $\frac{1}{8}$*th of the depth of the*



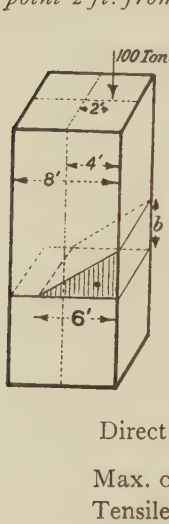
voussoirs from one edge; assuming no adhesion of the mortar, find the maximum intensity of the compressive stress in terms of the thrust T *and thickness of the cross-section* b.

acd represents the distribution of the stress.

$$T=\tfrac{3}{8}b\frac{x}{2}=\tfrac{3}{16}bx \qquad x=\frac{16T}{3b}$$

i.e. the max. intensity of stress is $5\frac{1}{3}$ times the mean.

20. *A pier is built up of well-bedded blocks of stone 8 ft. square without mortar. It supports a load of 100 tons at a point 2 ft. from one centre line and on the other centre line of the plan; the centre lines are parallel with the edges. Find the max. intensity of stress in the stone.*



The load acts at the c. of g. of the shaded triangular stress diagram, the base of which is therefore 6 ft. Then

$$\frac{6b}{2}=\frac{100}{8} \qquad b=\mathbf{4{\cdot}2\ tons\ sq.\ ft.}\ \text{(no cement)}.$$

If the joints were cemented and the adhesion perfect

$$100\times 2=f\frac{S^3}{6}=f\frac{512}{6}$$

$$f=2{\cdot}35 \text{ tons sq. ft. bending}$$

$$\text{Direct stress}=\frac{100}{8^2}=1{\cdot}56 \text{ tons sq. foot comp.}$$

Max. compressive stress=**3·91 tons sq. ft.**
Tensile stress=2·35−1·56=**0·79 ton sq. ft.**

21. *On the assumption that a hook may be treated in the same manner as a cranked tie bar, find the safe working load for a hook in which the principal section is 4·5″×1·8″, and the distance of the line of loading from the intrados is 2·7″. The maximum stress must not exceed 7 tons per sq. in. What is the intensity of the minimum stress?*

$$W(2{\cdot}7+2{\cdot}25)=f\frac{1{\cdot}8\times 4{\cdot}5^2}{6}$$

$$f=0{\cdot}815W \text{ (bending stress)}$$

$$t=\frac{W}{4{\cdot}5\times 1{\cdot}8}=0{\cdot}123W \text{ (tension all over section)}$$

$$W(0{\cdot}815+0{\cdot}123)=7 \qquad W=7{\cdot}46 \text{ tons working load.}$$

Min. stress=7·46(0·815−0·123)=**5·16 tons sq. in.**

The max. stress is more nearly $7 \times 1{\cdot}64 = 11{\cdot}5$ tons sq. in. (see below).

Taking $Y_t = 1{\cdot}9''$ and $R = 4{\cdot}6''$ $\quad n = 0{\cdot}414$

The stress on the intrados of a hook is always greater than the above method gives. The following empirical corrections are convenient (approximate only).

Let $m = \dfrac{\text{Stress at intrados}}{\text{Stress on a cranked tie bar}}$ $\qquad n = \dfrac{Y_t}{R}$

Y_t = Distance of the neutral axis of principal section of hook from the intrados.

R = Radius of curvature of neutral axis.

Principal section.	m
Rectangular	$1 + 0{\cdot}6n + 1{\cdot}5n^2$
Rectangular with rounded corners	$1 + 0{\cdot}7n + 1{\cdot}5n^2$
Triangular	$1 + 0{\cdot}95n + 1{\cdot}5n^2$
Circular	$1 + 0{\cdot}75n + 1{\cdot}5n^2$
Crane hook, roughly triangular with rounded sides	$1 + 0{\cdot}9n + 1{\cdot}5n^2$

See article by Morley in *Engineering*, September 11 and 25, 1914.

22. *A hook is required for lifting a load of* 15 *tons. The principal section through the back of the hook is rectangular. The breadth in the plane of bending is three times the thickness. The line of loading is* 3 *ins. from the nearest edge of the section, working stress* 10 *tons per sq. in. Find the required section.*

$$\text{Bending moment} = 15\left(3 + \frac{H}{2}\right) = 45 + 7{\cdot}5H$$

$$f = \frac{(45 + 7{\cdot}5H)6}{\frac{H}{3} \times H^2} = \frac{18(45 + 7{\cdot}5H)}{H^3} \qquad t = \frac{3 \times 15}{H^2}$$

$$\frac{18(45 + 7{\cdot}5H)}{H^3} + \frac{45}{H^2} = 10 \qquad \mathbf{H = 5{\cdot}7''} \qquad \mathbf{B = 1{\cdot}9''}.$$

The max. stress is in reality greater than 10 tons sq. in. (See p. 547, Vol. I.; see also Problem 21, Chap. XIV.)

$$n = \frac{2{\cdot}85}{5{\cdot}85} = 0{\cdot}49 \qquad m = 1 + 0{\cdot}7 \times 0{\cdot}49 + 1{\cdot}5 \times 0{\cdot}49^2 = 1{\cdot}7$$

The max. stress is about 17 tons sq. in.

Thus if the nominal stress had been taken at 6 tons sq. inch, the max. would have been nearer the intended 10 tons sq. in.

In the above put $f+t=6$. Then $\mathbf{H=7''}$ and $\mathbf{B=2{\cdot}33''}$ $m=1{\cdot}81''$. Max. stress $=6\times1{\cdot}81=10{\cdot}8$ tons sq. in. (approx.).

23. *Calculate the load for a drop forged crane hook in which the line of loading is 4" from the neutral axis of the principal section. Breadth of section, $b=2''$; depth, $h=3{\cdot}3''$. Nominal stress 6 tons sq. inch on intrados.*

For such crane hooks the following relations hold very nearly. Area $A=0{\cdot}753bh$. *Distance of N.A. from extrados* $y_e=0{\cdot}545h$, *from intrados* $y_i=0{\cdot}455h$. *Moment of inertia of section* $I=\frac{bh^3}{22}$. *See "Engineering," October* 18, 1901.

$$A=0{\cdot}753\times2\times3{\cdot}3=4{\cdot}97 \text{ sq. ins.}$$

$$y_e=0{\cdot}545\times3{\cdot}3=1{\cdot}8'' \qquad y_i=0{\cdot}455\times3{\cdot}3=1{\cdot}5''$$

$$I=\frac{2\times3{\cdot}3^3}{22}=3{\cdot}27$$

Bending moment $M=4W$

$$f_i=\frac{My_i}{I}=\frac{4W\times1{\cdot}5}{3{\cdot}27}=1{\cdot}83W$$

$$f_c=\frac{My_e}{I}=\frac{4W\times1{\cdot}8}{3{\cdot}27}=2{\cdot}20W$$

$$t=\frac{W}{4{\cdot}97}=0{\cdot}2W$$

$$1{\cdot}83W+0{\cdot}2W=6 \qquad W=\frac{6}{2{\cdot}04}=2{\cdot}94 \text{ tons.}$$

The max. stress is greater than 6 tons sq. in. (see p. 547, Vol. I., and example 21, Chap. XIV.).

$$n=\frac{1{\cdot}5}{4}=0{\cdot}375 \qquad m=1+0{\cdot}9\times0{\cdot}375+1{\cdot}5\times0{\cdot}375^2=1{\cdot}55$$

Thus the max. stress is about $6\times1{\cdot}55=9{\cdot}3$ tons sq. in. and the load for 6 tons sq. in. stress

$$=\frac{2{\cdot}94\times6}{9{\cdot}3}=\mathbf{1{\cdot}9 \text{ tons.}}$$

CHAPTER XV

STRUTS

1. MANY elaborate and complicated expressions have been put forward professing to reduce the strut problem to a rational basis, but they all rest on more or less questionable assumptions as to the straightness of the strut, the exact method of loading, the uniformity of the material, and other properties which cannot be relied upon to hold in actual practice, hence such refinements may be a source of danger in that they give the designer a false impression of the accuracy with which the buckling load of struts can be calculated, and thereby lead him to subject struts to dangerously high loads. Amongst designers the "straight-line" strut formula is in common use and for nearly all cases gives quite satisfactory results for practical purposes. Much useful data on the strength of columns will be found in the books of sections issued by structural steel firms.

2. *Calculate the thickness of metal required for a hollow circular cast-iron column* 10″ *extl. diam. Length*, 40′. *Ends rigidly held. Load*, 60 *tons. Factor of safety*, 5. $a=\frac{1}{1\,5\,0}$ (*Gordon*). S=80,000 *lb. sq. in.* E=6,000 *tons sq. in.*

Gordon: $l=\frac{40}{2}$ $r=\frac{40\times 12}{2\times 10}=24$

$$P=\frac{80000}{1+ar^2}=\frac{80000\times 150}{150+576}=16{,}500 \text{ lb. sq. in.}$$

$$\text{Area required}=\frac{60\times 5\times 2240}{16500}=40{\cdot}5 \text{ sq. in.}$$

$(100-d^2)0{\cdot}785=40{\cdot}5$ $d=7''$ approx. $t=\mathbf{1{\cdot}5''}$.

3. *Find the necessary sectional area of a hollow circular cast-iron column to support a load of* 240 *tons, factor of*

safety 5. *Length* 18 *ft. Both ends rigidly fixed.* S=36 *tons sq. in.* $a=\frac{1}{200}$.

$$\text{Buckling stress per sq. in.} = \frac{36}{1+\frac{1}{200}\left(\frac{L}{D_2}\right)^2} = \frac{36D_2^2}{D_2^2+58\cdot3}$$

Let $D_1=0\cdot8D_2$

$$\text{Working stress per sq. in.} = \frac{7\cdot2D_2^2}{D_2^2+58\cdot3} = \frac{240}{0\cdot785(D_2^2-D_1^2)}$$

Substituting for D_1 and solving by trial and error we get $D_2=12\cdot6''$ nrly. $D_1=10\cdot1''$

Area of section=**45 sq. ins.**

4. *A solid circular steel strut is required to support a load of* 20 *tons. The total length of the strut is* 30 *ft. Both ends are rigidly held. Taking the crushing strength of the material as* 28 *tons per sq. in., and the Elastic Limit as* 16 *tons per sq. in., Young's Modulus of Elasticity as* 12,000 *tons per sq. in., find the diam. Factor of safety,* 4.

By Gordon's formula (modified) (see p. 558, Vol. I.):

$$a=\frac{Afd}{5EZ}=\frac{\pi d^2\times16\times d\times32}{4\times5\times12000\times\pi d^3}=\frac{1}{470}$$

$$P=\frac{SA}{1+ar^2} \qquad 80=\frac{28\times0\cdot785d^2\times470}{470+\left(\frac{180}{d}\right)^2}$$

$$d^4-3\cdot64d^2=251. \quad d=\mathbf{4\cdot21''}.$$

By Rankine's formula (see p. 561, Vol. I.):

$$P=\frac{SA}{1+bR^2} \qquad b=\frac{f}{10E}=\tfrac{1}{7500}$$

$$80=\frac{28\times0\cdot785d^4\times7500}{7500d^2+32400\times16} \qquad R=\frac{l}{K}$$

$d=$ **4·21 inches** $\qquad K^2$ for a circle$=\frac{d^2}{16}$.

The strut is too long to be dealt with by the straight line formula.

By Euler's formula:

$$P=\frac{10EI}{l^2}=\frac{10\times12000\times d^4}{180^2\times20\cdot4}=80$$

$d=$**4·56''.**

5. *A solid round steel strut with pin ends is* 90 *ins. in length ; it has to support with a factor of safety of* 8 *a load of* 40,000 *lb. ; find the diameter. Take the numerator constant in Gordon's formula as* 36 *tons per sq. in. and the denominator constant as* $\frac{1}{500}$.

$$P=\frac{36}{1+\frac{1}{500}\left(\frac{L}{d}\right)^2}=\frac{36d^2}{d^2+16\cdot2}$$

$$\text{Working load}=\frac{4\cdot5d^2}{d^2+16\cdot2}=\frac{40000}{0\cdot785d^2}\times\frac{1}{2240}=\frac{22\cdot7}{d^2}$$

$$4\cdot5d^4=22\cdot7d^2+22\cdot7\times16\cdot2 \qquad d=\mathbf{3\cdot45\ ins.}$$

In American bridge building practice pin joints are commonly used for both ties and struts.

6. *Find the buckling load of a solid circular section strut rigidly held at both ends. Length,* 12 *ft. ; diameter,* 3 *ins. ; Gordon's* $\alpha=\frac{1}{3500}$; *numerator,* 30 *tons per sq. in.*

Sectional area 7·07 sq. ins.

Equivalent length 6 ft.=72 ins. $\qquad r=\frac{72}{3}=24$

$$P=\frac{30\times7\cdot07}{1+\frac{1}{3500}(24^2)}=\frac{30\times7\cdot07\times350}{350+576}=\mathbf{80\ tons.}$$

Safe load, say, 10 to 12 tons.

7. *Find the buckling load of a mild steel hollow circular column, one end rigidly fixed and the other pin jointed. Total length,* 12′ ; *external diam.,* 4″ ; *internal,* 3·5″. *Choose suitable constants.*

Gordon : $P=\frac{AS}{1+\alpha r^2}$ (See p. 558, Vol. I.)

$A=0\cdot785(4^2-3\cdot5^2)=2\cdot95$ sq. ins.

$l=\frac{144}{\sqrt{2}}=100''$ nrly. $\qquad \alpha=$say, $\frac{1}{625}$

$r=\frac{100}{4}=25$ $\qquad s=$say, 67,000 lb. sq. in.

$$P=\frac{2\cdot95\times67000}{1+\frac{1}{625}\times625}=99{,}000\text{ lb.}=\mathbf{44\ tons.}$$

Rankine (see p. 561, Vol. I.) :

$$W=\frac{AS}{1+bR^2} \qquad b=\frac{f}{10E}=\text{say, }\frac{45000}{10\times30{,}000{,}000}=\frac{1}{6700}$$

$$K^2=\frac{D_e^2+D_i^2}{4\times 4}=\frac{16+12\cdot 2}{16}=1\cdot 76 \qquad R^2=\frac{l^2}{K^2}=\frac{10000}{1\cdot 76}=5{,}680$$

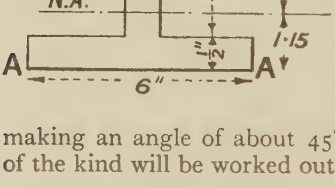
$$W=\frac{2\cdot 95\times 67000}{1+\frac{1}{6700}\times 5680}=\frac{2\cdot 95\times 67000\times 6700}{6700+5680}$$

$$=107{,}000 \text{ lb.}=\mathbf{48 \text{ tons.}}$$

Safe load, say, 6 to 8 tons.

8. *Find the buckling load of the Tee section shown. Length,* 11 *ft. Both ends fixed.* S=30 *tons sq. in.*, f=16 *tons sq. in.*, E=12,000 *tons sq. in.*

In the case of a strut section which is not symmetrical about the N.A., the question arises as to what is the diameter of the strut. It may be taken as either twice the distance of the A edge from the N.A. or twice the distance of the B edge; whichever is chosen the corresponding Z must be taken. Both methods will be found to give the same results. It is possible, however, that the strut may buckle diagonally about a N.A., making an angle of about 45° to the N.A. shown. A case of the kind will be worked out shortly.



$$A=5 \text{ sq. ins.}$$

$$I=\frac{0\cdot 5\times 3\cdot 35^3+6\times 1\cdot 15^3-5\cdot 5\times 0\cdot 65^3}{3}=8\cdot 8$$

$$Z_A=\frac{8\cdot 8}{1\cdot 15}=7\cdot 65 \qquad \alpha=\frac{Afd}{5EZ} \quad \text{(see p. 558, Vol. I.)}$$

$$\alpha_A=\frac{5\times 16\times 2(1\cdot 15)}{5\times 12000\times 7\cdot 65}=\frac{1}{2500} \qquad r_A^2=\left(\frac{66}{2\times 1\cdot 15}\right)^2=824$$

$$P=\frac{30\times 5}{1+\frac{824}{2500}}=\mathbf{113 \text{ tons}} \qquad \text{Safe load say 18 to 20 tons.}$$

Or $Z_B=\frac{8\cdot 8}{3\cdot 35}=2\cdot 63$ $\qquad r_B^2=\left(\frac{66}{2\times 3\cdot 35}\right)^2=98\cdot 5$

$$\alpha_B=\frac{5\times 16\times 2(3\cdot 35)}{5\times 12000\times 2\cdot 63}=\frac{1}{294} \qquad P=\frac{30\times 5}{1+\frac{98\cdot 5}{294}}=\mathbf{112 \text{ tons}}$$

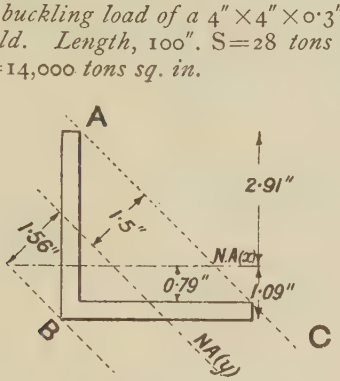
Or by Rankine: $b=\frac{f}{10E}=\frac{16}{120000}=\frac{1}{7500}$

$$K^2=\frac{8\cdot8}{5}=1\cdot76 \qquad R^2=\left(\frac{66}{1\cdot76}\right)^2=1,406$$

$$P=\frac{30\times5}{1+\frac{1406}{7500}}=\mathbf{126\ tons.}$$

For so short a strut Euler's formula gives absurd results.

9. *Find the buckling load of a* 4″×4″×0·3″ *angle. Both ends rigidly held. Length,* 100″. S=28 *tons sq. in.*, f=17 *tons sq. in.*, E=14,000 *tons sq. in.*



If the strut bends about the N.A. x we have:

$$Z_{xA}=\frac{0\cdot3\times2\cdot91^3+4\times1\cdot09^3-3\cdot7\times0\cdot79^3}{3\times2\cdot91}=1\cdot23$$

If the corners are taken as rounded this becomes 1·20.

$A=2\cdot31$ sq. ins. $\qquad \alpha_x=\frac{2\cdot31\times17\times2(2\cdot91)}{5\times14000\times1\cdot23}=\frac{1}{376}$

$$r^2=\left(\frac{50}{2\times2\cdot91}\right)^2=73\cdot8 \qquad P_x=\frac{28\times2\cdot31}{1+\frac{73\cdot8}{376}}=54\text{ tons}$$

If the strut bends about N.A. (y), which is the weakest direction,

$Z_{yB}=0\cdot91$ (obtained graphically)

$$\alpha_{yB}=\frac{2\cdot31\times17\times2(1\cdot56)}{5\times14000\times0\cdot91}=\frac{1}{520} \qquad r^2=\left(\frac{50}{2\times1\cdot56}\right)^2=257$$

$$P_y=\frac{28\times2\cdot31}{1+\frac{257}{520}}=\mathbf{43\ tons.}$$ Safe load, say, 6 to 8 tons.

Thus the strut is weaker about the axis y than about x. When struts are tested to destruction they nearly always fail by buckling about this axis and the angle opens, becoming greater than 90°.

When angles are made of thin sheet steel such as are used for aeroplane spars they are found to be stronger if loaded slightly outside the N.A.(y) and nearer B, to ensure that the points A and C shall be on the convex side when the strut buckles. When loaded on the N.A.(y) the points A and C buckle locally before the whole strut fails.

Struts of U section are frequently used for aeroplane purposes and give slightly higher buckling loads for the same sectional area than angle struts.

10. *Find the safe load for a* 12″×7″×1″ *channel section strut. Length* 16′, *ends fixed.* f=40,000 *lb. sq. in.* S=67,000 *lb. sq. in.* E=30,000,000 *lb. sq. in.*

Taking moments about the long outer edge :

$$10\times0{\cdot}5+14\times3{\cdot}5=24y \qquad y=2{\cdot}25''$$

$$I=\frac{2\times4{\cdot}75^3+12\times2{\cdot}25^3-10\times1{\cdot}25^3}{3}=110$$

$$Z=\frac{110}{4{\cdot}75}=23{\cdot}2 \quad \text{or} \quad \frac{110}{2{\cdot}25}=49$$

Taking Z as 23·2, and $d=2\times4{\cdot}75=9{\cdot}5$

$$a=\frac{Afd}{5EZ}=\frac{24\times40000\times9{\cdot}5}{5\times30,000,000\times23{\cdot}2}=\frac{1}{380}$$

$$r^2=\left(\frac{96}{9{\cdot}5}\right)^2=100 \text{ nearly}$$

$$W=\frac{SA}{1+ar^2}=\frac{67000\times24\times380}{2240\times480}=\mathbf{570 \text{ tons}}$$

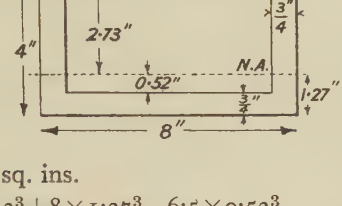
Or taking Z=49, $d=2\times2{\cdot}25=4{\cdot}5$ $r^2=456$

$$a=\frac{24\times40000\times4{\cdot}5}{5\times30,000,000\times49}=\frac{1}{1700}$$

$$W=\frac{67000\times24\times1700}{2240\times2156}=\mathbf{566 \text{ tons.}}$$

Safe load, say, **80** to **90** tons.

11. *Find the buckling load of the channel section shown. Both ends fixed.* S=24, f=16, E=12,000 *tons sq. in. Length* 14 *ft.*



A=10·87 sq. ins.

$$I=\frac{1\cdot5\times2\cdot73^3+8\times1\cdot27^3-6\cdot5\times0\cdot52^3}{3} \qquad I=15\cdot35 \text{ ins.}^4.$$

Rankine's formula: $b=\frac{f}{10E}=\frac{16}{120000}=\frac{1}{7500}$

$$K^2=\frac{15\cdot35}{10\cdot87}=1\cdot41 \qquad R^2=\frac{84^2}{1\cdot41}=5{,}000$$

$$W=\frac{10\cdot87\times24}{1+\frac{5000}{7500}}=\mathbf{156\ tons.}$$ Safe load, say, 22 to 25 tons.

12. *Calculate by Euler's and by Gordon's formulæ the safe load for a wide flanged rolled joist when used as a strut, with one end pivoted, but guided, and one end rigidly fixed. Total length,* 40 *ft. The depth of joist is* 14 *ins., breadth* 10 *ins., and the thickness* $1\frac{1}{8}$ *ins. throughout. Factor of safety,* 4. *The crushing strength of the material is* 30 *tons sq. in. Elastic limit,* 16 *tons sq. in.* E=13,000 *tons sq. in.*

$l=\frac{40}{\sqrt{2}}$ ft. $A=20\times1\cdot125+11\cdot75\times1\cdot125=35\cdot7$ sq. in.

Least $Z=\frac{100\times2\cdot25}{6}+\frac{1\cdot125^2\times11\cdot75\times1\cdot125}{6\times10}=37\cdot8.$

Least I=189.

$$a=\frac{35\cdot7\times16\times10}{5\times13000\times37\cdot8}=\frac{1}{430}$$

Gordon: $r^2=\frac{40\times40\times144}{2\times100}=1{,}152$

$$W=\frac{30\times35\cdot7\times430}{430+1152}=\mathbf{290\ tons.}$$ =8·2 tons. sq. in.

The same result is obtained by Rankine's formula

$$\textit{Euler}: \quad W = \frac{10 \times 13000 \times 37{\cdot}8 \times 5 \times 2}{480 \times 480} = \textbf{213 tons}$$

Safe Load: Gordon, **72 tons**; Euler, **53 tons.**

13. *Find the buckling load of a R.S.J. when rigidly held at one end and pivoted at the other but guided laterally. Total length*, 28′; *breadth of flanges*, 6″; *thickness*, 0·8″; *depth of section*, 10″; *thickness of web*, 0·6″; *Young's Modulus of Elasticity*, 13,000 *tons sq. in.; elastic limit of material*, 16 *tons sq. in.; crushing strength of material*, 28 *tons sq. in.*

By Gordon's formula:

$$\text{Least Z of section} = \frac{1{\cdot}6 \times 36}{6} + \frac{8{\cdot}4 \times 0{\cdot}36 \times 0{\cdot}6}{6 \times 6} = 9{\cdot}65.$$

NOTE.—If the web had been neglected Z=9·6. Thus the web is negligible for all practical strut purposes.

Area of section=9·6+5·04=14·64 sq. ins.

$$a = \frac{14{\cdot}64 \times 16 \times 6}{5 \times 13000 \times 9{\cdot}65} = \frac{1}{440}$$

$$r = \frac{28}{0{\cdot}5 \times \sqrt{2}} = 39{\cdot}7 \qquad r^2 = 1{,}580$$

$$P = \frac{28 \times 14{\cdot}64 \times 440}{2020} = \textbf{89 tons.}$$ Safe load, say, 12 to 15 tons.

By Euler's formula:

$$P = \frac{10EI}{l^2} = \frac{10 \times 13000 \times 9{\cdot}65 \times 3}{28^2} = \textbf{96 tons.}$$

2

The strut is too long to be dealt with by the straight line formula.

14. *Two R.S.Js.* 15″ *deep*, 6″ *wide, flanges* 0·88″ *thick, web* 0·5″ *thick, are fastened together to form a box section stanchion* (*with the two webs parallel*) *by riveting a plate* 12″ *wide* $\frac{5}{8}$″ *thick to each pair of flanges. When calculating the sectional area and the modulus of the section the rivet holes may be neglected. Length*, 36 *ft., both ends fixed. Crushing strength of material*, 30 *tons per sq. inch. Elastic*

limit, 16 *tons sq. in. Modulus of elasticity*, 12,000 *tons sq. in. Factor of safety*, 5. *Find the safe load.*

Taking the N.A. parallel to the webs :

$$Z_1=\frac{2(0{\cdot}625+0{\cdot}88)12^2}{6}+(15-2\times0{\cdot}88)0{\cdot}5\times6\times\tfrac{6}{12} \qquad Z_1=92$$

Taking the N.A. parallel to the plates :

$Z_2=12(0{\cdot}625+0{\cdot}88)\times(15+2\times0{\cdot}625)=293$ approx.

Sectional area $A=12\times2(0{\cdot}625+0{\cdot}88)+13{\cdot}24=49{\cdot}4$ sq. ins.

$$a=\frac{Afd}{5EZ}=\frac{49{\cdot}4\times16\times12}{5\times12000\times92}=\frac{1}{580}$$

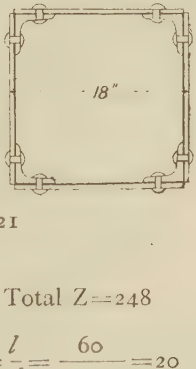
$$l=\frac{36\times12}{2}=216'' \qquad \left(\frac{l}{d}\right)^2=\left(\frac{216}{12}\right)^2=18^2=324$$

$$P=\frac{49{\cdot}4\times30\times580}{580+324}=950 \text{ tons} \quad \text{Safe load, } \mathbf{190 \text{ tons.}}$$

From the great difference in the two Zs of the section, it is evident that a more economical strut would be obtained by putting the two R.S.Js. further apart, thus making a rectangular box section.

Two channel sections with the webs outside give a more economical section, but the riveting is a difficulty except in very large sections and in comparatively short struts.

15. *A box column is built up of four* $\frac{1}{2}''$ *plates with internal angles* $4''\times4''\times0{\cdot}3$ (*British Standard, B.S.E.A.* 11) *at each corner. Length*, 60 *ft. Both ends rigidly held. Find the buckling load.* $E=30{,}000{,}000$, $S=67{,}000$, $f=45{,}000$ *lb. sq. inch. Neglect rivet holes.*



Area of section $=17\times\frac{1}{2}\times4+4\times2{\cdot}31$ $=43{\cdot}2$ sq. ins.

Distance of c. of g. from edge of angle $=1{\cdot}09''$

I angles $=\{3{\cdot}48+2{\cdot}31(8{\cdot}5-1{\cdot}09)^2\}4=521$

Z angles $=\frac{521}{9}=58$ nly.

$$Z \text{ plates}=\frac{17\times18^3-16\times17^3}{6\times18}=190 \qquad \text{Total } Z=248$$

$$a=\frac{43{\cdot}2\times45000\times18}{5\times30{,}000{,}000\times248}=\frac{1}{1060} \qquad r=\frac{l}{d}=\frac{60}{2\times1{\cdot}5}=20$$

$$W=\frac{67000\times43{\cdot}2\times1060}{1060+20^2}=2{,}100{,}000 \text{ lb.} = \mathbf{940 \text{ tons.}}$$

16. *Find the safe load—factor of safety 6—for a strut made up of two Tees 5″ × 2·5″ × ½″ with the 5″ sides riveted together back to back. Length 120″, both ends pivoted.* f=16, E=12,000 *tons sq. in.*, S=30 *tons sq. in.*

$$I=\frac{5^3}{24}+\frac{4\cdot5\times1}{12}=5\cdot58 \qquad Z=\frac{5\cdot58}{2\cdot5}=2\cdot23$$

A= 7 sq. ins. $\quad K^2=\frac{5\cdot58}{7}=0\cdot8 \quad K=0\cdot89$

$$R^2=\frac{120^2}{0\cdot8}=18{,}000 \qquad b=\frac{f}{10E}=\frac{1}{7500}$$

$$P=\frac{30\times7}{1+\frac{18000}{7500}}=62 \text{ tons, by Rankine.}$$ Safe load, **10 tons.**

An expression commonly used for the safe load is :

$$P=\left(6-\frac{l}{40K}\right)A \qquad P=\left(6-\frac{120}{40\times0\cdot89}\right)7$$

$$=(6-3\cdot37)7=\mathbf{18\cdot4 \text{ tons}}$$

but it gives a low factor of safety, viz. 3·3.

17. *A steel stanchion supports a roof principal which exerts a horizontal as well as a vertical thrust. The foot of the stanchion is rigidly held at the base. Height, 177″. Vertical load, 8·9 tons; horizontal thrust, 6 tons.* Z=200. *Sectional area, 51 sq. ins. Find the stress in the material, also find the safe load if there were no thrust.* $\alpha=\frac{1}{1230}$. S=25 *tons sq. in. Least diam. 12″.*

Bend. moment at foot=6 × 177=1,062 in.-tons

Bend. stress=$\frac{1062}{200}$=5·3 tons sq. in.

Comp. stress all over section $=\frac{8\cdot9}{51}=0\cdot2$ tons sq. in. (nearly)

Max. comp. stress=**5·5 tons sq. in.**

Max. ten. stress=5·1 tons sq. in.

Buckling load, assuming the top to be free,

$$l=2\times177=354 \text{ ins.} \qquad \left(\frac{l}{d}\right)^2=29\cdot5^2=870$$

$$P=\frac{51\times25\times1230}{1230+870}=750 \text{ tons.}$$ Safe load =150 tons.

Thus the horizontal thrust reduces the safe load from 150 tons to about 9 tons.

This example serves to show the weakening effect of a

horizontal thrust on a stanchion; but designers are very apt to overlook it and to trust to tables of safe loads for stanchions irrespective of the horizontal thrust.

18. *A column* 40 *ft. long is placed under the middle of a girder to which it is rigidly bolted and the base is firmly held. The distributed load on the girder is* 144 *tons. The column consists of four* $5'' \times 5'' \times \frac{1}{2}''$ *angles held together by diagonal bracing to form a square with an angle at each corner. Find the outside dimensions of the square to give a factor of safety of* 5 *against buckling. In Rankine's formula* S=28 *tons sq. in.,* $f=15$, E=12,000. *Neglect rivet holes.*

Load on column $=\frac{5}{8} \times 144 = 90$ tons (see p. 532, Vol. I.).
Sectional area $=4 \times 4{\cdot}75 = 19$ sq. ins. (neglecting holes).
The column buckles at $5 \times 90 = 450$ tons.
Taking moments about edge to find c. of g.

$$= \frac{4{\cdot}5 \times 0{\cdot}5 \times 0{\cdot}25 + 5 \times 0{\cdot}5 \times 2{\cdot}5}{4{\cdot}75} = 1{\cdot}434''$$

I about N.A. parallel to one edge

$$= \frac{0{\cdot}5 \times 3{\cdot}566^3 + 5 \times 1{\cdot}434^3 - 4{\cdot}5 \times 0{\cdot}934^3}{3} = \mathbf{11{\cdot}25.}$$

The above angle is British Standard (B.S.E.A. 13).

The moment of inertia is given for an angle 0·375″ thick, but not for 0·5″ thick. For the former I=8·51, for the latter the I may be found very nearly by increasing it in the ratio of the areas of the sections or the weights per foot, thus:

$$I = \frac{8{\cdot}51 \times 16{\cdot}15}{12{\cdot}27} = 11{\cdot}2\text{;}$$ the exact value being 11·15

Let d=side of square

$$\text{I for column} = \left\{11{\cdot}25 + 4{\cdot}75\left(\frac{d}{2} - 1{\cdot}43\right)^2\right\}4$$

$$= 83{\cdot}76 + 4{\cdot}76d^2 - 27{\cdot}16d$$

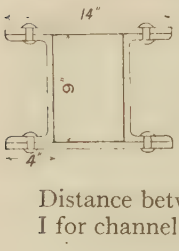
Let $\mathbf{d=15''}$ (arrived at by trial and error)

$$I = 743 \quad \text{and} \quad K_{min.}{}^2 = \tfrac{743}{19} = 39{\cdot}1$$

$$r^2 = \frac{l^2}{K^2} = \frac{240^2}{39{\cdot}1} = 1{,}470 \qquad b = \tfrac{15}{120000} = \tfrac{1}{8000}$$

$$P = \frac{28 \times 19 \times 8000}{1470 + 8000} = 450 \text{ tons.}$$

19. *A column built up of two* 9″×4″ (*B.S.C.* 18) *channels and two* ½″ *plates* 14″ *wide. Sectional area of each channel* 8·4 *sq. inches.* $I_{max.}=101\cdot7$, $I_{min.}=11\cdot6$. *Distance of c. of g. from base,* 1·15″. *Length* 20 *ft. with both ends fixed.* $f=16$ *tons sq. in.,* $S=26$, $E=12{,}000$. *Neglect rivet holes. Find the safe load; factor of safety* 5.



Distance between bases of channels $=14-2\times4=6''$

Distance between N.As. of channels $=6+2\times1\cdot15=8\cdot3''$

I for channels about N.A. parallel with bases

$$=2\left\{11\cdot6+8\cdot4\times\left(\frac{8\cdot3}{2}\right)^2\right\}=313$$

I for plates (on edge) $=\dfrac{2\times0\cdot5\times14^3}{12}=229$ I for section $=542$

I for channels about N.A. parallel with plates

$$=2\times101\cdot7=203$$

$$\text{I for plates}=\left\{\frac{14\times0\cdot5^3}{12}+7\left(\frac{9}{2}+\frac{1}{4}\right)^2\right\}2=316$$

I for section $=519$

Sectional area $=2\times8\cdot4+14=30\cdot8$ sq. ins.

$$K_{min.}=\sqrt{\frac{519}{30\cdot8}}=4\cdot1''$$

By Rankine's formula:

$$b=\frac{26}{120000}=\frac{1}{4620}\qquad\frac{l}{K}=\frac{120}{4\cdot1}=29\cdot3$$

$$P=\frac{26\times30\cdot8\times4620}{4620+856}=676 \text{ tons}\qquad \text{Safe load}=\mathbf{135\ tons.}$$

By the straight line formula:

$$P=\frac{(71000-1870\times\frac{120}{10})30\cdot8}{2240}=668 \text{ tons}\quad \text{Safe load}=\mathbf{134\ tons.}$$

20. *A column is loaded on a bracket as shown in the figure. Draw the bending moment diagram when the column is* (i) *pivoted at both ends,* (ii) *built in at both ends.*

Taking moments about x:

$$R_1 = \frac{5 \times 8}{70} = 0{\cdot}57 \text{ ton} \qquad R_2 = -0{\cdot}57 \text{ ton}$$

$$M_m = 0{\cdot}57 \times 10 = 5{\cdot}7 \text{ in. tons.}$$
$$M_n = -0{\cdot}57 \times 40 = -22{\cdot}8 \text{ in.-tons.}$$

In B, *abc* is the bending moment diagram with free ends for P_m only and *adc* for P_n only.

$$P_m = \frac{8 \times 5}{20} = 2 \text{ tons.}$$

In C, *abc* is the same as in B, and *aefc* is the pier moment

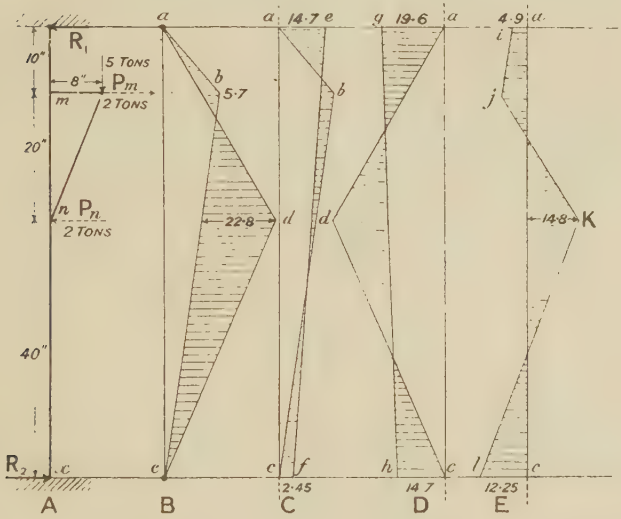


diagram. In Example 21, Chap. XII., we showed that the pier moments are:

$$ae = \frac{10 \times 60^2 \times 2}{70^2} = 14{\cdot}7 \text{ in.-tons}$$

$$cf = \frac{10^2 \times 60 \times 2}{70^2} = 2{\cdot}45 \text{ in.-tons.}$$

In D, *adc* is the same as in B, and *aghc* is the pier moment diagram :

$$ag = \frac{30 \times 40^2 \times 2}{70^2} = 19{\cdot}6 \text{ in.-tons}$$

$$hc = \frac{30^2 \times 40 \times 2}{70^2} = 14{\cdot}7 \text{ in.-tons.}$$

The two diagrams are superposed in E :

$$ai = ag - ae = 19{\cdot}6 - 14{\cdot}7 = 4{\cdot}9 \text{ in.-tons}$$

$$cl = hc - cf = 14{\cdot}7 - 2{\cdot}45 = 12{\cdot}25 \text{ in.-tons.}$$

The bending moment at d due to P_n only is 34·3 in.-tons.
The pier moment at $d = 14{\cdot}7 + (19{\cdot}6 - 14{\cdot}7)\frac{40}{70} = 17{\cdot}5$ in.-tons.
The bending moment at d due to P_m only is 11·4 in.-tons.
The pier moment at $d = 2{\cdot}45 + (14{\cdot}7 - 2{\cdot}45)\frac{40}{70} = 9{\cdot}4$ in.-tons.
The bending moment at $K = 34{\cdot}3 - 17{\cdot}5 - (11{\cdot}4 - 9{\cdot}4)$
$= 14{\cdot}8$ in.-tons.

Thus with built-in ends the bending moment is $\frac{14{\cdot}8}{22{\cdot}8} = 0{\cdot}65$ of the bending moment when the ends are pivoted.

Diagram E might have been obtained by plotting B to a straight base and putting in the pier moments *ia* and *lc*.

CHAPTER XVI

TORSION

1. The calculation of the strength and stiffness of shafts of sections other than circular, square, rectangular, elliptical, and triangular has never been accomplished by rigid mathematical methods. An extremely ingenious experimental-analytical method has recently been devised by Griffiths, which enables shafts of any section to be dealt with. Readers are referred to the following papers, published by H.M. Stationery Office:

Taylor and Griffith: "The use of Soap Films in solving Torsion Problems" (Reports and Memoranda, No. 333).

Griffith: "The Determination of the Torsional Stiffness and Strength of Cylindrical Bars of any Shape" (No. 334).

Griffith and Taylor: "The Application of Soap Films to the Determination of the Torsion and Flexure of Hollow Shafts" (No. 392).

Griffith and Taylor: "The Problem of Flexure and its Solution by the Soap Film Method" (No. 399).

2. *A rope is wrapped round a pulley* 7 *ft. diam. and from its end a weight of* 5 *tons is suspended. Find the diam. of the shaft* (i) *solid*, (ii) *hollow, in which the external diam. is three times the internal. Stress*, 3 *tons per sq. inch.*

(i) Twisting moment = Stress × Polar Modulus of the section

$$5 \times \frac{7 \times 12}{2} = 3 \frac{D^3}{5 \cdot 1} \qquad D = 7 \cdot 73''$$

(ii) $$5 \times \frac{7 \times 12}{2} = \frac{3}{5 \cdot 1} \left(\frac{D^4 - \left(\frac{D}{3}\right)^4}{D} \right) \qquad D = 7 \cdot 76''.$$

3. *Compare the strengths and weights of a solid wrought-iron shaft and a hollow steel shaft of the same external diam. —taking the internal diam. of the hollow shaft as half the external, the working stress of steel $1\frac{1}{2}$ times that of iron, and the densities of wrought iron and steel to be the same.*

The strength varies as the polar modulus of the section, hence :

$$\frac{\pi D^3}{16} Kf_i = \frac{\pi(D^4 - D_i^4)1{\cdot}5 f_i}{16D}$$

$$\frac{\pi D^3}{16} K = \frac{\pi \times \frac{15}{16} D^4 \times 1{\cdot}5}{16D} \qquad K = \frac{22{\cdot}5}{16} = 1{\cdot}4.$$

Thus the strength of the steel shaft is 1·4 times as great as the wrought-iron shaft. The weight varies as the area of the section. Hence :

$$c \cdot \frac{\pi D^2}{4} = \frac{\pi}{4}(\tfrac{3}{4}D^2) \qquad \therefore \quad c = \tfrac{3}{4}$$

$\therefore$ the steel shaft is $\frac{3}{4}$ weight of iron shaft.

4. *Find the relative weights of circular, square, and hollow circular shafts of the same strength. In the hollow shaft* $d = \frac{3}{4}D_e$.

Since the stress is the same in all, the strength varies as the polar modulus of the section.

Solid circular : $Z_p = 0{\cdot}196D^3$

Square : $Z_p = 0{\cdot}208S^3 \qquad \therefore S = D\sqrt[3]{\frac{0{\cdot}196}{0{\cdot}208}} = 0{\cdot}98D$

Hollow circular : $Z_p = \dfrac{0{\cdot}196\left(D_e^4 - \frac{3^4}{4^4}D_e^4\right)}{D_e} = 0{\cdot}134D_e^3$

$$\therefore D_e = \sqrt[3]{\frac{0{\cdot}196}{0{\cdot}134}}D = 1{\cdot}135D \qquad d = 0{\cdot}85D.$$

The *weights* vary as the sectional areas of the shafts.

Solid circular : Area $= 0{\cdot}785D^2$

Square : Area $= S^2 = (0{\cdot}98D)^2 = 0{\cdot}96D^2$

Hollow circular $= 0{\cdot}785(1{\cdot}135^2 - 0{\cdot}85^2)D^2 = 0{\cdot}44D^2$

Solid circle : Square : Hollow circle

:: 0·785 : 0·96 : 0·44 :: **1 : 1·22 : 0·56.**

5. *Find an expression for the weight* W *of a solid circular shaft required to transmit a given mean horse-power* P *at a given speed* N *revs. p. min. Ratio of max. to mean twisting moment* K. *Length of shaft,* l *ins. Diam.* d *ins. Stress* f_s *lb. per sq. in. Weight of* 1 *cu. in.* 0·28 *lb.*

Let the max. twisting moment be T pound-ins. Then

$$P=\frac{2\pi TN}{21\times 33000K} \qquad T=\frac{198000PK}{\pi N}=f_s\frac{\pi d^3}{16}=\frac{f_s d^2}{4}\times\frac{\pi d^2}{4}$$

$$\frac{\pi d^2}{4}=\frac{4\times 198000PK}{\pi N f_s d}$$

$$W=\frac{\pi d^2}{4}\times 0{\cdot}28l=\frac{0{\cdot}28\times 4\times 198000PlK}{\pi N f_s d} \qquad \mathbf{W=\frac{71000KPl}{f_s dN}}.$$

6. *Find the diam. of a hollow shaft required to transmit* 500 *horse-power at* 200 *r.p.m. The stress must not exceed* 6000 *lb. sq. in. The internal diam. is one-half the external.*

$$\text{Polar modulus for hollow shaft}=\frac{\pi(D^4-d^4)}{16D}=\frac{\pi}{16}\left(\tfrac{15}{16}D^3\right)$$

$$HP=\frac{2\pi^2 N\left(\frac{15}{16}D^3\right)f_s}{12\times 33000\times 16}$$

$$D^3=\frac{500\times 12\times 33000\times 16^2}{2\times 3{\cdot}14^2\times 200\times 15\times 6000}=143 \qquad D=\mathbf{5{\cdot}23''}.$$

7. *Find the diam. of a shaft to transmit* 40 *H.P. at* 90 *r.p.m. if the max. twisting moment exceeds the mean by* 33 *per cent., and the max. shear stress is not to exceed* 7000 *lb. sq. in.* G = 12,000,000 *lb. sq. in. Calculate the angular twist on a length of* 5 *feet.*

$$T_{max.}=\frac{12\times 33000\times \text{H.P.}\times 1{\cdot}33}{2\pi N}=37{,}240 \text{ lb.-ins.}$$

$$37{,}240=\frac{\pi D^3 f_s}{16}=\frac{\pi D^3\times 7000}{16} \qquad \therefore\ D=\mathbf{3 \text{ ins.}}$$

The angle of twist :

$$\theta=\frac{584TL}{GD^4}=\frac{584\times 37240\times 5\times 12}{12{,}000{,}000\times 81}=\mathbf{1{\cdot}34^\circ}.$$

8. *A vessel having a single propeller shaft* 12 *ins. diam., and running at* 160 *r.p.m., is re-engined with turbines*

driving two equal propeller shafts at 750 r.p.m. and developing 60 per cent. more horse-power. If the working stresses of the new shafts are 10 per cent. greater than that of the old shaft, find their diameters.

$$\text{Horse-power} = \frac{2\pi TN}{12 \times 33000}$$ where T is the twisting moment in inch-pounds.

$$T = \frac{f_s \pi D^3}{16} \qquad \text{Hence H.P.}_1 = \frac{2\pi^2 f_s D_1^3 N_1}{12 \times 16 \times 33000}$$

$$\text{H.P.}_2 = \frac{2\pi^2 (1{\cdot}1 f_s) D_2^3 N_2}{12 \times 16 \times 33000}$$

$$1{\cdot}1 D_2^3 N_2 = \frac{1{\cdot}6 D_1^3 N_1}{2} \qquad D_1^3 N_1 = 1{\cdot}375 D_2^3 N_2$$

$$D_2 = \sqrt[3]{\frac{12^3 \times 160}{1{\cdot}375 \times 750}} = \mathbf{6{\cdot}45''}.$$

9. *The outboard end of a propeller shaft of a ship whose engines developed 1,000 H.P. at 60 r.p.m., was originally 8" diam., but due to the action of the sea water it was found to be corroded to a depth of $\frac{3}{8}$" all over the surface. If the max. shearing stress is to be kept the same as before and the H.P. correspondingly reduced, find the reduced H.P. and the r.p.m., also find the max. working stress. Assume that the power varies as (speed)^3, and further that the ratio of max. to mean twisting moment remains the same as before.*

$$\text{Twisting moment } M_t = \frac{1000 \times 33000}{60 \times 2\pi} = 87{,}700 \text{ lb.-ft.}$$

$$f_s = \frac{5{\cdot}1 \times 87700 \times 12}{8^3} = 10{,}470 \text{ lb. sq. in.}$$

$$M_{t2} = \frac{10470 \times 7{\cdot}25^3}{12 \times 5{\cdot}1} = 65{,}200 \text{ lb.-ft.}$$

$$\text{H.P.}_2 = \frac{65200 \times 2\pi N_2}{33000} = 12{\cdot}4 N_2$$

$$\frac{\text{H.P.}_2}{\text{H.P.}_1} = \frac{N_2^3}{N_1^3} \qquad \therefore \quad \frac{12{\cdot}4 N_2}{1000} = \frac{N_2^3}{216000}$$

$$N_2^2 = 216 \times 12{\cdot}4 = 2680 \qquad N_2 = \mathbf{51{\cdot}8}$$

$$\text{H.P.}_2 = 51{\cdot}8 \times 12{\cdot}4 = \mathbf{643}.$$

10. *A flanged coupling on a propeller shaft is provided with nine 3 in. bolts on a 30 in. circle. Find the mean shear*

stress on the bolts when transmitting 6,000 *H.P. at* 110 *r.p.m.*

$$\text{Horse-power} = \frac{2\pi NT}{12 \times 33000}$$

$$T = \frac{6000 \times 12 \times 33000}{2 \times 3{\cdot}14 \times 110} = 3{,}440{,}000 \text{ lb.-ins.}$$

$$3{,}440{,}000 = f_s \times 9 \times 0{\cdot}785 \times 3^2 \times \tfrac{30}{2}$$

$$f_s = \mathbf{3{,}600} \text{ lb. per sq. in.}$$

11. *A cast-iron sleeve is attached at each end to a steel shaft. Find the stress in the shaft and sleeve. Length of sleeve,* 20″ ; *diam. of steel shaft and interior of sleeve* (D_s), 1·2″ ; *diam. of exterior of sleeve* (D_i), 2″ ; *angle of twist,* 0·8° ; G_s (steel), 6,000 *tons sq. in.* ; G_i (*cast-iron*), 3,000 *tons sq. in.*

The angle of twist is the same in both, hence :

$$\frac{584 T_s l}{G_s D_s^4} = \frac{584 T_i l}{G_i (D_i^4 - D_s^4)}$$

$$\frac{T_s}{T_i} = \frac{G_s D_s^4}{G(D_i^4 - D_s^4)} = \frac{6000 \times 1{\cdot}2^4}{3000(2^4 - 1{\cdot}2^4)} = \frac{2 \times 2{\cdot}07}{13{\cdot}93} = \frac{1}{3{\cdot}36}$$

Twist of shaft $0{\cdot}8 = \dfrac{584 T_s \times 20}{6000 \times 2{\cdot}07}$ $\qquad T_s = 0{\cdot}85$ inch-tons

$$f_s = \frac{5{\cdot}1 \times 0{\cdot}85}{1{\cdot}2^3} = \mathbf{2{\cdot}5 \text{ tons sq. inch}}$$

$$f_i = \frac{5{\cdot}1 \times 0{\cdot}85 \times 3{\cdot}36 \times 6000}{13{\cdot}93 \times 3000} = \mathbf{2{\cdot}09 \text{ tons sq. in.}}$$

In all cases $\dfrac{f_s}{f_i} = \dfrac{G_s D_s}{G_i D_i} = \dfrac{2 \times 1{\cdot}2}{2} = \mathbf{1{\cdot}2}.$

12. *Find the diam. required for a crankshaft having an overhung crank to satisfy* (a) *Rankine hypothesis, tensile stress* 8,000 *lb. sq. in.* ; (b) *Guest hypothesis, shear stress* 5,000 *lb. sq. in. Throw of crank,* 14″ ; *distance from plane of connecting rod to centre of nearest bearing,* 17″ ; *diameter of piston,* 30″ ; *steam pressure,* 120 *lb. sq. in.*

Total load on piston $= 120 \times 0{\cdot}785 \times 30^2 = 84{,}800$ lb.
Bending moment $M = 84{,}800 \times 17 = 1{,}440{,}000$ lb.-inches
Twisting moment $T = 84{,}800 \times 14 = 1{,}186{,}000$ lb.-inches

(*a*) $f_t Z_p = M + \sqrt{M^2+T^2}$

$$\frac{8000D^3}{5\cdot1} = 1{,}440{,}000(1+\sqrt{1+0\cdot824^2}) = 3{,}310{,}000$$

$$D^3 = 2{,}110 \qquad D = \mathbf{12\cdot8''}$$

(*b*) $f_s Z_p = \sqrt{M^2+T^2}$

$$\frac{5000D^3}{5\cdot1} = 1{,}440{,}000\sqrt{1+0\cdot824^2} = 1{,}865{,}000$$

$$D = \mathbf{12\cdot4''}.$$

13. *In an engine crank-shaft the max. twisting moment is* 1·4 *times the mean, and the bending moment is* 0·875 *times the mean twisting moment. Find the diameter of shaft required to transmit* 100 H.P. *at* 100 *r.p.m. Maximum stress on Rankine's hypothesis,* 9,000 *lb. sq. in.*

$$\text{H.P.} = \frac{2PR\pi N}{12\times33000}$$

$$PR = \frac{12\times33000\times100}{2\times100\times3\cdot14} = 63{,}000 \text{ lb.-ins. (mean)}$$

Maximum twisting moment $= 1\cdot4\times63{,}000 = 88{,}000$

Bending moment $= 0\cdot875\times63000 = 55{,}000$

$$T_e = 55{,}000(1+\sqrt{1+1\cdot6^2}) = 160{,}000$$

$$\frac{d^3}{5\cdot1} = \frac{160000}{9000} \qquad d^3 = 91 \quad \text{and} \quad d = \mathbf{4\cdot5''}.$$

14. *A shaft overhangs* 4 *feet from the nearest bearing and has a belt pulley* 5 *feet diameter at the free end. The pulley weighs* 180 *lb. The tension on the tight side of a vertical belt is* 580 *lb., and on the slack side* 110 *lb. Find the size of shaft to satisfy the condition that the maximum tensile stress shall not exceed* 3 *tons sq. in., and the maximum shear stress* 2 *tons sq. in.*

$$\left.\begin{aligned} M &= (180+690)4 = 3{,}480 \text{ lb.-feet} \\ T &= 470\times2\cdot5 = 1{,}175 \quad ,, \end{aligned}\right\} \text{ Ratio } 2\cdot96$$

$$T_e = \sqrt{M^2+T^2} = 1{,}175\sqrt{2\cdot96^2+1} = 3{,}670 \text{ lb.-feet}$$

$$\frac{12\times3{,}670\times5\cdot1}{2\times2240} = d_1{}^3 \qquad d_1 = 3\cdot69'' \text{ for shear conditions}$$

Similarly for tensile conditions :

$$d = 4\cdot03''$$

Hence the diameter must not be less than **4″**.

15. *A shaft of a vertical engine with overhung cranks at each end supports a flywheel midway between the bearings. Find the diam. in the necks and middle of the shaft to satisfy the condition that the stress shall not exceed 2 tons per sq. in. Throw of cranks, 20″ ; load on each connecting rod, 24 tons ; distance between bearing centres, 12′ 6″ ; distance from centre of bearings to central planes of crank pins, 1′ 6″ : weight of flywheel, 36 tons.*

In the necks—

$T = 24 \times 20 = 480$ tons-ins.

$M = 24 \times 18 = 432$ tons-ins. due to steam pressure.

$T_e = 432\{1 + \sqrt{1 + (\frac{480}{432})^2}\} = 432 \times 2{\cdot}5 = 1{,}080$ tons-ins.

$$D \text{ (in necks)} = \sqrt[3]{\frac{5{\cdot}1 \times 1080}{2}} = \mathbf{14\ ins.} \text{ in necks.}$$

In the middle of the shaft :

$M = \frac{36 \times 150}{4} + 24 \times 18 = 1{,}782$ tons-in. due to wheel and steam

$T = 480$ tons-inches $\qquad T_e = 1{,}782(1 + \sqrt{1 + 0{\cdot}27^2}) = 3{,}650$

$$D \text{ (middle)} = \sqrt[3]{\frac{5{\cdot}1 \times 3650}{2}} = \mathbf{20\ ins.} \text{ (nearly).}$$

16. *A propeller shaft is subjected to twisting and bending. The twisting moment is due to the transmission of 2,500 H.P. at 90 revolutions per minute. The ben. mom. amounts to 18 foot-tons. Ext. diam. of shaft, 12 ins. ; int., 6 ins. Find the maximum intensity of direct and shear stress.*

$$\frac{2\pi PRN}{12 \times 33000} = \frac{2\pi TN}{12 \times 33000} = \text{H.P.}$$

$$T = \frac{12 \times 33000 \times 2500}{6{\cdot}28 \times 90 \times 2240} = 781 \text{ in.-tons twisting mom.}$$

$$Z_p = \frac{\pi}{16}\left(\frac{12^4 - 6^4}{12}\right) = 318, \quad f_s = \frac{781}{318} = 2{\cdot}46 \text{ tons sq. in. (shear)}$$

$$f_c = \frac{18 \times 12 \times 2}{318} = 1{\cdot}36 \text{ tons sq. in. (tension or compression)}$$

$$f_{sc} = \frac{f_c}{2} + \sqrt{\frac{f_c^2}{4} + f_s^2}$$

$$= \frac{1{\cdot}36}{2} + \sqrt{\frac{1{\cdot}85}{4} + 6{\cdot}05} = \mathbf{3{\cdot}23} \text{ tons sq. in. tensile stress}$$

$$f_{s\,max.} = \sqrt{\frac{1{\cdot}85}{4} + 6{\cdot}05} = \mathbf{2{\cdot}55} \text{ tons sq. in. shear stress.}$$

17. *A propeller shaft is* 12 *ins. in diameter. It transmits* 2,000 *H.P. at* 90 *r.p.m. It is subjected in addition to an endlong thrust of* 14 *tons and at a certain section there is a bending moment of* 18 *foot-tons. Find :* (1) *the max. and min. principal stresses ;* (2) *the max. shear stress ;* (3) *the direct stress which will produce a strain equal to that produced by the principal stresses if Poisson's ratio* n=4.

Twisting moment $T=\frac{12\times33000\times2000}{2\times3\cdot14\times90\times2240}=625$ in.-tons.

Shear stress $f_s=\frac{625\times16}{3\cdot14\times12^3}=1\cdot84$ tons sq. in.

End thrust $\frac{14}{0\cdot785\times12^2}=0\cdot12$ ton sq. in.

Bending stress $f=\frac{18\times12\times32}{3\cdot14\times12^3}=1\cdot27$ tons sq. in.

Total longitudinal compression $=0\cdot12+1\cdot27=1\cdot4$ tons sq. in.

(1) $f_{sc}=\frac{f_c}{2}\pm\sqrt{\frac{f_c^2}{4}+f_s^2}=\frac{1\cdot4}{2}\pm\sqrt{\frac{1\cdot4^2}{4}+1\cdot84^2}$

$=$**2·67 tons sq. in.** and $-$**1·27 tons sq. in.**

(2) $T_e=\sqrt{M^2+T^2}=\sqrt{216^2+625^2}=661$ in.-tons

$f_s=\frac{661\times16}{3\cdot14\times12^3}=$**1·95 tons sq. in.**

(3) $\frac{f}{E}=\frac{f_{sc}}{E}+\frac{f}{nE}$ $\qquad f=f_{sc}+\frac{f}{4}$

$f=2\cdot67+\frac{1\cdot27}{4}=$**2·99 tons sq. in.**

18. *A hollow circular shaft is required for a* 1,000 *H.P. steam engine. The max. twisting moment on the shaft is* 1·8 *times the mean. The shaft rests in bearings* 15 *ft. apart and carries a flywheel weighing* 90 *tons in the middle. Find the ext. diam. of the shaft. Stress in tons per sq. inch,* 4 ; *ratio of int. to extl. diam. of shaft,* 0·6 ; *r.p.m.,* 80. *The ben. mom. due to the steam pressure is of the same sign as that due to the flywheel and is* 0·8 *times the mean twisting moment.*

$H.P.=\frac{2\pi NT_m}{12\times33000}$ where T_m=Mean twisting moment

$$T_m = \frac{1000 \times 12 \times 33000}{6{\cdot}28 \times 80} = 788{,}000 \text{ inch-lb.} = 352 \text{ in.-tons}$$

$$T_{max.} = 352 \times 1{\cdot}8 = 630$$

$$\text{M due to flywheel} = \frac{90 \times 180}{4} = 4{,}050 \text{ in.-tons}$$

$$\text{M due to steam} = 352 \times 0{\cdot}8 = \frac{282}{4{,}332} \text{ ,,}$$

$$\text{Total} \quad \text{,,}$$

$$T_e = 4{,}332(1 + \sqrt{1 + 0{\cdot}146^2}) = 8{,}700 \text{ in.-tons}$$

$$Z = \frac{D^4 - (0{\cdot}6D)^4}{5{\cdot}1D} = 0{\cdot}17D^3$$

$$D^3 = \frac{8700}{4 \times 0{\cdot}17} = 12{,}800 \qquad \mathbf{D = 23{\cdot}4''.}$$

19. *A gas engine flywheel weighing* 2 *tons is keyed to an overhanging shaft* 6 *ins. diam. and at a distance of* 15 *ins. from the middle of the nearest bearing. The fluctuation in the twisting moment is such that a point at the radius of gyration of the wheel is accelerated* 12 *f.p.s.p.s. Find the maximum intensity of tensile and shear stresses in the shaft. Radius of gyration,* 30 *ins.*

$$\text{Twisting moment} \quad T = \frac{2 \times 12 \times 30}{32{\cdot}2} = 22{\cdot}4 \text{ in.-tons}$$

$$\text{Bending moment} \quad M = 2 \times 15 = 30 \text{ in.-tons}$$

$$T_e = 30 + \sqrt{30^2 + 22{\cdot}4^2} = 67{\cdot}5 \text{ inch-tons}$$

$$f_{es} = \frac{67{\cdot}5 \times 16}{3{\cdot}14 \times 216} = \mathbf{1{\cdot}59 \text{ tons sq. in. tension}}$$

$$T_e' = \sqrt{30^2 + 22{\cdot}4^2} = 37{\cdot}5 \text{ inch-tons}$$

$$f_s = \frac{37{\cdot}5 \times 16}{3{\cdot}14 \times 216} = \mathbf{0{\cdot}89 \text{ ton sq. in shear stress.}}$$

20. *A circular shaft* 3″ *diam. and* 420′ *long springs* 51° *when transmitting power at* 180 *r.p.m. Find the H.P. transmitted.*

$$\Theta = \frac{584Tl}{Gd^4}$$

$$T = \frac{51 \times 12.000{,}000 \times 81}{584 \times 420 \times 12} = 16{,}900 \text{ lb.-ins.} = 1{,}410 \text{ lb.-ft.}$$

$$\text{H.P.} = \frac{P2\pi RN}{33000} = \frac{1410 \times 3{\cdot}14 \times 2 \times 180}{33000} = \mathbf{48{\cdot}2.}$$

21. *A square shaft is required to transmit* 60 *H.P. at* 130 *r.p.m. Find the side of the square, and also the twist on a length of* 80 *ft.* f_s=6,000 *lb. sq. in.;* G=12,000,000 *lb. sq. in.*

$$M_t=\frac{12\times33000\times H.P.}{2\pi N}=f_sZ_p$$

$$\frac{12\times33000\times60}{2\pi\times130}=6{,}000\times0{\cdot}208S^3 \qquad S=\mathbf{2{\cdot}86\ ins.}$$

$$\Theta=\frac{410M_tl}{S^4G}=\frac{410\times6000\times0{\cdot}208\times2{\cdot}86^3\times80\times12}{2{\cdot}86^4\times12{,}000{,}000}=\mathbf{14{\cdot}3^\circ.}$$

22. *Find the angle of twist of a tapered shaft, length* (L), 100″; *diam. at large end* (D), 6″; *taper, where* l=*distance from large end*=$\frac{l}{50}=nl$; *diam. at small end* (d), 4″; *twist. mom.* (M_t), 150,000 *lb.-ins.;* G=12,000,000 *lb. per sq. inch.*

The diam. at any distance l from large end

$$d=D-nl$$

Then $$d\theta=\frac{584M_t}{G}\cdot\frac{dl}{(D-nl)^4}$$

$$\theta=\frac{584M_t}{G}\int_{l=0}^{l=L}(D-nl)^{-4}dl=\frac{584M_t}{G}\left[\frac{(D-nl)^{-3}}{3n}\right]_{l=0}^{l=L}$$

$$=\frac{584M_t}{3Gn}\left\{\frac{1}{(D-nl)^3}-\frac{1}{D^3}\right\}=\frac{584M_t}{3Gn}\left(\frac{1}{d^3}-\frac{1}{D^3}\right)$$

$$=\frac{M_t}{G}\times107=\mathbf{1{\cdot}3\ degree.}$$

If the shaft were 6″ diam. throughout θ=0·56 degree.
,, ,, 4″ ,, ,, θ=2·9 ,,

23. *Calculate the maximum intensity of stress in a shaft of elliptical section* 8″×5″ *when subjected to a twisting moment of* 400,000 *inch-pounds.*

Also find the angle of twist on a length of 30 *feet. Modulus of rigidity* 11,000,000 *lb. sq. inch.*

$$400{,}000=f_s\frac{8\times25}{5{\cdot}1} \qquad f_s=\mathbf{10{,}200\ lb.\ sq.\ in.}$$ (See Vol. I. p. 578)

$$\theta=\frac{292\times400{,}000\times360(25+64)}{11{,}000{,}000\times125\times512}=\mathbf{5{\cdot}32^\circ.}$$

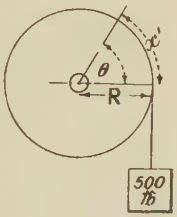
24. *A square shaft is used for transmitting power to an overhead travelling crane which is capable of lifting a load of 20 tons at the rate of 16 ft. per min. Efficiency of gearing 71 per cent. Find the size of shaft required. R.p.m. 160. Stress 4,000 lb. sq. in.*

$$T = fZ_p = 4000 \times 0{\cdot}208S^3 = 832S^3$$

$$\text{H.P.} = \frac{20 \times 2240 \times 16}{0{\cdot}71 \times 33000} = 30{\cdot}5 = \frac{2\pi NT}{12 \times 33000}$$

$$30{\cdot}5 = \frac{2 \times 3{\cdot}14 \times 832 \times 160S^3}{12 \times 33000} \qquad S = \mathbf{2{\cdot}44''}.$$

25. *A shaft 20 ft. long and 2 ins. diam. is held at one end, on the other end there is a winding drum 3 ft. diam. The shaft is supported in suitable bearings. A weight of 500 lb. is suspended by a rope wrapped round the winding drum. Find the height from which this weight must be dropped in order to produce a momentary stress in the shaft of 20,000 lb. per sq. in. The stretch of the rope, the bending and inertia of the shaft and drum also the friction of the bearings may be neglected. Modulus of rigidity* G = 12,000,000 *lb. sq. in.*



Let the equivalent instantaneous load = W′

Work done in twisting shaft $= \frac{W'x'}{2}$

$$\frac{W'x'}{2} = \frac{W'}{2} \times \frac{2\pi R\theta}{360} = \frac{W'\pi R}{360} \times \frac{584Tl}{Gd^4}$$

also $$W' \times 18 = \frac{20000 \times \pi \times 2^3}{16} = T \qquad W' = 1{,}743 \text{ lb.}$$

$$x' = \frac{2 \times 3{\cdot}14 \times 18 \times 584 \times 20000 \times 3{\cdot}14 \times 2^3 \times 240}{360 \times 12{,}000{,}000 \times 2^4 \times 16} = 7{\cdot}2 \text{ ins.}$$

$$\frac{1743 \times 7{\cdot}2}{2} = 500(h + 7{\cdot}2) \qquad h = \mathbf{5{\cdot}3 \text{ ins.}}$$

26. *A close-coiled helical spring is required for a given purpose in which it is necessary for it to stretch to four times its original length. Find from the following data the ratio of the mean diameter of the coils to the diameter of the*

wire ; shear stress in the wire, lb. sq. in., 40,000 ; *coefficient of rigidity* (G), *lb. sq. in.*, 11,000,000.

$$\delta=\frac{8D^3Wn}{Gd^4}=3nd \qquad WD=\frac{f_s\pi d^3}{8}$$

Substitute in the above $\delta_1=\frac{f_s\pi D^2}{Gd}=3d$ per coil

$$\frac{D}{d}=\sqrt{\frac{3G}{\pi f_s}}=\sqrt{\frac{3\times 11,000,000}{3\cdot 14\times 40000}}=\mathbf{16\cdot 2}.$$

27. *A conical helical spring is loaded axially. Find the deflection under a load* (W) *of* 500 *lb. Diam. of wire* (*d*), 0·5″ ; *mean diam. of coils at one end* (D_1), 3″ ; *mean diam. of coils at other end* (D_2), 5″ ; *number of free coils* (n), 12 ; *modulus of rigidity* (G), 12,000,000 *lb. sq. in.*

For a cylindrical helical spring we have $\delta=\frac{8WLd^2}{\pi Gd^4}$ (see p. 587, Vol. I.).

In a conical spring the diam. D varies with the distance from the end of the spring, consequently it varies with the distance x measured along the centre line of the coils. It is convenient to start measuring this length from the smaller end of the spring where the diam. is D_1. When the larger diam. D_2 is reached the distance x becomes equal to L, *i.e.* the length of wire in the spring. Let the larger diam. be D_2. Then the diam. D at any distance x is :

$$D=D_1+\frac{x}{L}(D_2-D_1)$$

For a conical helical spring we have

$$\delta=\frac{8W}{\pi Gd^4}\int_0^L D^2dx=\frac{8W}{\pi Gd^4}\int_0^L\left[D_1+\frac{x}{L}(D_2-D_1)\right]^2dx$$

$$=\frac{8W}{\pi Gd^4}\left[D_1^2L+D_1(D_2-D_1)L+\frac{(D_2-D_1)^2}{3}L\right]$$

$$\delta=\frac{8W}{\pi Gd^4}\cdot\frac{L}{3}(D_1^2+D_1D_2+D_2^2)$$

Inserting the given quantities we have :

$$\delta=\frac{8\times 500\times 12\times\frac{5+3}{2}\times 3\cdot 14}{3\cdot 14\times 12,000,000\times 0\cdot 5^4\times 3}(25+15+9)=\mathbf{4\cdot 2}''.$$

If the cylindrical spring expression be used, taking the mean diameter D as $\frac{D_1+D_2}{2}$, we get as an approximation

$$\delta=\frac{8\times500\times12\times4\times3{\cdot}14\times16}{3{\cdot}14\times12{,}000{,}000\times0{\cdot}5^4}=\mathbf{4{\cdot}1''}.$$

28. *A conical helical spring is axially loaded; find the amount it will compress under a load* (W) *of* 100 *lb. Diam. of wire* (d), 0·3″; *mean diam. of coils at base* (D_2), 4·5″; *number of free coils*, 8; *modulus of rigidity*, 12,000,000 *lb. sq. in.*

(See Example 27.) In this case $D_1=0$

$$\delta=\frac{8W}{\pi Gd^4}\cdot\frac{l}{3}(D_2^2+D_2D_1+D_1^2)$$

$$\delta=\frac{8\times100\times8\times2{\cdot}25\times3{\cdot}14}{3{\cdot}14\times12{,}000{,}000\times0{\cdot}0081\times3}(20{\cdot}25)=\mathbf{1''}.$$

29. *A helical spring will safely store* 60 *ft.-lb. of energy per pound of spring. A tramcar weighing* 8 *tons and moving with a speed of* 12 *m.p.h. is pulled up by brakes which store* 90 *per cent. of the kinetic energy of the car in the springs. Find the weight of springs required if the efficiency of the mechanism is* 84 *per cent.*

$$12 \text{ miles per hour} = \frac{12\times5280}{3600}=17{\cdot}6 \text{ ft. per sec.}$$

$$\text{Kinetic energy of car} = \frac{8\times17{\cdot}6^2}{2\times32{\cdot}2}=38{\cdot}5 \text{ foot-tons}$$

$$\text{Kinetic energy stored in springs}=38{\cdot}5\times0{\cdot}9\times2{,}240 \text{ foot-lb.}$$

$$\text{Weight of springs required} = \frac{38{\cdot}5\times0{\cdot}9\times2240}{60\times0{\cdot}84}=\mathbf{1{,}540 \text{ lb.}}$$

30. *A weight of* 50 *lb. falls on to a helical compression spring. Find the height it must fall before striking the spring in order to compress it* 6 *ins. Number of coils*, 18; *diam. of wire in coils*, 1 *in.*; *mean diameter of coils*, 9″. G=11,000,000 *lb. per sq. inch.*

$$\text{Deflection } \delta=\frac{8D^3W_1n}{Gd^4}\Big\{\begin{array}{l}\text{where } W_1 \text{ is the maximum}\\ \text{instantaneous load.}\end{array}$$

$$W_1=\frac{\delta Gd^4}{8D^3n}=\frac{6\times11{,}000{,}000\times1}{8\times9^3\times18}=629 \text{ lb.}$$

$$W(h+\delta)=\frac{W_1\delta}{2} \qquad 50(h+\cdot6)=\frac{629\times6}{2} \qquad h=\mathbf{31\cdot7\ ins.}$$

31. *A weight of* 600 *lb. falls through a height of* 5 *ft. on to a spring which it compresses a further* 8 *ins. Find the max. load on the spring. The mean diam. of the coils is* 9 *ins. The diam. of the wire is* 1·5 *ins. Find the number of free coils required.* G=12,000,000 *lb. sq. in.*

Work done by weight $=600\times5+\frac{600\times8}{12}=3{,}400$ ft.-lb.

Work absorbed by spring

$$=\text{Max. load}\times\frac{8}{2\times12}=\frac{P_{max.}}{3}\text{ ft.-lb.}$$

$$3400=\frac{P_{max.}}{3} \qquad P_{max.}=\mathbf{10{,}200\ lb.}$$

$$\delta=\frac{8D^3P_{max.}N}{Gd^4} \qquad 8=\frac{8\times D^3\times10200\times n\times6^4}{12{,}000{,}000\times D^4}$$

$$n=\mathbf{8\cdot2\ coils}$$

The stress in the wire f_s

$$=\frac{10200\times\frac{9}{2}\times5\cdot1}{1\cdot5^3}=\mathbf{69{,}300\ lb.}\text{ sq. in.}$$

32. *A helical spring is formed of steel tubing* $\frac{3}{4}$ *in. ext. and* $\frac{1}{2}$ *in. int. diam. The mean diam. of the coils is* 8 *ins. and the number of free coils* 20. *Find:* (a) *the extension for a load of* 100 *lb.;* (b) *the safe load;* (c) *the work stored per pound of spring when stressed to* 70,000 *lb. per sq. in. Modulus of rigidity,* 11,000,000 *lb. per sq. in.*

(*a*) $\delta=\frac{8D^3Wn}{G(d_e^4-d_i^4)}=\frac{8\times8^3\times100\times20}{11{,}000{,}000(0\cdot75^4-0\cdot5^4)}=\mathbf{2\cdot93\ ins.}$

(*b*) $\frac{4W_s}{70000}=\frac{0\cdot75^4-0\cdot5^4}{0\cdot75\times5\cdot1}$ $\qquad W_s=\mathbf{1{,}160\ lb.}$

(*c*) Weight of spring = 34·5 lb.

Work stored $=\frac{W_s\delta_s}{2}=\frac{1160\times2\cdot93\times1160}{2\times12\times100}=1{,}640$ foot-lb.

Work stored per pound of metal $=\frac{1640}{34\cdot5}=\mathbf{48\ ft.\text{-}lb.}$

33. *A helical spring of square wire is used as a flexible shaft: find the power transmitted and the amount the shaft will twist. Side of square* (d), 0·25″; *mean diam. of coils* (2ρ), 1·5″; *bending stress in wire* (f), 60,000 *lb. sq. in.*; *modulus of elasticity* (E), 30,000,000 *lb. sq. in.*; *r.p.m.* (N), 1,000; *number of free coils* (n_1), 250.

$$\text{Then } M=\frac{EI}{\rho}=\frac{fI}{y} \qquad \text{Hence } f=\frac{E}{\rho}=\frac{Ey2\pi}{L}(n_1-n_2)$$

$$n_1-n_2=\frac{60000\times3\cdot14\times1\cdot5\times250}{0\cdot125\times2\times3\cdot14\times30,000,000}=\mathbf{3}$$

That is the shaft twists 3 complete turns.

Twisting moment

$$M_t=\frac{E\theta_r I}{L}=\frac{30,000,000\times2\times3\cdot14\times3\times0\cdot25^4}{3\cdot14\times1\cdot5\times250\times12}=\mathbf{156\ in.\text{-}lb.}$$

Or it might have been arrived at thus:

$$M=\frac{fI}{y}=\frac{60000\times0\cdot25^4}{0\cdot125\times12}=\mathbf{156\ in.\text{-}lb.}$$

$$\text{H.P.}=\frac{M_t\pi N}{198000}=\frac{156\times3\cdot14\times1000}{198000}=\mathbf{2\cdot5\ nearly.}$$

34. *If the flexible shaft in Example 33 had consisted of two concentric helical springs, the inner one being of wire* 0·15″ *square in section slack fitting inside the outer spring, find the stress in the inner spring, the twisting moment, and the horse-power transmitted.*

The inside diam. of the outer spring is 1·5″−0·25″=1·25″, the mean diam. of the inner spring is 1·25″−0·15″=1·1″.

The length of the outer spring, assuming close coiling, is 250×0·25=62·5″. The length of the inner spring is the same, hence the number of coils $=\frac{62\cdot5}{0\cdot15}=417$.

The angle of twist is the same in both.

For the inner coil:

$$f=\frac{Ey2\pi}{L}\times3=\frac{30,000,000\times0\cdot075\times2\times3\cdot14\times3}{3\cdot14\times417\times1\cdot1}$$

$$=\mathbf{29,500\ lb.\ sq.\ in.}$$

$$\text{Twisting moment } M=\frac{fI}{y}=\frac{29500\times0\cdot15^4}{0\cdot075\times12}=17 \text{ inch-lb.}$$

Total twisting moment $= 156 + 17 = \mathbf{173\ inch\text{-}lb.}$

$$\text{H.P.} = \frac{173 \times 3{\cdot}14 \times 1000}{198{,}000} = \mathbf{2{\cdot}7.}$$

It will be seen that the inner spring contributes but very little to the strength of the shaft.

If the outer spring is a tight fit on the inner one the problem becomes much more complex. The coils of flexible shafts are usually wound tightly on one another with a tight-fitting core, the power transmitted is then much greater than when each coil is free, as assumed in the example.

35. *A pair of buffers is required to stop a vehicle, which has no buffer springs, in a distance of* 12 *ins. The vehicle weighs* 2 *tons and is moving at* 5 *m.p.h. Design a spring of square section steel in which the mean diam. of coils* D *is* 6 *times the side of the square section* S. *Stress* f_s *not to exceed* 50,000 *lb. sq. in.*

Energy stored in moving vehicle

$$= \frac{W_1 V^2}{2g} = \frac{2 \times 2240 \times 7{\cdot}3^2}{64{\cdot}4} = 3{,}710 \text{ foot-lb.}$$

Energy stored in both springs $= \frac{2}{12} \times \frac{f_s^2 D n S^2}{24{,}700{,}000}$ foot-lb.

(See p. 589, Vol. I.)

$$3{,}710 = \tfrac{2}{12} \times \frac{50{,}000^2 D^3 n}{24{,}700{,}000 \times 36} = 0{\cdot}47 D^3 n$$

$$\text{also } W = \frac{2{,}135{,}000 S^4 \delta}{D^3 n} \qquad D^3 n = 7{,}900$$

Energy stored in both springs $= W\delta = \frac{2{,}135{,}000 S^4 \delta^2}{D^3 n}$

$$\delta = 1 \text{ foot} \qquad 3{,}710 = \frac{2{,}135{,}000 \times D^4}{D^3 n \times 1296} = 1650 \frac{D}{n}$$

$$\frac{D}{n} = 2{\cdot}25 \qquad n = \frac{D}{2{\cdot}25}$$

Substituting in the above expression:

$$7900 = \frac{D^4}{2{\cdot}25} \qquad D^4 = 17{,}800$$

$$\mathbf{D = 11{\cdot}6'',\quad n = 5{\cdot}2\ coils,\quad S = 1{\cdot}93''.}$$

If the diam. of the coils had been specified as four times the side of the square, we should have had :

$D=6{\cdot}32''$, $n=14$ nrly., $S=1{\cdot}58''$.

36. *Find the extension of an open-coiled helical spring :* (i) *Regarded as a close-coiled spring.* (ii) *Neglecting the bending of the coils.* (iii) *Taking the twisting and the bending of the coils into account. Load* (W), 90 *lb.; number of free coils* (n), 15; *diameter of wire* (d), $0{\cdot}2''$; *modulus of rigidity* (G), 12,000,000 *lb. sq. in.; mean diameter of coils* (D), $1{\cdot}8''$; *angle of coils* (α), 15°.

(i) $$\delta=\frac{8D^3Wn}{Gd^4}=\frac{8\times1{\cdot}8^3\times90\times15}{12{,}000{,}000\times0{\cdot}2^4}=\mathbf{3{\cdot}28''}$$

(ii) $$\delta=\frac{8D^3Wn}{Gd^4\cos\alpha}=\frac{3{\cdot}28}{0{\cdot}966}=\mathbf{3{\cdot}4''}$$

(iii) $$\delta=\frac{8D^3Wn}{Gd^4\cos\alpha}\left(\cos^2\alpha+\frac{\sin^2\alpha}{1{\cdot}25}\right)$$

$$=3{\cdot}4\left(0{\cdot}966^2+\frac{0{\cdot}259^2}{1{\cdot}25}\right)=\mathbf{3{\cdot}35''}.$$

CHAPTER XVII

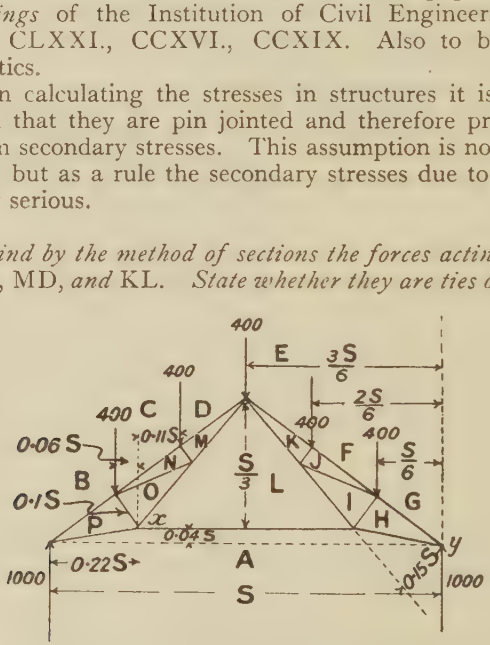
STRUCTURES

WIND PRESSURE

1. FOR recent work readers should refer to papers in the *Proceedings* of the Institution of Civil Engineers, Vols. CLVI., CLXXI., CCXVI., CCXIX. Also to books on aeronautics.

When calculating the stresses in structures it is always assumed that they are pin jointed and therefore practically free from secondary stresses. This assumption is not always justified, but as a rule the secondary stresses due to this are not very serious.

2. *Find by the method of sections the forces acting on the bars* LA, MD, *and* KL. *State whether they are ties or struts.*



Taking moments about apex for one-half of the truss

$$400\frac{S}{6}+400\frac{2S}{6}-1000\frac{3S}{6}=LA\times\frac{S}{3}$$

$$LA=\mathbf{900\ lb.\ (tie).}$$

Taking moments about x

$$1000\times0{\cdot}22S-400\times0{\cdot}06S+400\times0{\cdot}11S=MD\times0{\cdot}1S$$

$$MD=\mathbf{2400\ lb.\ (strut).}$$

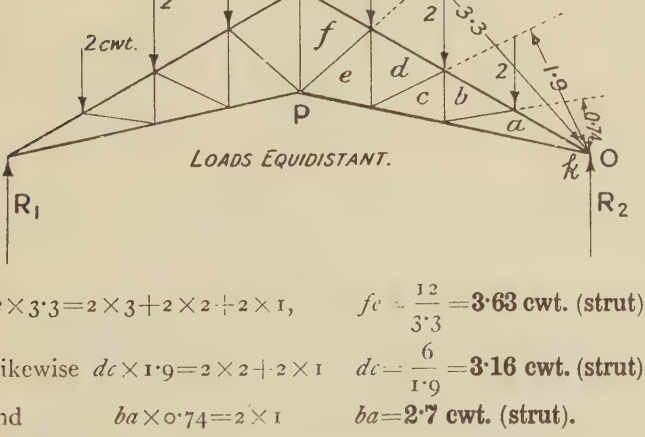
Taking moments about y

$$900\times0{\cdot}04S+400\times\frac{S}{6}+400\times\frac{2S}{6}=KL\times0{\cdot}15S$$

$$KL=\mathbf{1570\ lb.\ (tie).}$$

3. *Calculate, by taking sections, the forces acting on the bars* fe, dc, ba, *in the structure shown in the figure.*

Let the horizontal distance between the loads be unity. The perpendicular distances of the forces from O are given to the same scale. If a section be taken through the panels *g fe*P the structure to the right of this section is supported by the forces *fg*, *e*P, and *fe*, which is the only one which has a moment about the point O. The moment of *fe* about O is therefore equal to the moment of the external loads about O. Hence



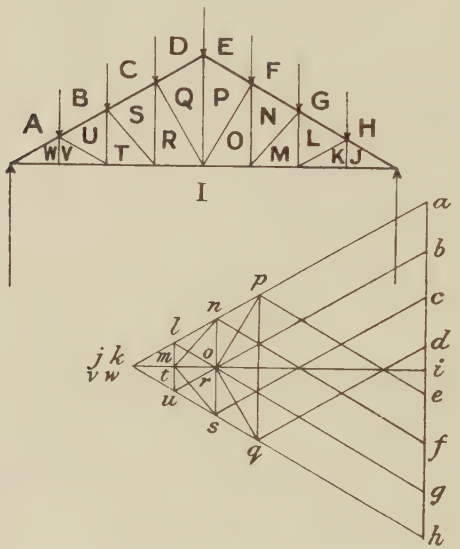
$fe\times3{\cdot}3=2\times3+2\times2+2\times1$, $\quad fe=\frac{12}{3{\cdot}3}=\mathbf{3{\cdot}63\ cwt.\ (strut)}$

Likewise $dc\times1{\cdot}9=2\times2+2\times1$ $\quad dc=\frac{6}{1{\cdot}9}=\mathbf{3{\cdot}16\ cwt.\ (strut)}$

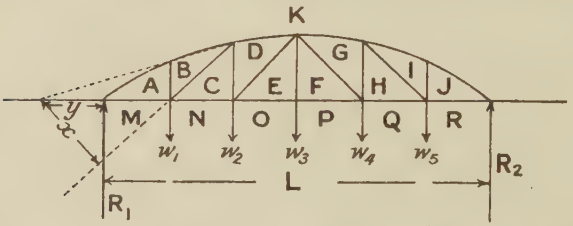
and $\quad ba\times0{\cdot}74=2\times1$ $\quad ba=\mathbf{2{\cdot}7\ cwt.\ (strut).}$

4. *Construct a reciprocal diagram for the structure shown in the figure and tabulate the forces acting in each bar. Each load is* 1200 *lb.*

Bar.	Compression (lb.)	Bar.	Tension (lb.).
AW and HJ	8400	WI and JI	7260
BU and GL	7200	VI and KI	7260
CS and FN	6000	TI and MI	6240
DQ and EP	4800	RI and OI	5190
QP	3600	VU and LK	1200
SR and ON	1200	ST and NM	1580
UT and ML	600	RQ and OP	2080
WV and KJ	0		



5. *A bowstring girder bridge of* 120 *ft. span, supports an*



evenly distributed load of 1·5 *tons per foot run. The verticals are* 20 *feet apart. Find by the method of sections the stress*

that occurs in the diagonal of the second panel; the heights of the two verticals at this panel are 18 *ft. and* 12 *ft.*

Since there are two girders side by side the load at each joint when fully covered $(w)=\frac{1\cdot5\times20}{2}=15$ tons.

Let T=force in BC. L=length of span. Then

$$Tx+R_1y-w_1\left(y+\frac{L}{6}\right)=0$$

$$T=\frac{w_1\left(y+\frac{L}{6}\right)-R_1y}{x}$$

$$R_1=\tfrac{5}{6}w_1+\tfrac{4}{6}w_2+\tfrac{3}{6}w_3+\tfrac{2}{6}w_4+\tfrac{1}{6}w_5$$

$$T=\frac{1}{x}\left\{\frac{w_1(6y+L)}{6}-\frac{5w_1y}{6}-\frac{4w_2y}{6}-\frac{3}{6}w_3y-\frac{2}{6}w_4y-\frac{1}{6}w_5y\right\}$$

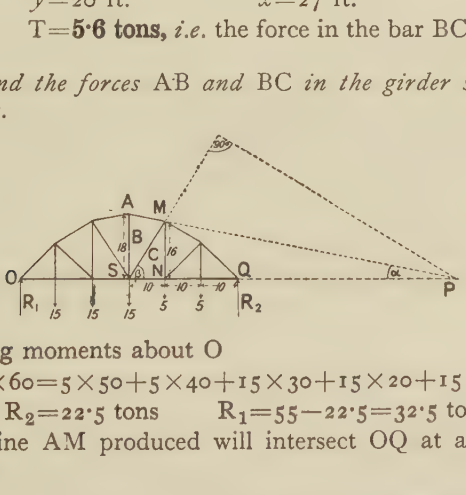
$$T=\frac{w}{x}\left\{\left(\frac{6y+L}{6}\right)-\frac{y}{6}(5+4+3+2+1)\right\}$$

where the loads are all equal.

$y=20$ ft. $x=27$ ft.

T=**5·6 tons,** *i.e.* the force in the bar BC.

6. *Find the forces* AB *and* BC *in the girder shown in the figure.*



Taking moments about O

$$R_2\times60=5\times50+5\times40+15\times30+15\times20+15\times10$$

$R_2=22\cdot5$ tons $R_1=55-22\cdot5=32\cdot5$ tons.

The line AM produced will intersect OQ at a point P such that

$\frac{PN}{16}=\frac{10}{18-16}$ $\therefore$ PN=80 ft. and PQ=60 ft.

$\tan\alpha=0\cdot2$ $\tan\beta=1\cdot6$ $\therefore\ \beta=58°$.

Taking moments about P

Force BC $\times$ PS sin $\beta = 22{\cdot}5 \times 60 - 5 \times 70 - 5 \times 80 = 600$

$$\text{Force BC} = \frac{600}{90 \times 0{\cdot}848} = \mathbf{7{\cdot}86\ tons.}$$

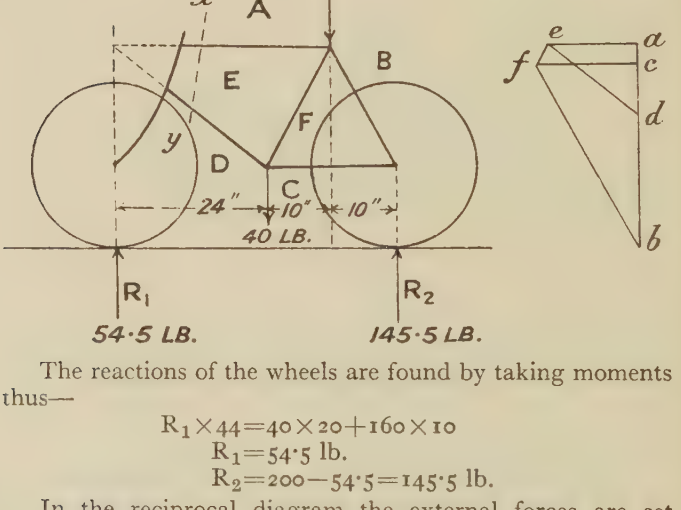
Taking moments about S we have

PS sin $\alpha \times$ Force AB $= 22{\cdot}5 \times 30 - 5 \times 20 - 5 \times 10 = 525$

Also $\alpha = 11^\circ\ 18'$ $\quad \sin \alpha = 0{\cdot}196$

$$\therefore \text{Force AB} = \frac{525}{90 \times 0{\cdot}196} = \mathbf{29{\cdot}8\ tons.}$$

7. *Find the forces acting on the members of the bicycle frame shown in the diagram. Weight of rider,* 200 *lb.; effort exerted on crank axle,* 40 *lb.*



The reactions of the wheels are found by taking moments thus—

$$R_1 \times 44 = 40 \times 20 + 160 \times 10$$
$$R_1 = 54{\cdot}5 \text{ lb.}$$
$$R_2 = 200 - 54{\cdot}5 = 145{\cdot}5 \text{ lb.}$$

In the reciprocal diagram the external forces are set down $ab = 160$ lb. $\quad bc = 145{\cdot}5 = R_2$. $\quad cd = 40$ lb. $\quad da = 54{\cdot}5$ lb. $= R_1$.

Lines are then drawn parallel to the members of the frame in the usual way from which by measurement we find—

Top tube AE = **70 lb.** Bottom tube ED = **90 lb.** Seat

struts FB=**81 lb. each.** Down tube EF=**18 lb.** Bottom back fork FC=**40 lb. each.**

If a section be taken through the frame on the line *xy* the forces to the left must balance those to the right, and to secure the condition that there is no bending in the tubes the lines of action of these forces must coincide with the axes of the tubes. The reaction R_1 is the only external force acting on the front part of the frame. To secure equilibrium the lines of action of this force and the two forces acting on the tubes must meet in a point. If they do not meet there will be a bending moment at the junction of the tubes and the steering head. When the front wheel meets an obstacle or a hollow in the road a bending moment will be set up in any case, but in general it is not sufficient to do any damage to the frame.

8. *A wheel consists of a rim which may be regarded as rigid, connected to the hub by means of thirty-two radial spokes. If the spokes have each an ultimate tensile strength of* 500 *lb., estimate the load on the axle which will crush the wheel.*

Vert. load carried by a spoke making an angle θ with the vert.$=500\cos\theta$.

The strength of spokes in comp. may be neglected. Hence total vert. load carried by wheel—

(*a*) When one spoke is vertical.

$$W_1=500(1+2\cos 11\tfrac{1}{4}+2\cos 22\tfrac{1}{2}+2\cos 33\tfrac{3}{4}+2\cos 45 +2\cos 56\tfrac{1}{4}+2\cos 67\tfrac{1}{2}+2\cos 78\tfrac{3}{4})=\mathbf{5050}\text{ lb.}$$

(*b*) When spokes are "off centres": $W_2=\mathbf{5100}$ lb.

9. *An arch of* 120 *ft. span and* 15 *ft. rise, is loaded with* 20 *tons in the middle,* 20 *tons* 70 *ft. from the springing* B, *and* 20 *tons* 80 *feet from* B. *Find the horizontal thrust at the crown due to these loads.*

$$\text{Vert. reaction at A} = \frac{20\times 60+20\times 70+20\times 80}{120}=35\text{ tons.}$$

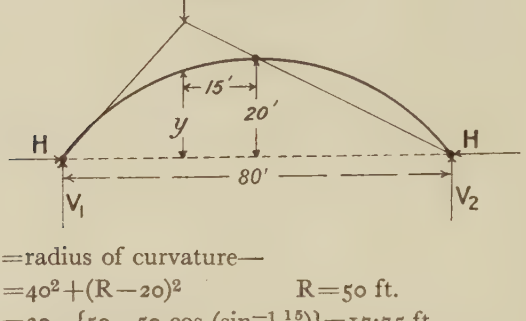
Vert. reaction at B $=60-35=25$ tons.

Hor. thrust at crown, which is the same all along the arch

$$=\frac{\text{Bending moment at crown}}{\text{Rise}}$$

$$=\frac{35\times 60-20\times 20-20\times 10}{15}=\mathbf{100\text{ tons.}}$$

10. *In a circular arch the span is* 80 *ft., and the rise at the centre is* $\frac{1}{4}$ *of the span. The arch is hinged at the ends and at the crown, and is loaded with* 10 *tons at a point* 15 *ft. to the left of the crown measured horizontally. Find the maximum bending moment.*



Let R=radius of curvature—

$R^2=40^2+(R-20)^2$ $R=50$ ft.

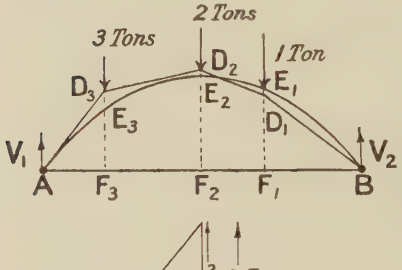
$y=20-\{50-50\cos(\sin^{-1}\frac{15}{50})\}=17{\cdot}75$ ft.

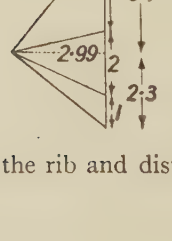
$V_1=10\times\frac{55}{80}=6{\cdot}88$ tons $V_2=10\times\frac{25}{80}=3{\cdot}12$ tons

$H\times 20=V_2\times 40$ $H=6{\cdot}24$ tons

Bend. mom. under load $=6{\cdot}88\times 25-6{\cdot}24\times 17{\cdot}75=$ **61 tons-ft.**

11. *Find the bending moments under the loads in the case of a parabolic arch hinged at both ends.*





Span, 50 *ft. Rise,* 15 *ft. Loads:* 1 *ton at* 15 *ft. from one end,* 2 *tons at* 25 *ft.,* 3 *tons at* 40 *ft. Horizontal thrust,* 2·99 *tons.* (Edinburgh University.)

$V_1=\frac{1\times 15+2\times 25+3\times 40}{50}$

$=3{\cdot}7$ tons

$V_2=6-3{\cdot}7=2{\cdot}3$ tons

The height y of any point below a hor. tangent to the rib and distant x from the middle is given by

$y=mx^2$, where m is a constant. At the end $y=15$ ft. and $x=25$ ft.

$$m=\frac{15}{25^2}=\frac{3}{125}$$

Under the 3-ton load $y=\frac{3}{125}\times 15^2=5{\cdot}4$ and $E_3F_3=15-5{\cdot}4=9{\cdot}6$ ft.

Under the 2-ton load $y=\frac{3}{125}\times 0=0$ and $E_2F_2=15$ ft.

Under the 1-ton load $y=\frac{3}{125}\times 10^2=2{\cdot}4'$ and $E_1F_1=12{\cdot}6$ ft.

$D_3F_3=\frac{3{\cdot}7\times 10}{2{\cdot}99}=12{\cdot}36'$
$D_3E_3=12{\cdot}36-9{\cdot}6=2{\cdot}76'$
and $M_3=2{\cdot}76\times 2{\cdot}99=$ **8·26 foot-tons.**

$D_2F_2=\frac{3{\cdot}7\times 25-3\times 15}{2{\cdot}99}=15{\cdot}88'$
$D_2E_2=15{\cdot}88-15=0{\cdot}88'$
$M_2=0{\cdot}88\times 2{\cdot}99=$ **2·63 ft.-tons.**

$D_1F_1=\frac{2{\cdot}3\times 15}{2{\cdot}99}=11{\cdot}54'$
$D_1E_1=11{\cdot}54-12{\cdot}6=-1{\cdot}06'$
$M_1=-1{\cdot}06\times 2{\cdot}99=$ **−3·17 ft.-tons.**

12. *The dead load on a bridge of* 80 *ft. span is* 1 *ton per ft. run and the live load* 2 *tons per ft. run due to a train which is not less than* 80 *ft. in length. Calculate the maximum shear which can occur at the ends and at* 35 *ft. from the abutments. What is the focal distance of the bridge?*

The distribution of the shear for this case is shown in Fig. 593, Vol. I.

Max. shear at ends $=\frac{1\times 80}{2}+\frac{2\times 80}{2}=$ **120 tons.**

Shear at 35 ft. from ends—

Due to dead load $=40-35\times 1=5$ tons

Due to live load when (80−35) feet are covered by train $=\frac{2\times 45^2}{2\times 80}=25{\cdot}3$ tons

Maximum shear $=$ **30·3 tons.**

Shear due to live load when 35 ft. are covered by train $=\frac{2\times 35^2}{2\times 80}=15{\cdot}3$ tons

Shear due to dead and live loads $=15{\cdot}3-5=10{\cdot}3$ tons.

Focal distance—

$x=80(1-2\sqrt{0{\cdot}5+0{\cdot}25}+2\times 0{\cdot}5)$ $\quad x=$ **21·4 ft.**

13. *Calculate the maximum forces which occur in the flanges of a girder bridge with parallel booms at sections 20 ft. and 30 ft. from the ends. Span, 60 ft.; depth of girders, 5 ft.; dead load, 0·6 ton per ft. run; live load 3 tons per ft. run.*

Total load per ft. run, 3·6 tons.

Total load on each girder, 1·8 tons.

Max. ben. mom. at any section distant a and b from the ends $= \frac{1}{2} \times 1{\cdot}8 \times a \times b$

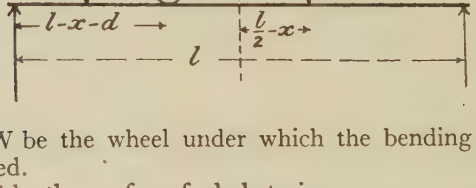
At 20 ft. from end $= 0{\cdot}9 \times 20 \times 40 = 720$ tons-ft.

Force in flange $= \frac{720}{5} =$ **144 tons.**

Max. ben. mom. at 30 ft. from end $= 0{\cdot}9 \times 30 \times 30 = 810$ tons-ft.

Force in flange $= \frac{810}{5} =$ **162 tons.**

14. *Show that when a train passes over a freely supported bridge the maximum bending moment under any given wheel occurs when the wheel in question is at the same distance from the middle of the bridge as the centre of gravity of the whole train.*



Let W be the wheel under which the bending moment is required.

Let G be the c. of g. of whole train.

Let P be the c. of g. of the loads to the left of W.

Left-hand reaction due to train $= \frac{Gx}{l}$

Ben. mom. M under W $= \frac{Gx}{l}(l-x-d) - Pc$

$$M = \frac{G}{l}(lx - x^2 - dx) - Pc$$

$\frac{dM}{dx} = l - 2x - d = 0$ for max. value or $\frac{l}{2} - x = \frac{d}{2}$

i.e., the distance of G from mid span $=\frac{d}{2}$, or the middle of the span is midway between G and W.

15. *A traction engine passes over a bridge of* 50 *ft. span and* 6 *feet deep. The load on the leading wheels is* 4 *tons and on the driving wheels* 14 *tons, the wheel base is* 12 *feet. Find the maximum bending moment under the driving wheels and the required moment of inertia of the bridge section. Stress* 5 *tons per sq. inch.*

In Example 14 it is shown that the maximum bending moment occurs under the driving wheels when they and the c. of g. of the engine are equidistant from the middle of the bridge. Taking moments about the driving wheels we have $4\times12=18x$, $x=2\frac{2}{3}$ ft. where x is the distance of the c. of g. of the engine from a vertical through the wheels. $\therefore$ the max. bending moment occurs when the driving wheels are $1\frac{1}{3}$ ft. beyond the middle of the bridge. The leading wheels are then $12-1\frac{1}{3}=10\frac{2}{3}$ ft. from the middle. Taking moments about the abutment nearest the leading wheels we get the reaction R on the side of the driving wheels

$$R\times50=14\times26\tfrac{1}{3}+4\times14\tfrac{1}{3}=8{\cdot}5 \text{ tons}$$

Max. ben. mom. under driving wheel $=8{\cdot}5\times23\frac{2}{3}=$ **201 ft.-tons.**

Mom. of inertia in inch units $=\frac{201\times12\times36}{5}=$ **17,400.**

16. Deflection of Framed Structures

The deflection can be found either by a graphical construction or by the application of the principle of work, viz. that the external work done by the load on the structure is equal to the work done in straining the various members of the structure.

Let L = length of member in ins.

x = strain due to a force of F tons acting on the member in ins.

E = Young's Modulus of Elasticity in tons sq. in.

A = sectional area of bar in sq. ins.

Then $x=\frac{FL}{AE}$ (see Chap. X., Vol. I.)

The work done in straining a member $\Big\} = \frac{F_1x_1}{2}=\frac{F_1^2L_1}{2A_1E}$, $\frac{F_2^2L_2}{2A_2E}$, etc.

The work done by a single load W tons on the structure which produces a deflection Δ underneath the load is $\frac{W\Delta}{2}$, then

$$\frac{W\Delta}{2} = \frac{1}{2E}\left(\frac{F_1^2L_1}{A_1} + \frac{F_2^2L_2}{A_2} + \text{etc.}\right)$$

$$\Delta = \frac{1}{WE}\left(\frac{F_1^2L_1}{A_1} + \frac{F_2^2L_2}{A_2} + \text{etc.}\right)$$

The forces acting on each member can be obtained by the methods given in Chap. IV. or XVII., Vol. I.

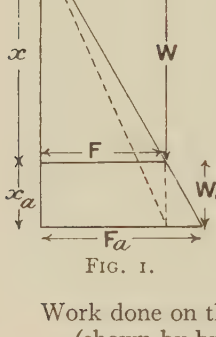


FIG. 1.

When a second load W_a is placed on the structure the force F is increased to F_a and the strain is increased by an amount x_a and the total work done in straining the bar is

$$\frac{F_a}{2}(x+x_a)$$

The load W sinks an additional amount x_a. Then

$$F_a = F\left(\frac{x+x_a}{x}\right) \quad \text{and} \quad F = \frac{F_a x}{x+x_a}$$

Work done on the bar by W (shown by broken line) $= \frac{F(x+x_a)}{2} = \frac{F_a x(x+x_a)}{2(x+x_a)}$

$$= \frac{F_a x}{2} = \frac{F_a FL}{2AE}$$

The total internal work done by the load W at the point where the deflection is to be measured

$$= \frac{W\Delta}{2} = \frac{1}{2E}\left(\frac{F_{a1}F_1L_1}{A_1} + \frac{F_{a2}F_2L_2}{A_2} + \text{etc.}\right)$$

$$\Delta = \frac{1}{WE}\left(\frac{F_{a1}F_1L_1}{A_1} + \frac{F_{a2}F_2L_2}{A_2} + \text{etc.}\right)$$

The above reasoning holds for any position of W_a, hence it holds for any system of loading.

Since F_1 is proportional to W it will save arithmetic in tabulating the work if W be taken as 1 ton acting at the point where the deflection is to be measured. Then F_1, F_2, etc., are the forces acting on the members due to the single load W. All forces should be given their appropriate sign +

or — for tension and compression; with certain distribution of the load some members may be in compression when the structure is normally loaded and in tension under the single load W, hence the necessity for paying attention to signs.

After describing a graphic method due to Professor Kernot, of Melbourne University, some cases will be worked out by both methods. When the load W is applied to a simple cantilever the bar AB will extend an amount x (greatly exaggerated in the figure).

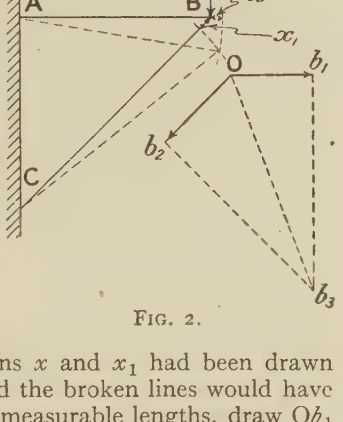


FIG. 2.

Let L be the original length of the bar. With A as centre describe an arc of radius $L+x$. The bar BC will shorten an amount x_1, with c as centre describe an arc of radius L_1-x_1, the loaded structure will then take up the position shown in broken lines. If the strains x and x_1 had been drawn to their true scales the full and the broken lines would have nearly coincided. To obtain measurable lengths, draw Ob_1 parallel to AB and representing x on an exaggerated scale, also draw Ob_2 parallel to BC and representing x_1 to the same scale and in the correct direction. From b_1 and b_2 draw lines at right angles to Ob_1 and Ob_2 respectively. This is where the approximation comes in. Since the movement of B is extremely small the arc and the tangent practically coincide. Join O to the point b_3 where the above-mentioned lines cut, then Ob_3 gives to an exaggerated scale the approximate path of the point B when the structure is loaded, and b_1b_3 is the vertical deflection (very nearly) to the same scale.

To compare the results obtained by the principle of work method and the graphical. Let W=5 tons, AB=10 ft., $BC=10\sqrt{2}$, $F_{ab}=5$ tons, $F_{bc}=5\sqrt{2}$, E=14,000 tons sq. in. Sectional area of bar AB. $A_{ab}=1$ sq. in. $A_{bc}=1{\cdot}76$ sq. in.

By the principle of work

$$\Delta=\frac{1}{5\times14000}\left(\frac{5^2\times120}{1}+\frac{5^2\times2\times120\sqrt{2}}{1{\cdot}76}\right)=0{\cdot}11''.$$

By the graphic method

$$\text{Extension of AB} = \frac{fL}{E} = \frac{5 \times 120}{14000} = 0{\cdot}043''$$

$$\text{Compression of BC} = \frac{5\sqrt{2} \times 120\sqrt{2}}{1{\cdot}76 \times 14000} = 0{\cdot}049''.$$

In the deflection diagram these are taken ten times full size, then by measurement $b_1b_3 = 1{\cdot}1''$ or the deflection $= 0{\cdot}11''$.

When finding the deflection of such a bridge as that shown in the figure it is convenient to take the loaded joint as stationary and regard the supports as moving upwards relatively to it.

Let W = 5 tons. Then by methods already given the forces in each member are readily obtained.

Bar.	Force in tons. F	Area sq. ins. A	Length, feet.	Change in length (ins.).
(Ties) AE & BD	$2{\cdot}5\sqrt{2}$	$\frac{\sqrt{2}}{2}$	$10\sqrt{2}$	0·060
(Struts) CD & BC	5	1·25	10	0·034
(Struts) BE & AB	2·5	0·625	10	0·034
(Tie) ED	2·5	0·5	10	0·043

In the diagram the change of length is exaggerated ten times. The full lines represent the movement at the joint, the suffix *d* indicates a diagonal, *h* a horizontal, and *v* a vertical shift.

Set off $CD_v = 0{\cdot}34''$ to represent the upward vertical movement of D due to the shortening of the bar. From D_v set off D_vB_d parallel to the bar BD and length $= 0{\cdot}6''$ to represent the lengthening of the diagonal bracing bar; from D_v set off $D_vB_h = 0{\cdot}34''$ to represent the shortening of the bar BC. From their extremities draw lines (broken) at right angles to them which meet in B, then BB_h represents the vertical movement of the joint B due to the strain of BC and BD. B has also moved vertically upwards an amount B_1B_h due to the shortening of DC, thus the total shift of B is B_1B. The point E moves vertically up $0{\cdot}34''$ relative to B due to the compression of BE represented by BE_v; it also moves horizontally $0{\cdot}43''$ relative to B due to the extension of ED, represented by D_vE_h. Thus E has moved to the left to E_h and vertically by an amount B_1E_v, from the extremities of these lines draw lines (broken) at right angles to them

meeting in E, the position of which relative to C represents

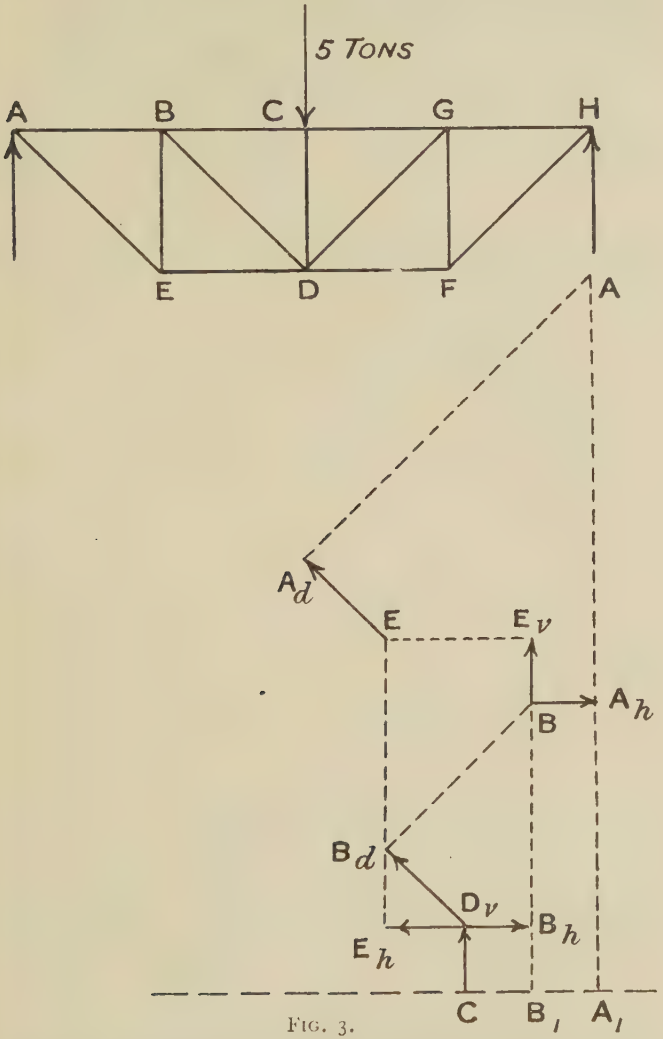


FIG. 3.

the shift of the joint E. From E set off the stretch of the diagonal AE, viz. 0·6″ represented by EA_d. From B set off

$BA_h = 0{\cdot}34''$ to represent the movement of A towards B. From A_d and A_h draw lines (broken) at right angles to them meeting in A. Then AA_h represents the vertical movement of A relative to B, but B has moved vertically upwards a distance BB_1, hence the total vertical movement of A is AA_1, which is the deflection of the girder or the amount C has sunk relatively to the supports. (Ten times full size owing to the scale of exaggeration.)

$$AA_1 = 3{\cdot}8'' \text{ or a deflection of } 0{\cdot}38''$$

In the structure the following bars are of the same length and subjected to the same stress, hence they are strained to the same amount.

4 ties, AE, BD, GD, HF.
3 struts, CD, BC, CG.
4 struts, BE, GF, AB, GH.
2 ties, ED, DF.

The deflection calculated by the principle of work is—

$$\Delta = \frac{1}{5 \times 14000}\left(4\frac{2{\cdot}5^2 \times 2 \times 120\sqrt{2} \times 2}{\sqrt{2}} + 3\frac{5^2 \times 120}{1{\cdot}25} + 4\frac{2{\cdot}5^2 \times 120}{0{\cdot}625} + 2\frac{2{\cdot}5^2 \times 120}{0{\cdot}5}\right)$$

$$\Delta = 0{\cdot}386''.$$

If the girder were loaded with 5 tons at B, C, G, we should get :

Bars.	F tons due to distributed load.	F_a tons due to 1 ton at C.	L feet.	A sq. ins.	$F_a F L / A$	Ditto times the number of similar bars.
AE & FH	$-7{\cdot}5\sqrt{2}$	$-0{\cdot}5\sqrt{2}$	$10\sqrt{2}$	2·26	46·8	93·6
BD & GD	$-2{\cdot}5\sqrt{2}$	$-0{\cdot}5\sqrt{2}$	$10\sqrt{2}$	0·85	41·5	83·0
CD	+5	+1	10	1·5	33·3	33·3
BC & CG	+10	+1	10	2·75	36·4	72·8
BE & GF	+7·5	+0·5	10	2·00	18·8	37·6
AB & GH	+7·5	+0·5	10	2·00	18·8	37·6
ED & DF	−7·5	−0·5	10	1·6	23·4	46·8
						404·7

+ denotes a strut.

$$\Delta = \frac{1}{1 \times 14000} \times 404{\cdot}7 \times 12 = 0{\cdot}35'' \text{ (nearly).}$$

The defl. is less than with a single load, but it must not be forgotten that the sections of the bars have been increased to correspond with the greater loads.

For more complicated cases readers should refer to books on structures. Fully worked out examples will be found in "Structural Engineering," by Husband and Harby; "Engineering Construction in Steel and Timber," by Warren.

17. *In a pin-jointed frame* ABCD *one of the diagonals was made* 0·12″ *short, but the rest of the frame was forced out of shape to enable the pin to be inserted. Find the stress in each of the members.*

Member.	*Length (unstrained).*	*Sectional area.*
AB *and* CD	14 *ft.*	30 *sq. ins.*
AD *and* BC	18 *ft.*	20 *sq. ins.*
AC	22·80 *ft.*	10 *sq. ins.*
DB	22·79 *ft.*	10 *sq. ins.*

E = 12,000 *tons sq. in.*

(See Example 16.)

$$W = \frac{1}{\Delta E}\left(\frac{2F_1^2L_1}{A_1} + \frac{2F_2^2L_2}{A_2} + \frac{2F_3^2L_3}{A_3}\right)$$

Taking the stress in the bar DB as 1 ton, the stress in

AB and DC is $\frac{14}{22{\cdot}80} = 0{\cdot}614$ ton

AD and BC is $\frac{18}{22{\cdot}80} = 0{\cdot}789$ ton

$$W = \frac{2}{0{\cdot}01 \times 12000}\left(\frac{0{\cdot}614^2 \times 14}{30} + \frac{0{\cdot}789^2 \times 18}{20} + \frac{22{\cdot}80}{10}\right) = \tfrac{1}{20} \text{ nrly.}$$

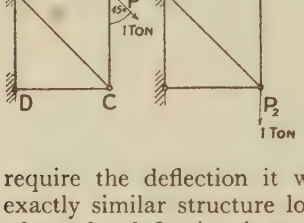
But this is for a stress of 1 ton in the bar DB, hence the total stress in the bar is **20 tons,** or 2 tons per sq. inch in DB and AC (tension).

Stress in AB and CD = 0·614 × 20 = **12·28 tons,** or 0·41 tons sq. inch (compression).

Stress in AD and BC = 0·789 × 20 = **15·78 tons,** or 0·79 ton sq. inch (compression).

These stresses are in addition to stresses due to external loading, attention must be paid to signs.

18. *The hinged structure* ABCD *is loaded as shown. The length of the vertical and horizontal bars is* 5 *feet. Find the vertical deflection of the point* C *assuming that* A *and* D *are fixed. Sectional area of all members,* 0·5 *sq. inch. Young's Modulus,* 13,000 *tons sq. in.*



Since the force does not act in the direction in which we require the deflection it will be convenient to assume an exactly similar structure loaded with unit load at the point where the deflection is required, the sectional areas of the members being such that the stress in them is the same as in the actual structure, then the deflection of the imaginary structure will be the same as in the actual.

Member.	Force due to P_1.	Force due to P_2.	Length L (ins.).	P_1P_2L.
AB	$+\frac{1}{\sqrt{2}}$	0	60	0
BC	$-\frac{1}{\sqrt{2}}$	0	60	0
CD	$-\frac{1}{\sqrt{2}}$	−1	60	42·4
CA	+1	$+\sqrt{2}$	$60\sqrt{2}$	120
				162·4

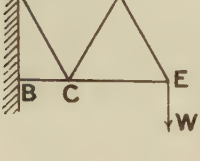
— denotes a strut.

$$\delta = \frac{1}{WE} \times \frac{162{\cdot}4}{A} = \frac{1}{1 \times 13000} \times \frac{162{\cdot}4}{0{\cdot}5} = \mathbf{0{\cdot}025\ in.}$$

19. *The members of the frame shown in the figure are all, with the exception of* BC, 10 *ft. in length, and they are all subjected to the same intensity of stress when loaded as shown.*

The deflection of E *for a load* W *is observed to be* $\frac{1}{3}$ *in. What is the stress in the members?*

Young's Modulus of Elasticity, 13,000 *tons per sq. in.*



(See Example 16.)

$$\tfrac{1}{2}(W\times\tfrac{1}{3}'')=\tfrac{1}{2}\sum\frac{(fA)fL}{E}=\tfrac{1}{2}\sum\frac{(KW)fL}{E}$$

where K is the total stress on the member for the load W. Let f be the intensity of stress. Then the total stress in each of the bars AD, DE, DC, CE, AC is $\frac{2}{\sqrt{3}}f$, and in the bar BC it is $\frac{3}{2\sqrt{3}}f$.

Then $\frac{1}{3}''=\sum\frac{KfL}{E}=\left(5\times\frac{2}{\sqrt{3}}+\frac{3}{2\sqrt{3}}\right)\frac{f\times 120}{E}$

$$\tfrac{1}{3}=\frac{6{\cdot}64\cdot f\times 120}{13{,}000}\qquad f=\mathbf{5{\cdot}44\ tons}\text{ per sq. in.}$$

20. *A rolled joist* $12''\times 6''\times\frac{7}{8}''$. *Span* 15 *ft. A weight of* 0·5 *ton is dropped on to the middle of the beam and produces an instantaneous stress of* 7 *tons sq. in. Find the height from which the weight is dropped.* E=12,000 *tons sq. in.*

Z approx. $=12\times 6\times\frac{7}{8}=63\qquad I=63\times 6=378$

$$M=7\times 63=441\text{ in.-tons}$$

$$\delta=\frac{Ml^2}{12EI}=\frac{441\times 180^2}{12\times 12000\times 378}=0{\cdot}26''$$

also $M=\frac{Wl}{4}$ where W is the equivalent dead weight

$$W=\frac{4M}{l}=\frac{4\times 441}{180}=9{\cdot}8\text{ tons.}$$

Energy of falling weight=Work done in deflecting beam

$$\tfrac{1}{2}(H+\delta)=\frac{0{\cdot}26\times 9{\cdot}8}{2}\qquad H+\delta=2{\cdot}55''$$

$$H=2{\cdot}55''-0{\cdot}26''=\mathbf{2{\cdot}29''}.$$

21. *A weight of* 1 *ton is dropped on to the middle of an elastic beam from a height of* 4 *ft. Calculate the momentary stress in the beam. Length of beam,* 20 *ft.; depth of beam,* 12 *ins.; moment of inertia of beam section,* 500 (*in. units*); E 12,000 *tons per sq. in.*

$W(h+\delta)=\frac{W_0\delta}{2}$ where W_0 is the equivalent momentary load.

$$\delta=\frac{W_0 l^3}{48EI}=KW_0 \qquad \text{where } K=\frac{l^3}{48EI}$$

$$Wh+WKW_0=\frac{KW_0^2}{2}$$

$$W_0^2-\frac{2WKW_0}{K}+W^2=\frac{2Wh}{K}+W^2$$

$$W_0=W\left(1+\sqrt{\frac{2h}{WK}+1}\right)=W\left(1+\sqrt{\frac{2h\times 48EI}{Wl^3}+1}\right)$$

$W_0=45{\cdot}7$ tons

Momentary stress $=\frac{M}{Z}=\frac{W_0 ly}{4I}=$ **33 tons** sq inch.

This is far beyond the elastic limit of the material, and since the above expressions only hold up to the elastic limit the result is incorrect.

If the weight were 0·1 ton the stress comes out to 14·2 tons per sq. inch, which is probably near the elastic limit of the material in the beam, hence this is about the biggest load which might safely be dropped on to the beam from the given height.

22. *A compression helical spring is placed in the middle of a freely supported beam and a load is dropped on to the spring from a height of* 3 *ins. above the top of the spring when unstrained. Find the load required to produce a stress of* 7 *tons per sq. in. in the beam. Span of beam* (L), 5′ ; *modulus of section* (Z), 14·5 *inch units ; compression of spring per ton dead load*, 2″ ; *deflection of beam per ton dead load*, 0·0086″ ; *modulus of elasticity* (E), 12,000 *tons sq. inch.*

$$I=\frac{WL^3}{48E\delta}=\frac{1\times 60^3}{48\times 12000\times 0{\cdot}0086}=43{\cdot}6$$

Let W_e=the equivalent dead load.

Maximum instantaneous bending moment

$M=\frac{W_e l}{4}$ $\qquad 7\times 14{\cdot}5=\frac{W_e\times 60}{4}$ $\qquad W_e=6{\cdot}77$ tons

Deflection of beam $6{\cdot}77\times 0{\cdot}0086=0{\cdot}06$″

Deflection of spring $=2\times 6{\cdot}77=13{\cdot}54$″

Energy stored in spring $= \frac{6{\cdot}77 \times 13{\cdot}54}{2} = 45{\cdot}8$ in.-tons

Energy stored in beam $= \frac{6{\cdot}77 \times 0{\cdot}058}{2} = 0{\cdot}2$ in.-ton nearly

Ext. work done by weight $W = W(3 + 13{\cdot}54 + 0{\cdot}06)$ in.-tons

Then $W \times 16{\cdot}6 = 45{\cdot}8 + 0{\cdot}2$ **W = 2·77 tons.**

If there had been no spring a very much smaller weight would have produced the same stress in the beam.

23. *A vertical tie bar with a collar on the lower end is provided with a sliding weight of* 3000 *lb. The weight is lifted* 3 *ins. and is then allowed to drop on to the collar. Find the momentary stress in the bar which remains elastic. Diameter of bar,* 1·8 *ins.; length of bar,* 12 *ft.;* E, 30,000,000 *lbs. sq. in.*

Let W_0 be the maximum momentary load on the bar.

$$W(h+x) = W_0 \frac{x}{2}, \quad x = \frac{W_0 l}{AE}, \quad \frac{W}{A} = f, \quad \frac{W_0}{A} = f_0, \quad f(h+x) = \frac{f_0^2 l}{2E}$$

$$f_0 = \sqrt{\frac{2Ef^2h}{fl} + f^2} + f = f\left(1 + \sqrt{\frac{2Eh}{fl} + 1}\right)$$

$$= \frac{3000}{2{\cdot}54}\left(1 + \sqrt{\frac{2 \times 30{,}000{,}000 \times 3 \times 2{\cdot}54}{3000 \times 144} + 1}\right) = \textbf{39,500 lb.} \text{ sq. in.}$$

24. *A sliding weight of* 2000 *lb. drops down a vertical rod which is suspended from the top and is fitted with a collar at the bottom. A helical buffer spring rests on this collar which compresses* 1 *inch for* 1000 *lb. Find the height, measured from the free position of the top of the spring, from which the weight must be dropped in order to produce a momentary stress in the bar of* 10,000 *lb. sq. in.* E = 30,000,000 *lb. sq. in. Also find the height for the case in which there is no spring. Length of rod,* 10 *ft. Diameter,* 2 *ins.*

Work done by falling weight $= 2000(h + \delta + x)$

Stretch of bar $x = \frac{fl}{E} = \frac{10000 \times 120}{30{,}000{,}000} = 0{\cdot}04''$

Work done in stretching bar (usually a negligible quantity)

$$= \frac{Afx}{2} = \frac{3{\cdot}14 \times 10000 \times 0{\cdot}04}{2} = 628 \text{ in.-lb.}$$

Compression of spring $= \delta = 31{\cdot}4''$

Work done on spring $\frac{W\delta}{2} = \frac{31400 \times 31{\cdot}4}{2} = 493{,}000$ in.-lb.

$2000(h + \delta + x) = 493{,}000 + 628 =$ say, 494,000 in.-lb.

$$h + \delta + x = 246{\cdot}8''$$

Height weight falls $= 246{\cdot}8 - 0{\cdot}04 - 31{\cdot}4 = 215{\cdot}4''$ say **18 ft.**

With no spring the height is **0·28 in.**, which shows clearly the beneficial effect of the spring.

25. *A vertical bar, 3 ins. diameter, is suspended from its top end and is fitted with a collar on its bottom end, a sliding weight of 1000 lb. is allowed to drop on to a helical spring which rests on the collar, the spring is of such a degree of stiffness that it compresses 1 in. for every 6,180 lb. The weight is lifted 3 ft. clear of the spring. Find the momentary stress in the bar.*

Total work done by falling weight $= 1000(36 + x)$ inch-lb., where x is the comp. of the spring in ins.

Work taken up by spring $= \frac{6180x^2}{2}$ inch-lb.

$$36000 + 1000x = 3090x^2 \qquad x = 3{\cdot}58 \text{ inches}$$

Momentary load on bar $= 3{\cdot}58 \times 6180 = 22{,}100$ lb.

Momentary stress in bar $= \frac{22100}{0{\cdot}785 \times 3^2} =$ **3,130 lb.** sq. in.

26. *A falling weight sliding in vertical guides strikes a helical spring on the top of a column which is prevented from buckling. Find the maximum momentary stress in the column. Length, 144"; diameter, 2·5"; sliding weight, 1500 lb.; height of fall before the weight comes into contact with the spring, 120"; compression of spring for 1000 lb. static load, 0·75"; E, 30,000,000 lb. sq. in.*

Sectional area of column A $= 4{\cdot}91$ sq. ins.

Load P required to compress spring 1 in. $= \frac{1000 \times 4}{3} = 1333$ lb.

$$\delta = \frac{4{\cdot}91f}{1333} = 0{\cdot}00368f$$

$$\frac{Af^2l}{2E} + \frac{P\delta^2}{2} = W\left(h + \delta + \frac{fl}{E}\right)$$ (See p. 631, Vol. I.)

$$\frac{4{\cdot}91f^2\times 144}{60{,}000{,}000}+\frac{1333(0{\cdot}00368f)^2}{2}$$

$$=1500\left(120+0{\cdot}00368f+\frac{144f}{30{,}000{,}000}\right)$$

$$0{\cdot}009f^2-5{\cdot}53f=180{,}000=\mathbf{4{,}780\ lb.}\ \text{sq. in.}$$

If a cast-iron column be struck in this way, but without a spring, it may be fractured by *tension* when the column rebounds, provided the stress f exceeds the tensile strength of the cast iron.

27. *A weight is attached to one end of a chain the other end of which is fixed. The weight is allowed to fall suddenly. Find the weight required to just break the chain. Length of chain,* 5′ ; *fall of weight,* 8′ ; *elastic limit of chain,* 9 *tons sq. in. ; breaking stress of chain,* 16 *tons sq. in. ; diameter of links,* $\frac{3}{4}$″. *Extension of chain before fracture,* 32 *per cent. The stress-strain diagram has the same form as that for a plain tension bar.*

Assuming that the work done by the falling weight = work done in fracturing the chain by gradual loading.

(See p. 382, Vol. I.)

$$W(96+60\times 0{\cdot}32)=\frac{60\times 32\times 0{\cdot}88}{300}(9+2\times 16)\quad \mathbf{W=1{\cdot}98\ tons.}$$

The safe load for a chain in tons is (number of $\frac{1}{8}$ths of an inch in the link)$^2\div 10$; in this case there are 6 $\frac{1}{8}$ths. Hence safe load is 3·6 tons. The breaking load will be about $4\times 3{\cdot}6$ = **14·4 tons** steadily applied.

28. *A winding rope of length* L *feet and sectional area* A *sq. ft., carrying a wt.* W *lb. at its bottom end, is descending vertically with uniform velocity* V *feet per sec. The top end of the rope is suddenly stopped. Show that the increase of stress due to the stoppage is*

$$f=\mathrm{V}\sqrt{\frac{\mathrm{WE}}{g\mathrm{AL}}}$$

where E *is Young's modulus of elasticity for the rope. The weight of the rope itself may be neglected, and it may be assumed to be truly elastic.*

$$\text{Work done on rope} = \frac{fAx}{2} = \frac{f^2AL}{2E}$$

$$\frac{WV^2}{2g} = \frac{f^2AL}{2E} \qquad f^2 = \frac{2EWV^2}{2gAL} \qquad f = V\sqrt{\frac{WE}{gAL}}.$$

29. *Bridge members are subjected to the following stresses. Find the equivalent static stresses on the assumptions made in Chap. XVII., Vol. I.*

Stress when the bridge is unloaded.	Stress when a train is standing on the bridge.
(1) 40 *tons tension*	70 *tons tension*
(2) 40 *tons compression*	70 *tons tension*
(3) 40 *tons tension*	70 *tons compression*
(4) *No stress in tension*	70 *tons tension*

(1) $40+2(70-40)$ = **100 tons tension.**
(2) $-40+2(70+40)$ = **180 tons tension.**
(3) $40-2(70+40)$ = **180 tons compression.**
(4) $0+2\times70$ = **140 tons tension.**

According to some authorities these values are too high, but there is a great difference of opinion upon this question.

30. *A bar supports a dead load of* 40 *tons in tension which at times is increased to* 70 *tons. Find the section required.* $f=7$ *tons per sq. inch.*

Maximum equivalent load $=40+2(70-40)=100$ tons

Section $\frac{100}{7}$ = **14·3 sq. ins.**

31. *A bar is subjected to a load of* 40 *tons in tension which may be supplemented by a live load of* 15 *tons. Find the section.* $f=7$ *tons sq. in.*

Max. equivalent load $=40+2\times15=70$ tons

Section **10 sq. ins.**

32. *A member of a structure is* 14 *ft. long with pin joints, and is subjected to a dead load of* 20 *tons in tension. Under certain conditions of loading it is under compression to the extent of* 5 *tons. Find the required section. Tensile stress not to exceed* 7 *tons sq. in. As a strut the factor of safety must be not less than* 4. *Mild steel, solid circular section.*

Max. equivalent load $=20-2(20+5)=-30$ tons (*i.e.* compression).

The strut must be designed for 120 tons.

Let S=67,000 lb. sq. in. $\alpha=\frac{1}{400}$

$$\frac{120\times 2240}{0{\cdot}785d^2}=\frac{67000}{1+\frac{1}{400}\left(\frac{l}{d}\right)^2}=\frac{67000d^2\times 400}{400d^2+168^2}$$

$$d^4-5{\cdot}12d^2=361 \qquad d=\mathbf{4{\cdot}66}''\ \text{(diam. of strut)}.$$

33. *A bar of circular cross-section* 16 *ft. long is subjected to* 25 *tons in tension, due to the dead weight of the structure. When a vehicle stands on the structure the bar is under compression to the extent of* 6 *tons. Find the diameter to fulfil the condition that the factor of safety as a strut shall be not less than* 4. S=67,000 *lb. sq. in.* $\alpha=\frac{1}{400}$. *Both ends pivoted.*

Max. equivalent load $=2(25+6)-25=37$ tons compn.

Buckling load $=4\times 37=148$ tons.

$$148\times 2240=\frac{67000\times 0{\cdot}785d^2}{1+\frac{1}{400}\left(\frac{192}{d}\right)^2}$$

From which we get

$$d^4-6{\cdot}3d^2=581 \qquad d=\mathbf{5{\cdot}25}''.$$

34. The question of the effect of repeated stresses on materials has been very ably dealt with by J. E. Gough in "The Fatigue of Metals," published by Scott Greenwood. Readers interested in the subject should consult this work.

Gough shows that the Dynamic Theory which we have adopted (but it is not original on our part) does not agree in many instances with the results of experiments on repeated stresses. In order to bring the two more into accord he proposes a "Modified Goodman" formula, which undoubtedly gives better results, but it is still far from perfect. Other expressions are also critically examined by him, but none are regarded as being satisfactory.

All agree that the Dynamic Theory is by far the simplest to use and remember. This, in our opinion, is a very great point in favour of its retention for designers, and, moreover, whatever error it may lead to, it nearly always errs on the

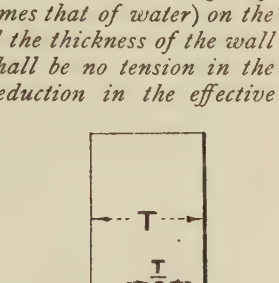
safe side. Hence, while freely acknowledging its shortcomings, we consider that on the whole it is well to retain it until a more satisfactory expression is found.

With some materials the Dynamic Theory agrees very closely with experimental results, but with others it is distinctly in error, hence if greater accuracy is desired a different expression must be used for each material. Possibly the Dynamic Theory with a coefficient of correction for each material or for certain physical properties of structural material may lead to a satisfactory expression, but until this or some other reasonably simple method is devised, we consider that the Dynamic Theory, with all its faults, may continue to be used with safety.

CHAPTER XVIII

HYDRAULICS

1. *A vertical wall* 30 *ft. in height, of uniform thickness, is subjected to water pressure on the one side to a height of* 20 *ft., and to a fluid* (*density* 2·1 *times that of water*) *on the other side to a height of* 6 *ft. Find the thickness of the wall to fulfil the condition that there shall be no tension in the masonry. You may neglect the reduction in the effective weight of the masonry due to its immersion. Weight of masonry,* 125 *lb. per cu. ft.*



Pressure on wall per ft. run due to water

$=62{\cdot}4\times10\times20=12{,}480$ lb.

Moment of water pressure about base

$$=\frac{12480\times20}{3}=83{,}200 \text{ ft.-lb.}$$

Moment of dense fluid pressure about base

$$=62{\cdot}4\times2{\cdot}1\times3\times6\times2=4{,}700 \text{ ft.-lb.}$$

Resultant moment, $83{,}200-4{,}700=78{,}500$ ft.-lb.

Weight of masonry per ft. run $=125\times30\text{T}=3{,}750\text{T}$ lb. where T is the thickness of wall in ft.

Direct comp. stress due to masonry $f=3{,}750$ lb. sq. ft. Since the tensile stress is to be zero, the ben. stress due to the overturning moments must be equal to the direct stress :

$$78{,}500=fz=\frac{f\text{BT}^2}{6} \qquad \text{B}=1 \qquad \text{T}=\mathbf{11{\cdot}2 \text{ ft.}}$$

This result could also have been arrived at thus : The resultant of the hor. and vert. pressures must cut the wall section at a point distant $\frac{\text{T}}{6}$ from the centre line of the section.

Let the height of the resultant hor. pressure be H feet above the base.

$$ab=\frac{78500}{H} \qquad ac=3{,}750T$$

$$\frac{ab}{ac}=\frac{T}{6H}=\frac{78500}{H\times 3750T}=\frac{T}{6H} \qquad T=\mathbf{11{\cdot}2\ ft.}$$

2. *An open channel of triangular cross-section is closed by a vertical sluice gate which is supported at the three corners of the triangle* ABC. *Find the fraction of the thrust taken by each support. If the triangle is equilateral of* 15 *ft. sides, what is the load at each corner?*

(See p. 641, Vol. I.)

$$\text{Depth of centre of pressure}=\frac{K^2\sin^2\theta+H_0^2}{H_0}=\frac{\frac{H^2}{18}+\frac{H^2}{9}}{\frac{H}{3}}=\frac{H}{2}$$

Taking moments about the top edge AB :

$$T\times\frac{H}{2}=T_c\times H \qquad T_c=\frac{T}{2} \qquad T_A=T_B=\frac{T}{4}$$

Height of triangle $=\sqrt{15^2-7{\cdot}5^2}=12{\cdot}95$ feet

Depth of c. of g. of triangle $=\frac{12{\cdot}95}{3}=4{\cdot}32'$

Total thrust $T=4{\cdot}32\times 62{\cdot}4\times 15\times 6{\cdot}47=26{,}200$ lb.

Thrust at apex $T_c=\mathbf{13{,}100\ lb.}$ $T_A=T_B=\mathbf{6{,}550\ lb.}$

3. *A flat triangular plate is immersed in water in a vertical position. The base is horizontal and is* 6 *ft. below the surface, the vertex is* 2 *ft. below the surface. Find the depth of immersion of the centre of pressure.*

H = height of plate H_0 = head above c. of g.

$$K^2=\frac{H^2}{18}=\tfrac{16}{18} \qquad H_0=2+\tfrac{2}{3}\times 4=\tfrac{14}{3}=4{\cdot}67'$$

$$H_c=\frac{K^2}{H_0}+H_0=\frac{16\times 3}{18\times 14}+\frac{14}{3}=\mathbf{4{\cdot}86\ ft.}$$

4. *The submerged upper end of a vertical pipe is fitted with a hinged door. The slope of the door when shut is* 30°

to the surface of the water. Find the pull on a vertical chain attached to the centre of the door required to open it when the head of water above the point of attachment is 4 ft., and the hinge is at the highest point. Neglect the weight of the door. Diameter of pipe, 4 ft.

The door is elliptical: Major axis $= \frac{4}{\cos 30^\circ} = 4{\cdot}62'$

Area of door $= 0{\cdot}785 \times 4{\cdot}62 \times 4 = 14{\cdot}4$ sq. ft.
Total press. on door $= 62{\cdot}4 \times 4 \times 14{\cdot}4 = 3{,}600$ lb.
(See p. 641, Vol. I.)
Depth of centre of press.

$$= H_c = \frac{K^2 \sin^2\theta + H_0^2}{H_0} = \frac{0{\cdot}25\frac{4{\cdot}62^2}{16} + 4^2}{4} = 4{\cdot}08'$$

Or the c. of p. is 0·08′ vert. below the centre of the door, or 0·16′ below when measured on the surface of the door; thus the centre of press. is $2{\cdot}31 + 0{\cdot}16 = 2{\cdot}47'$ from the hinge measured on the door. Taking moments about the hinge:

$3{,}600 \times 2{\cdot}47 = P \times 2$ $\quad P = \mathbf{4{,}450\ lb.}$ pull on a vertical chain.

5. *A flat equilateral triangular door hinged at the base covers an opening in a surface sloping at 45°. The hinge is submerged 12 ft. The apex is on the surface of the water. Find the pull required to open the door by means of a vertical chain attached to the apex.*

Depth of immersion of c. of g. $= 8$ ft.
Height of triangle $H = 0{\cdot}866B = 17$ ft.
Base of triangle $B = 19{\cdot}6$ ft. $\qquad K^2 = \frac{H^2}{18} = \frac{289}{18}$
Area of triangle $= 167$ sq. ft.

$$H_c = \frac{\sin^2\theta K^2 + H_0^2}{H_0} = \frac{0{\cdot}707^2 \times \frac{289}{18} + 8^2}{8} = 9 \text{ feet.}$$

Distance from hinge 4·25 feet.
Total press. on door $P = 8 \times 62{\cdot}4 \times 167 = 83{,}000$ lb.
Taking moments about hinge:

$$83{,}000 \times 4{\cdot}25 = P_n \times 17 \quad P_n = 20{,}750 \text{ lb.}$$

where P_n is the pull normal to the door.

Vert. pull on chain $\quad P_v = \frac{P_n}{0{\cdot}707} = \mathbf{29{,}400\ lb.}$

6. *Find the total pressure and the depth of immersion of the centre of pressure of a vertical trapezium-shaped door. The top and bottom edges are horizontal. Width at top,* 3′ ; *width at bottom,* 6′ ; *height,* 4′ ; *bottom edge,* 8′ *below surface.*

C. of g. of trap. below top edge $=\frac{4}{3}\left(\frac{2\times6+3}{6+3}\right)=2\cdot22$ ft.

(See p. 60, Vol. I.)

Depth of c. of g. below surface $=6\cdot22$ ft. $=H_0$

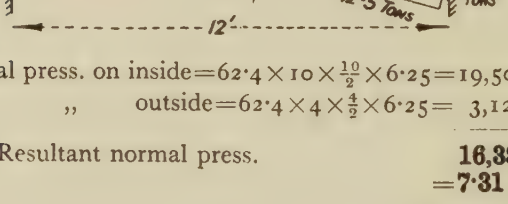
Total pressure on surface $=4\times4\cdot5\times6\cdot22\times62\cdot5=7{,}000$ lb.

$$K^2=\frac{2(B_1{}^2+4BB_1+B^2)H^2}{36(B_1+B)^2}=\frac{2(9+72+36)16}{36\times81}=1\cdot28$$

(See p. 86, Vol. I.)

Depth of c. of p. below surface $=\frac{K^2}{H_0}+H_0=6\cdot42$ ft.

7. *A canal lock is closed by two flat gates which are hinged vertically at the lock walls. Find* (i) *the resultant normal fluid pressure on each gate when closed ;* (ii) *the force acting along the plane of the gate ;* (iii) *the outward overturning force the gates exert on the dock walls. Width of lock,* 12′ ; *length of each gate,* 6·25′ ; *depth of water on outside,* 4′ ; *depth of water on inside,* 10′.



Total press. on inside $=62\cdot4\times10\times\frac{10}{2}\times6\cdot25=19{,}500$ lb.

„ „ outside $=62\cdot4\times4\times\frac{4}{2}\times6\cdot25=3{,}120$ lb.

(i) Resultant normal press. **16,380 lb. = 7·31 tons**

Taking moments about apex $7\cdot31\times\frac{6\cdot25}{2}=P\times1\cdot75$

(iii) Outward press. $P=$ **13·0 tons**

(ii) Force acting along plane of gates :

$T^2=13^2-3\cdot66^2$ $T=$ **12·5 tons.**

8. *A horizontal pipe 6 ft. diam. is closed by a door hinged about a horizontal diameter. The pipe is just full of water. Find the force* P_1 *acting 1·2 ft. above the hinge required to keep the door in equilibrium.*

Depth of centre of press. $H_c = \frac{\sin^2\theta K^2}{H_c} + H_c$

$H_c = \frac{2D^2}{16 \times D} + \frac{D}{2} = \frac{5}{8}D = 3{\cdot}75$ ft. Area $= 28{\cdot}3$ sq. ft.

Press. $P = 62{\cdot}4 \times 3 \times 28{\cdot}3 = 5{,}300$ lb. $= 2{\cdot}37$ tons

$P_1 \times 1{\cdot}2 = 2{\cdot}37(3{\cdot}75 - 3)$ $P_1 = \mathbf{1{\cdot}48\ tons.}$

9. *An open semicircular channel is closed by a vertical gate. Find the depth of immersion of the centre of pressure when the channel is full of water, also find the total pressure on the gate. Diameter of channel, 8 ft.*

Position of c. of g. (see p. 66, Vol. I.) :

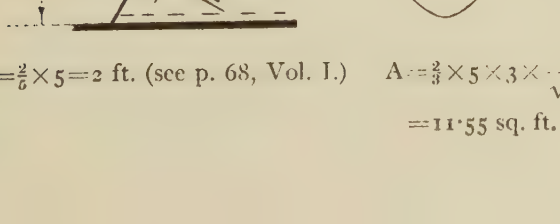
$$H = \frac{4R}{3\pi} = \frac{16}{3 \times 3{\cdot}14} = 1{\cdot}7 \text{ ft.}$$

Press. $P = \frac{0{\cdot}785 \times 64}{2} \times 62{\cdot}4 \times 1{\cdot}7 = 2{,}670$ lb.

$I = \frac{\pi D^4}{128} - \frac{\pi D^2}{8} \times 1{\cdot}7^2 = 28$ about a hor. axis thro. c. of g.

$$H_c = \frac{I + AH^2}{AH} = \frac{28 + 25{\cdot}12 \times 1{\cdot}7^2}{25{\cdot}12 \times 1{\cdot}7} = \mathbf{2{\cdot}35\ ft.}$$

10. *Find the pressure* P_1 *required to balance the water pressure on the door shown in the figure.*



$h_o = \frac{2}{6} \times 5 = 2$ ft. (see p. 68, Vol. I.) $A = \frac{2}{3} \times 5 \times 3 \times \frac{2}{\sqrt{3}}$

$= 11{\cdot}55$ sq. ft.

$K_o^2 = \frac{8}{35} \times 5{\cdot}77^2 = 7{\cdot}6$ about *oo* measured on slope (see p. 94, Vol. I.) $\sin^2\theta I_o = A h_c h_p$

$$h_c = \frac{K_o^2 \sin^2\theta}{h_c} = \frac{0{\cdot}866^2 \times 7{\cdot}6}{2} = 2{\cdot}85 \text{ ft. measured vertically}$$

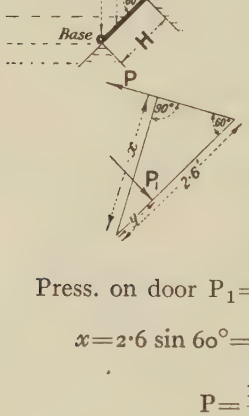
$$y = \frac{2{\cdot}85}{0{\cdot}866} = 3{\cdot}3 \text{ ft.} \qquad x = 4 \sin 50^\circ = 4 \times 0{\cdot}766 = 3{\cdot}06$$

$$P = 11{\cdot}55 \times 62{\cdot}4 \times 2 = 1{,}440 \text{ lb.}$$

$$P_1 \times 3{\cdot}06 = 1{,}440 \times 3{\cdot}3 \qquad P_1 = 1{,}550 \text{ lb.}$$

In the above case, where the hinge is on the surface of the water, it is more convenient to take the radius of gyration about the hinge rather than the radius about a line passing through the c. of g.

11. *The submerged door shown in the sketch is an equilateral triangle with the base hinged and level. Find the pull* P *at the apex required to open the door. Sides of door, 3 ft.; depth of immersion of hinge, 5 ft.*



Height of triangle H.

$$H = \sqrt{3^2 - 1{\cdot}5^2} = 2{\cdot}6'$$

Area of door $= 3{\cdot}9$ sq. ft.

Depth of immersion of c. of g.

$$H_0 = 4{\cdot}39'$$

Depth of immersion of c. of p.

$$H_c = \frac{K^2 \sin^2\theta}{H_0} + H_0$$

$$= \frac{0{\cdot}5 \times 0{\cdot}375}{4{\cdot}39} + 4{\cdot}39 = 4{\cdot}43'$$

Press. on door $P_1 = 3{\cdot}9 \times 62{\cdot}4 \times 4{\cdot}39 = 1{,}070$ lb.

$$x = 2{\cdot}6 \sin 60^\circ = 2{\cdot}25' \qquad y = \frac{5 - 4{\cdot}43}{\sin 45^\circ} = 0{\cdot}81$$

$$P = \frac{1070 \times 0{\cdot}81}{2{\cdot}25} = \mathbf{385 \text{ lb.}}$$

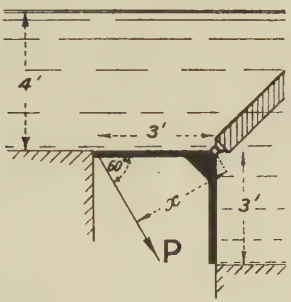
12. *Find the force* P *required to open the door shown in the figure. The length of the door in the direction of the hinge is 7 ft.*

Centre of pressure on the vertical portion.

$$H_c = \frac{K^2}{H_0} + H_0 = \frac{H^2}{12H_0} + H_0 = \frac{3^2}{12 \times 5{\cdot}5} + 5{\cdot}5 = 5{\cdot}64'$$

Press. on hor. portion $= 62{\cdot}4 \times 4 \times 3 \times 7 = 5{,}240$ lb

„ vertical „ $= 62{\cdot}4 \times 5{\cdot}5 \times 3 \times 7 = 7{,}200$ lb.



Taking moments about the hinge :

$$P \times 2{\cdot}6 + 5{,}240 \times 1{\cdot}5 = 7{,}200 \times 1{\cdot}64$$

$$P = \frac{11820 - 7860}{2{\cdot}6} = \mathbf{1{,}520\ lb.}$$

13. *A bridge consists of a series of masonry arches of* 30 *ft. span and* 10 *ft. rise, supported on piers* 8 *ft. thick running the whole width of the bridge, which is* 30 *ft. If the springings are* 10 *ft. above normal water-level, find the upward thrust on each pier due to a flood when the water rises* 3 *ft. above the crowns of the arches, but without flooding the bridge.*

Volume of water displaced during flood

$$= 30\left(8 \times 23 + 30 \times 13 - \frac{\pi}{2} \times 15 \times 10\right) \text{ cu. ft.}$$

Upward thrust due to this $= \frac{62{\cdot}4}{2240} \times 30 \times 338 = \mathbf{282\ tons.}$

14. *A vertical pipe with the upper end blanked is lowered into water, find the height* (y) *the water rises in the pipe when the lower end is immersed to a depth of* 40 *ft. Length of pipe* (V) 12 *ft. Atmospheric pressure* (h) 34 *ft. of water. Isothermal compression of air*

Since the process is isothermal we have :

Original press. × Original vol. = Final press. × Final vol.

$$h V = h_1 V_1$$

$$34 \times 12 = (34 + 40 - y)(12 - y)$$

$$408 = 408 + 480 - 86y + y^2 \qquad y = \mathbf{6\ ft.}$$

15. *Calculate the quantity of water, in excess of that required to fill the pipe at atmospheric pressure, which is contained in a hydraulic main when under a pressure* (p) *of* 2,000 *lb. per sq. in. above the atmosphere. Diam. of main* (d), 3″ ; *thickness* (T), $\frac{7}{8}$″ ; *length of main* (L), 3,000′ ; *coefficient of bulk elasticity for water* (K), 300,000 *lb. sq. in. ; modulus of elasticity of pipe* (E), 21,000,000 *lb. sq. in.*

(See p. 700, Vol. I.)

$$\text{Extra vol.} = \frac{pL\pi d^2}{4}\left(\frac{1}{K} + \frac{d}{TE}\right)$$

$$= \frac{2000 \times 3000 \times 3{\cdot}14 \times 3^2}{4 \times 144}\left(\frac{1}{300000} + \frac{3 \times 8}{7 \times 21{,}000{,}000}\right)$$

$$= \mathbf{1{\cdot}02\ cu.\ ft.}$$

16. *In a closed hydraulic system of transmitting power, the pump cylinder is the same diam. as the motor cylinder, but owing to the compression of the water and stretch of the mains the stroke of the motor plunger is less than that of the pump plunger. Find the ratio of the length of stroke of the motor plunger to that of the pump plunger. Diameter of cylinders,* 10″ ; *stroke of pump,* 3′ ; *diameter of mains* (d), 3″ ; *thickness of mains* (t), $\frac{7}{8}$″ ; *length of mains* (L), 2,000′ ; *excess of water pressure during working stroke over return stroke* (f), 1,000 *lb. sq. in. ; modulus of elasticity of pipes* (E), 20,000,000 *lb. sq. in.* K = 300,000 *lb. sq. in.*

$$\text{Shortening of water column in pipe} = fL\left(\frac{1}{K} + \frac{d}{tE}\right)$$

$$= 1000 \times 2000 \times 12\left(\frac{1}{300000} + \frac{3 \times 8}{7 \times 20{,}000{,}000}\right) = 84{\cdot}3''$$

$$\text{Corresponding traverse of piston} = 84{\cdot}3 \times \tfrac{9}{100} = 7{\cdot}58''$$

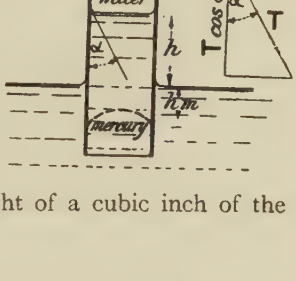
$$\text{Comp. of water-length in cylinder} = \frac{1000 \times 36}{300000} = 0{\cdot}12''$$

$$\text{Total shortening of stroke} = 7{\cdot}58 + 0{\cdot}12 = 7{\cdot}70''$$

$$\text{Ratio } \frac{\text{Motor stroke}}{\text{Pump stroke}} = \frac{36 - 7{\cdot}7}{36} = \mathbf{0{\cdot}79.}$$

17. Capillarity

All liquids behave as though their surfaces were covered with an elastic skin under tension, this tension being known as the "surface tension" of the liquid. It varies with the liquid and its temperature, also on the nature of the surface it is in contact with. For methods of measuring the surface tension, books on Physics should be consulted. From an engineering standpoint, readers cannot do better than refer to Gibson's "Hydraulics and its Applications."



One instance in which it concerns the engineer is the level at which the liquid stands in a vertical glass tube when its lower end is immersed in the liquid. Let T be the surface tension per inch; w be the weight of a cubic inch of the liquid.

All dimensions in inches.

$$\pi d \mathrm{T} \cos \alpha = \frac{w \pi d^2 h}{4} \qquad h = \frac{4 \mathrm{T} \cos \alpha}{wd}$$

T = 0·000462 lb. per inch for water in a clean glass tube
$\alpha = 25° 32'$ for clean glass at 68° F.

For water $$h = \frac{4 \times 0{\cdot}000462 \times 0{\cdot}9023 \times 12}{0{\cdot}434 d} = \frac{0{\cdot}0462}{d}$$

For mercury T = 0·003082 $\alpha = 128° 52'$

$$h_m = \frac{(4 \times 0{\cdot}003082 \times 12) - 0{\cdot}6276}{0{\cdot}434 \times 13{\cdot}6 d} = -\frac{0{\cdot}0157}{d}$$

d (inches) . .	0·01	0·10	0·15	0·20	0·25	0·5	1·0
h (water) . .	46·2	0·46	0·31	0·23	0·19	0·09	0·05
h_m (mercury).	−15·7	−0·16	−0·11	−0·08	−0·06	−0·03	−0·016

The − sign indicates that the level of the liquid in the tube is below the surrounding liquid.

In the same way the pressure inside a jet of water discharging from an orifice or notch is greater than the atmospheric pressure by an amount corresponding to

$$2\mathrm{T} = dp$$

The effective head is therefore reduced by :

$$h = \frac{2T}{wd} = \frac{0{\cdot}025}{d} \text{ inches for water.}$$

For example, the pressure inside a jet 0·25 inch diameter is $\frac{1}{10}''$ greater than outside.

18. Coefficient of Discharge

It has been known for a long time that the coefficient of discharge for an orifice, notch, or Venturi meter is by no means constant. In the case of a sharp-edged orifice differences amounting to 10 per cent. may occur due to a change in head. Such variations have been largely responsible for a general distrust of diaphragms and orifices as an accurate form of flow meter for liquids and gases. On the other hand, results obtained from an orifice under constant conditions are so consistent that these variations may be regarded as definitely systematic and not "accidental." The general nature of the variations is discussed in Vol. I., p. 646.

The law of variation of the coefficient of discharge has been the subject of a recent investigation by H. W. Swift, which shows that so far as the engineer is concerned the factors which determine these variations are : capillarity and viscosity of the fluid, head, shape, and size of the orifice.

Capillarity.—In submerged discharge or in the case of large orifices or notches under ordinary heads the effect of capillarity is negligible ; but when the head over a notch is less than about $1\frac{1}{2}$ inches, and in the case of small orifices, the effect is not unimportant.

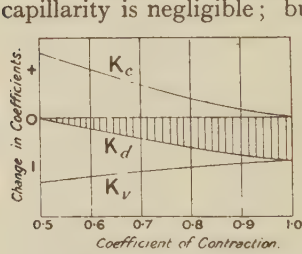


Fig. 1.—Effect of the capillarity of water on the coefficients for jets having the coefficients of contraction as indicated.

Owing to capillarity the pressure inside the jet is greater than that on the outside by an amount which can be calculated, see 17 above. The effect of this is to reduce the effective head and therefore the coefficient of velocity K_v, but at the same time it increases the section of the jet; the one effect thus partially counteracts the other. With a Borda mouthpiece, for which the coefficient of contraction K_c is 0·5, it can be shown that the one effect exactly neutralises

the other, so that the discharge is independent of capillarity. With a perfectly rounded orifice in which $K_c = 1$ the section of the jet cannot increase, hence the effect of capillarity is to reduce K_v to its full extent. With orifices having coefficients between these extreme values the value of K_d diminishes as the coefficient of contraction increases in the manner indicated in Fig. 1.

In all cases the effect depends simply on the liquid and size of orifice, and can therefore be eliminated for different rates of discharge by a simple correction of the head :

$$h = H - h_s$$

where h_s is calculable, and for cold water flowing through a sharp orifice d ins. in diameter has the value $h_s = \frac{0{\cdot}008}{d}$ inches.

Viscosity.—Viscosity depends for its effect on the value of a product $\frac{\nu}{d\sqrt{H}}$ which is analogous to the Reynolds number in pipe flow. Changes in head or diameter of orifice are therefore included in the treatment of changes in viscosity.

Its primary effect is to increase the resistance to flow and reduce the value of K_v according to a fairly simple law. But as in the case of surface tension this change reacts on the area of the jet, which will always tend to increase with increasing viscosity towards the limit of unity. Hence as the effect of viscosity becomes greater, K_v decreases and K_c increases, and the resultant effect on K_d depends upon which of these predominates. For a Borda mouthpiece it is easy to show that K_c must increase at a rate more than twice that at which K_v falls off. With a perfectly rounded orifice $K_d = K_v$ and the coefficients of velocity and discharge fall away together. For intermediate forms the general nature of the change is as shown in Fig. 2. If K_c is less than a certain value, represented by the point P, K_d will increase with increasing viscosity ; if greater K_d will fall off. An orifice designed to operate at or near the point P gives a coefficient of discharge which is almost independent·

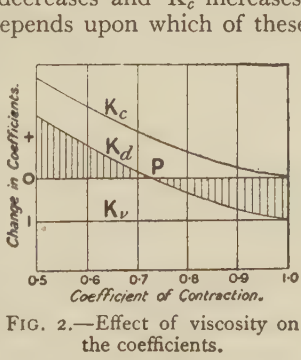


FIG. 2.—Effect of viscosity on the coefficients.

of either viscosity or head over the whole of the useful hydraulic range.

With a sharp-edged orifice under hydraulic conditions the value of K_c is between 0·6 and 0·7, and the coefficient of discharge always increases as viscosity increases or head falls off. Moreover, the effect is under these conditions comparatively small, and it can be shown that the coefficient bears a linear relation to the product $\frac{\nu}{d\sqrt{H}}=\eta$:—

$$K_d=K_0+\alpha\eta$$

where K_0 is the theoretical coefficient for a " perfect " fluid. The value of K_0 depends on the shape and exact sharpness of the orifice, and on the conditions of approach, and can only be found by calibration of any individual orifice plate.

Normally the engineer is concerned mainly with changes in head, and for these changes under hydraulic conditions the coefficient follows the law :

$$K_d=K_0+\frac{c}{\sqrt{h}} \text{ where } c \text{ is a constant.}$$

It has been found that apart from irregularities in the flow a correction for the effects of surface tension and viscosity enables a very close degree of accuracy to be attained.

Firstly, the *head* is corrected, if necessary, for surface tension : $h=H-h_s$

Next the *quantity*, as deduced from the corrected head, is in its turn corrected for the effects of viscosity :

$$Q=\left(K_0+\frac{c}{\sqrt{h}}\right)A\sqrt{2gh}=q_0+K_0A\sqrt{2gh}.$$

The values of q_0 and K_0 in this expression can be found by calibration at two different rates of flow, under similar conditions to those which will obtain in service, and so long as the orifice does not become worn or fouled the law of calibration will hold well within ½ per cent.

With very viscous liquids the value of K_c approaches unity and K_d therefore falls away with K_v towards zero. With crude petroleum values of K_c up to 1·0 have been obtained from a sharp orifice, and simultaneous values of K_d as low as 0·20.

In the case of the notch a similar linear relation holds between K and the viscosity, and the discharge increases

with an increase in viscosity to a certain point (about 0·66) beyond which it falls towards zero. Calibration at two rates of discharge will again enable results to be obtained as close as $\frac{1}{2}$ per cent. Surface tension becomes important at heads below about $1\frac{1}{2}$ ins., but owing to uncertainties in reading the head, and the consequent large effect on the calculated discharge, the notch is not of much practical value at such low heads.

For further information, see *Phil. Mag.*, Oct. 1926, p. 852.

19. *The mixture in a* 50 *H.P. oil engine consists of* 10 *parts by weight of air to one part of oil. The fuel consumption is* 0·5 *lb. per b.h.p. hour. Specify the diameter of a sharp-edged diaphragm to measure the air supply in such a way as to give a reading of about* 8 *ins. water gauge at full load.*

A cubic foot of air weighs 0·08 lb.

Oil consumed = 25 lb. per hr.
Air required = 250 lb. per hr. = 0·87 cusecs

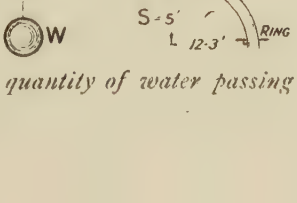
$$8 \text{ ins. water gauge} = \frac{8}{12} \times \frac{62{\cdot}4}{0{\cdot}08} = 520 \text{ feet-air}$$

Hence diameter of orifice is given by :

$$0{\cdot}87 = 0{\cdot}62 \times \frac{\pi}{4} d^2 \sqrt{64{\cdot}4 \times 520}$$

$$\therefore\ d^2 = \frac{0{\cdot}87 \times 4 \times 144}{0{\cdot}62 \times \pi \sqrt{64{\cdot}4 \times 520}} \text{ sq. ins.} \qquad d = \mathbf{1{\cdot}19 \text{ ins.}}$$

20. *A tank is freely balanced about a horizontal axis on knife edges; when the tank sides are vertical the stream of water from the orifice passes through a ring situated as shown in the figure. Find the weight* W *required to keep the tank sides in a vertical position. Also find the coefficients of velocity, contraction, and discharge. Diameter of orifice,* 2"; *quantity of water passing per minute,* 1,100 *lb.*



$$s=\tfrac{1}{2}gt^2=16\cdot1t^2 \qquad t=\sqrt{\frac{5}{16\cdot1}}=0\cdot558 \text{ sec.}$$

$$V=\frac{12\cdot3}{0\cdot558}=22\cdot0 \text{ feet per sec.}$$

$$K_v=\frac{V}{\sqrt{2gH}}=\frac{22\cdot0}{\sqrt{64\cdot4\times8}}=\mathbf{0\cdot969}$$

$$Q=\frac{3\cdot14}{144}\times22\cdot7\times K_d=\frac{1100}{60\times6\cdot24}$$

$$K_c=\frac{K_d}{K_v}=\frac{0\cdot59}{0\cdot969}=\mathbf{0\cdot609} \qquad K_d=\mathbf{0\cdot59}$$

$$P=\frac{W_w V}{g}=\frac{1100\times22}{32\cdot2\times60}=12\cdot5 \text{ lb.}$$

$$W=\frac{12\cdot5\times7\cdot5}{6}=\mathbf{15\cdot7 \text{ lb.}}$$

21. *In the orifice, Fig.* 628, *Vol. I., find the ratio of* A_b *to* A_a *to secure the maximum discharge for the heads given below. Height of water barometer* 34 *ft.*

$$h+\frac{V_a^2}{2g}+\frac{p_a}{w}=h+\frac{V_b^2}{2g}+\frac{p_b}{w}$$

$$\frac{V_a^2-V_b^2}{2g}=34-\frac{p_a}{w}$$

$$\frac{\frac{V_b^2A_b^2}{A_a^2}-V_b^2}{2g}=34-\frac{p_a}{w}$$

$$\frac{V_b^2}{2g}\left(\frac{A_b^2}{A_a^2}-1\right)=h\left(\frac{A_b^2}{A_a^2}-1\right)=34$$

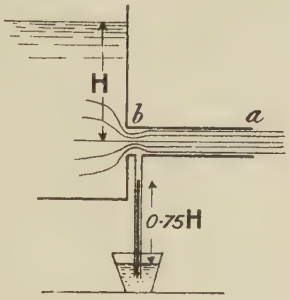
h (feet)	$\frac{A_b}{A_a}$
1	5·92
2	4·25
3	3·50
4	3·08
5	2·80
10	2·10
20	1·64
34	1·41
100	1·16

when water is about to break up.

When $A_a=1$ $\qquad A_b=\sqrt{\frac{34}{b}+1}$

22. *Show that the vacuum at the inlet of a horizontal pipe orifice when running a full stream is* 0·75H, *where* H *is the head of water over the centre of the orifice. Coefficient of discharge when running a clear stream,* 0·62 ; *coefficient of discharge when running a full stream,* 0·82. *Explain why*

the stream is transparent in the one case and not in the other.



$$\text{Loss of head} = \frac{(V_b - V_a)^2}{2g}$$

$$H = \frac{V_a^2}{2g} + \frac{(V_b - V_a)^2}{2g}$$

$$V_a = \frac{1}{\sqrt{1 + \left(\frac{1}{K} - 1\right)^2}} \sqrt{2gH}$$

$$= K_D \sqrt{2gH}$$

$Q = K_d A \sqrt{2gH}$ clear stream not touching pipe

$Q_1 = K_D A \sqrt{2gH}$ full stream—

$$Q_1 = K_d A \sqrt{2gH\left(\frac{K_D}{K_d}\right)^2}$$

$$\text{Vacuum at } b = H\left(\frac{K_D}{K_d}\right)^2 - H = H\left\{\left(\frac{0{\cdot}82}{0{\cdot}62}\right)^2 - 1\right\} = \mathbf{0{\cdot}75H.}$$

When the jet springs clear of the sides the flow is stream line, but when the jet expands and fills the pipe the flow is sinuous, due to eddying after the sudden expansion of the stream.

When the pipe orifice has a very sharp edge the coefficients may be 0·60 and 0·85, which gives a vacuum of 1·0H instead of 0·75H.

23. *Find the quantity of water which will pass through a horizontal external pipe orifice in the side of a tank when it is running full bore. Diam. of pipe,* 1·5″ ; *head above centre of orifice,* 28″ ; *coefficient of contraction for clear stream,* 0·61. *The discharge is* 97 *per cent. of the calculated discharge.*

$$V_1 = \frac{1}{\sqrt{1 + \left(\frac{1}{K_c} - 1\right)^2}} \sqrt{2gH} = 0{\cdot}843 \sqrt{2gH}$$

$$V = 0{\cdot}843 \times 0{\cdot}97 \times 8{\cdot}02 \sqrt{\tfrac{28}{12}} = 10 \text{ f.p.s.}$$

$$Q = \frac{1{\cdot}767}{144} \times 10 = \mathbf{0{\cdot}123 \text{ cusec.}}$$

24. *An external vertical pipe orifice has a slightly rounded inlet which gives* $K_c = 0{\cdot}8$ *when running a clear stream. Find the quantity of water which will pass when the pipe runs full bore. Length of pipe,* 5″ ; *diameter of pipe,* 1·2″ ; *head of water above inlet,* 31″. *Allow for the friction in the pipe.*

Let all quantities referring to the bottom of the pipe take the suffix *a*, and the top *b*.

(See p. 647, Vol. I.) When running full bore.

$$V_a = \frac{\sqrt{2gh_a}}{\sqrt{1+\left(\frac{1}{K_c}-1\right)^2}} = \frac{8{\cdot}02\sqrt{3}}{\sqrt{1+\left(\frac{1}{0{\cdot}8}-1\right)^2}} = 13{\cdot}5 \text{ f.p.s.}$$

$$Q = \frac{13{\cdot}5 \times 1{\cdot}13}{144} = 0{\cdot}106 \left\{ \begin{array}{l} \text{cusecs neglecting friction in} \\ \text{the pipe} \end{array} \right.$$

$$\frac{1}{K} = \frac{1}{3200}\left(1+\frac{12}{12\times 1{\cdot}2}\right) = \frac{1}{1750}$$

$$\frac{LV^2}{KD} = \frac{0{\cdot}42 \times 13{\cdot}47^2}{1750 \times 0{\cdot}1} = 0{\cdot}43 \text{ ft.}$$

Effective head $= 3 - 0{\cdot}43 = 2{\cdot}57$ ft.

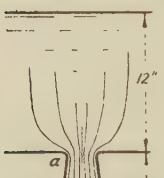
Assuming that the laws of friction apply where the turbulence is very great as in a pipe orifice.

$$Q = 0{\cdot}106\,\frac{\sqrt{2{\cdot}57}}{\sqrt{3}} = \mathbf{0{\cdot}98}\ \textbf{cusecs}$$ allowing for friction

$$K_v = \frac{0{\cdot}106 \times 144}{13{\cdot}89 \times 1{\cdot}13} = 0{\cdot}97$$ neglecting friction

$$K_v = \frac{0{\cdot}98 \times 144}{13{\cdot}89 \times 1{\cdot}13} = 0{\cdot}90$$ allowing for friction.

25. *Find the ratio of* A_b *to* A_a *in the orifice shown in order that the discharge may be a maximum.*



$$V_b^2 = 2g \times 1{\cdot}5$$

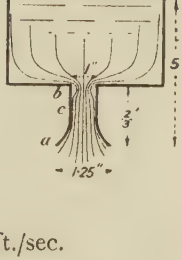
$$\frac{p_a}{w} + \frac{V_a^2}{2g} + h_a = \frac{p_b}{w} + \frac{V_b^2}{2g} + h_b$$

$$0 + \frac{n^2 V_b^2}{2g} + 0{\cdot}5 = \frac{14{\cdot}7}{0{\cdot}434} + \frac{V_b^2}{2g} + 0$$

$$\frac{V_b^2}{2g}(n^2-1) = 33{\cdot}4 = 1{\cdot}5(n^2-1)$$

$$n = \mathbf{4{\cdot}83} \qquad d_b = 2{\cdot}2\,d_a.$$

26. *Find the quantity of water which will flow through the bell-mouthed pipe orifice shown in the figure.* $K_d = 0{\cdot}6$ *when running a clear stream neglecting friction.*



$$\frac{V_a}{V_b} = \frac{K_d \times 1}{1{\cdot}25^2} = 0{\cdot}384 \qquad \frac{V_a}{V_c} = \frac{1}{1{\cdot}25^2} = 0{\cdot}64$$

$$H = h_a + \frac{p_a}{w} + \frac{V_a^2}{2g} + \frac{(V_b - V_c)^2}{2g}$$

$$5 = 0 + 0 + \frac{V_a^2}{2g} + \frac{\left(\frac{V_a}{0{\cdot}384} - \frac{V_a}{0{\cdot}64}\right)^2}{2g}$$

$$\frac{V_a^2}{2g} = 2{\cdot}4 \qquad V_a = 12{\cdot}4 \text{ ft./sec.}$$

When running a full stream $\left.\right\}$ $Q = \frac{1{\cdot}227 \times 12{\cdot}4}{144} = 0{\cdot}106$ cusec

With a clear stream

$$Q = \frac{0{\cdot}6 \times 0{\cdot}785 \times 8{\cdot}02 \times \sqrt{4{\cdot}33}}{144} = 0{\cdot}055 \text{ cusec}$$

Will the stream break up at *b* ?

$$5 = \tfrac{2}{3} + \frac{\left(\frac{12{\cdot}4}{0{\cdot}384}\right)^2}{2g} + \frac{p_b}{w} \qquad \frac{p_b}{w} = \mathbf{-11{\cdot}8 \text{ ft.}}$$

Hence no breaking up will occur at *b*.

27. *A tank standing on a level floor has an orifice in one of its vertical sides through which a jet of water issues. Find the depth of the centre of this orifice below the surface of the water to satisfy the condition that the jet shall strike the floor at the maximum distance from the side of the tank ; also find the distance.*

Let h = head of water above the orifice in feet.
y = height of orifice above the floor in feet.
V = velocity of efflux in feet per sec.
x = distance from side of tank in feet.
T = time taken by a particle of water in reaching the floor after leaving orifice in seconds.

$$T = \frac{x}{v} = \frac{x}{\sqrt{2gh}} \qquad x = \sqrt{2gh}\,T \text{ which must be a max.}$$

$$y = \tfrac{1}{2}gT^2 \qquad T = \sqrt{\frac{2y}{g}} \qquad x = \frac{\sqrt{2gh} \times \sqrt{2y}}{\sqrt{g}} = 2\sqrt{hy}$$

Let $h+y=H$ and $hy=N$

Then when x is a max., $h(H-h)$ must also be a max., and

$$N=hH-h^2 \qquad \frac{dN}{dh}=H-2h=0$$

when this expression has its maximum value,

$$\text{or } h=\frac{H}{2} \qquad x=2\sqrt{\frac{H^2}{4}}=H.$$

28. *Water flows through an orifice in the bottom of a tank; diameter* 2″, *head* 4′ 6″, $K_d=0{\cdot}62$. *The water discharges into a second tank, which is provided with a rectangular notch* 1·5″ *wide, also a right-angled Vee notch, the bottoms of which are at the same level. Find the head over each.* K_d *for both notches* $=0{\cdot}6$.

$$Q=0{\cdot}62\times 0{\cdot}0218\times 8{\cdot}02\sqrt{4{\cdot}5}=0{\cdot}23 \text{ cusec}$$

$$Q=0{\cdot}6\times\tfrac{2}{3}\times 0{\cdot}125\times H\times 8{\cdot}02\sqrt{H}+\tfrac{4}{15}\times 0{\cdot}6\times 2H^2\times 8{\cdot}02\sqrt{H}$$

$$0{\cdot}23=0{\cdot}401H\sqrt{H}+2{\cdot}57H^2\sqrt{H} \qquad 1=1{\cdot}74H^{\frac{3}{2}}+11{\cdot}2H^{\frac{5}{2}}$$

By trial and error: $H=0{\cdot}325'=\mathbf{3{\cdot}9''}$.

29. *The water flowing through an orifice,* 6″ *diameter in a horizontal plate, afterwards passes over a rectangular notch* 4′ *wide. The head of water over the orifice is* 10′. *Find the head of water over the sill of the notch.*

Area of orifice $=0{\cdot}196$ sq. ft.

Flow through orifice $=0{\cdot}62\times 0{\cdot}196\times 8{\cdot}02\times 3{\cdot}16=3{\cdot}08$ cusecs

Flow over weir $=0{\cdot}62\times\frac{2}{3}\times 4\times 8{\cdot}02\times H^{\frac{3}{2}}=13{\cdot}26H^{\frac{3}{2}}$

$$13{\cdot}26H^{\frac{3}{2}}=3{\cdot}08 \qquad H^{\frac{3}{2}}=0{\cdot}232 \qquad H=\mathbf{0{\cdot}377 \text{ feet.}}$$

Taking Barnes' expressions (see Barnes' "Hydraulic Flow Reviewed") for both the orifice and weir—

Orifice:

$$3{\cdot}73\times 0{\cdot}5^{1{\cdot}985}\times 10^{0{\cdot}5}=3{\cdot}73\times 0{\cdot}253\times 3{\cdot}16=2{\cdot}98 \text{ cusecs.}$$

Weir: $3{\cdot}324\times H^{1{\cdot}49}\times B^{1{\cdot}11}(B+2H)^{0{\cdot}11}=2{\cdot}98$ cusecs

$$H^{1{\cdot}49}(4+2H)^{0{\cdot}11}=\frac{2{\cdot}98}{3{\cdot}324\times 4^{1{\cdot}11}}=0{\cdot}192$$

$$H=\mathbf{0{\cdot}296'} \text{ (By trial and error).}$$

It will be seen that the commonly used expressions for the flow through orifices and over weirs give very inaccurate results in some cases.

30. *It is required to find the coefficient of discharge for an orifice of unusual shape, but no measuring tanks are available ; it is, therefore, decided to obtain it by observing the time taken by the water in falling from one level to another in the tank containing the orifice. Calculate the coefficient of discharge from the following data : constant horizontal section of tank,* 25 *sq. ft. ; area of orifice,* 0·04 *sq. ft. ; head of water above orifice at given instant,* 8·2 *ft. ; head of water after* 328 *seconds,* 2·7 *ft.*

$$K_d = \frac{2A(\sqrt{H_1} - \sqrt{H_2})}{Ta\sqrt{2g}} \qquad \text{(See p. 660, Vol. I.)}$$

$$K_d = \frac{2 \times 25(\sqrt{8{\cdot}2} - \sqrt{2{\cdot}7})}{328 \times 0{\cdot}04 \times 8{\cdot}02} = \mathbf{0{\cdot}58.}$$

31. *The head of water over a rectangular notch is measured by a glass water gauge in the side of the tank, the diameter of the tube is* 0·5 *inch, the zero reading on this tube is exactly level with the sill of the weir. Find the error in the discharge due to capillarity in the glass tube. Breadth of notch,* 3″ ; *true head over notch,* 2″ ; *coefficient of discharge,* 0·61 ; *angle between edge of meniscus and tube,* 25°.

$$dH = \frac{4T \cos \theta}{wd} = \frac{4 \times 0{\cdot}0056 \times 0{\cdot}906}{0{\cdot}434 \times 0{\cdot}5} = 0{\cdot}094''$$

$$Q = KH^{\frac{3}{2}} \qquad \therefore \frac{dQ}{Q} = \frac{3}{2} \cdot \frac{dH}{H}.$$

Hence fractional error in discharge due to error of 0·094″ is

$$\frac{3}{2} \times \frac{0{\cdot}094}{2} = \mathbf{7 \text{ per cent.}}$$

If both readings had been taken from the tube there would have been no error due to capillarity.

32. *In a triangular notch :*

Head ins.	Delivery lb./min.
2·0	85
6·0	1,330

Assuming an exponential law, find its equation in foot-second units, and calculate the delivery under a head of 4″.

$$Q = KH^n \qquad \log Q = \log K + n \log H$$

$H_1=\frac{1}{6}$ $\log H_1=1\cdot2218$ $Q_1=\frac{85}{60\times62\cdot5}$ $\log Q_1=\bar{2}\cdot3553$

$H_2=\frac{1}{2}$ $\log H_2=\bar{1}\cdot6990$ $Q_2=\frac{1330}{60\times62\cdot5}$ $\log Q_2=\bar{1}\cdot5698$

$n(\log H_2-\log H_1)=\log Q_2-\log Q_1$ $\therefore n=\frac{1\cdot1945}{0\cdot4772}=2\cdot50$

$\log K=2\cdot50\log H_1-\log Q_1=0\cdot3008$ $K=2\cdot0$

Equation for notch: $Q=\mathbf{2\cdot0H^{2\cdot5}}$

When $H=\frac{1}{3}$ foot: $Q=\mathbf{0\cdot128}$ **cusec.**

33. *A river* 50 *ft. wide discharges* 80 *million gallons per day. To what height above the existing water-level must a dam be built to raise the level by* 18 *inches?*

$$Q=\frac{80\times10^6}{24\times60\times60}\times\frac{10}{62\cdot5}=148\cdot5 \text{ cusecs}$$

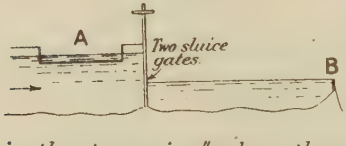
Head over dam will therefore be H ft., where:

$148\cdot5=3\cdot33\times50\times H^{\frac{3}{2}}$ $H=0\cdot926$ ft.

Height of dam above existing water-level $=1\cdot50-0\cdot926$

$=\mathbf{0\cdot574}$ **ft.**

34. *The flow of water through a mill pond is regulated by two sluice gates, each* 4′ *wide. When the gates are closed the supply stream spills into a backwater over the weir* A (20′ *wide*) *whose sill is* 5′ *above the bottom of the gate. The pond loses water over the weir* B (40′ *wide*) *whose level is* 2′ *above the bottom of the gates. If the initial level of water in the stream is* 9″ *above the weir* A *with the sluices closed, and it is required to reduce this level by* 3″, *find how far the gates must be opened. Take the coefficient of delivery as* 0·66.



Initially: $Q_0=3\cdot33\times20\times(\frac{3}{4})^{\frac{3}{2}}=43\cdot25$ cusecs.

After opening: $Q_1=3\cdot33\times20+(\frac{1}{2})^{\frac{3}{2}}=23\cdot5$ cusecs.

Water passing through sluices and over weir B

$=43\cdot25-23\cdot5=19\cdot75$ cusecs.

Hence head above weir B is given by:

$19\cdot75=3\cdot33\times40\times H^{\frac{3}{2}}$ $\therefore H=0\cdot28$ ft.

Effective head at sluices $=5\cdot5-2\cdot28$ ft. $=3\cdot22$ ft.

Height x the two gates must be opened :

$$2(x\times 4)\times 0{\cdot}66\times\sqrt{2g\times 3{\cdot}22}=19{\cdot}75 \qquad x=\mathbf{0{\cdot}26\ ft.}$$

35. *After heavy rain, the discharge of a river is liable to increase to ten times the dry weather flow. If the difference in water-level above a weir must not exceed* 1 *ft., find the minimum breadth of weir per million gallons daily of dry weather flow.*

$$Q_1=\tfrac{10}{3}BH_1^{\frac{3}{2}} \qquad 10Q_1=\tfrac{10}{3}B(H_1+1)^{\frac{3}{2}}$$

$$\left(\frac{H_1+1}{H_1}\right)^{\frac{3}{2}}=10 \qquad \frac{H_1+1}{H_1}=(10)^{\frac{2}{3}}=4{\cdot}64$$

$H_1=0{\cdot}275$ foot. $\qquad B=\mathbf{3{\cdot}87\ feet.}$

36. *Find the quantity of water that will flow per day over a broad-crested weir,* 16′ *in length, the height of the approach stream being* 2′ 2″ *above the top of the weir.*

$$Q=0{\cdot}385BH\sqrt{2gH} \qquad \text{(See p. 659, Vol. I.)}$$

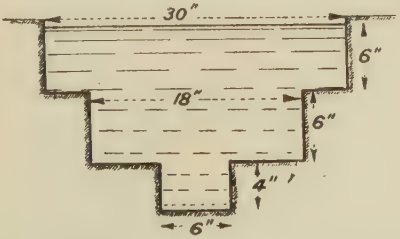
$$=0{\cdot}385\times 16\times 2{\cdot}17\times 8{\cdot}02\times 1{\cdot}47=158 \text{ cusecs}$$

$158\times 60\times 60\times 24=\mathbf{13{,}600{,}000}$ c. ft. p. day.

Although the reasoning on which the above expression is arrived at is not quite satisfactory, the calculated flow agrees well with experiment.

See a paper " Measurement of Discharge over a Rock-faced Dam," by Dixon and Macaulay, *Proceedings*, I.C.E., Vol. 220.

37. *Find the quantity of water flowing over a stepped notch as shown in the figure.* $K_d=0{\cdot}62$.



Method (i) Taking vertical strips—

Central strip :

$$Q=0{\cdot}62\times\tfrac{2}{3}\times\tfrac{1}{2}\times\tfrac{1}{3}\sqrt{64{\cdot}4\times\tfrac{1}{3}}=2{\cdot}55 \text{ cusecs}$$

Intermediate strips :

$$Q=0{\cdot}62\times\tfrac{2}{3}\sqrt{64{\cdot}4}=3{\cdot}32 \text{ cusecs}$$

Outside strips :

$$Q=0{\cdot}62\times\tfrac{2}{3}\times0{\cdot}5\sqrt{64{\cdot}4\times0{\cdot}5}=1{\cdot}17 \text{ cusecs}$$

Total **7·04**

Or (ii) Horizontal strips—

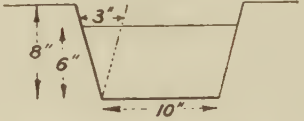
Top $Q=0{\cdot}62\times\frac{2}{3}\times2{\cdot}5\times8{\cdot}02\times0{\cdot}5^{\frac{3}{2}}$ $=2{\cdot}92$ cusecs

Middle $Q=0{\cdot}62\times\frac{2}{3}\times1{\cdot}5\times8{\cdot}02(1^{\frac{3}{2}}-0{\cdot}5^{\frac{3}{2}})$ $=3{\cdot}21$ cusecs

Bottom $Q=0{\cdot}62\times\frac{2}{3}\times0{\cdot}5\times8{\cdot}02(1{\cdot}33^{\frac{3}{2}}-1^{\frac{3}{2}})$ $=0{\cdot}90$ cusecs

Total **7·03**

38. *Find the quantity of water which will flow over the notch shown in the figure. Head* 6″ ; $K_d=0{\cdot}6$.



Taking it as a rectangle 10″ × 6″ and a triangle 3″ × 6″

$$Q=K(\tfrac{2}{3}BH\sqrt{2gH}+\tfrac{4}{15}B_1H\sqrt{2gH})$$

$$=0{\cdot}6\times0{\cdot}5\sqrt{64{\cdot}4\times0{\cdot}5}(\tfrac{2}{3}\times\tfrac{10}{12}+\tfrac{4}{15}\times\tfrac{3}{12})=\mathbf{1{\cdot}06 \text{ cusecs.}}$$

39. *A notch whose bottom edge slopes* 1 *in* 100 *is used to enable ready estimates to be made of the water flowing. Find the delivery when water is flowing over a length of* 30 *ft., and the law of calibration.*

Let x be the length of the notch over which flow is taking place, and C the coefficient of discharge.

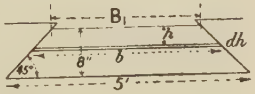
Then over a length dx the flow is :

$$dQ=C\times5{\cdot}35\times dx\times h^{\frac{3}{2}} \qquad Q=\frac{C\times5{\cdot}35}{1000}\int_0^x x^{\frac{3}{2}}dx$$

If $C=0{\cdot}62$ $\quad Q=0{\cdot}00133x^{\frac{5}{2}}$, which is the law of calibration.

If $x=30$, then $Q=0{\cdot}00133\times30^{\frac{5}{2}}=\mathbf{6{\cdot}53 \text{ cusecs.}}$

40. *A notch in a thin vertical plate is of the form shown in the figure. Find the quantity of water passing per second.* $K_d=0{\cdot}6$; *Head* $=8''$.



$$b=B_1+2h$$

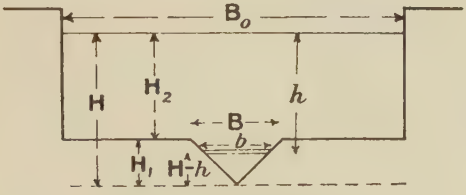
$$B_1=5-2(\tfrac{8}{12})=3{\cdot}66 \text{ feet}$$

Quantity passing strip $=K\ b.dh\sqrt{2gh}=K\sqrt{2g}(B_1+2h)h^{\frac{1}{2}}dh$

Quantity passing over notch $=Q=0{\cdot}6\sqrt{2g}\int_0^H(B_1h^{\frac{1}{2}}+2h^{\frac{3}{2}})dh$

$$Q=0{\cdot}6\times 8{\cdot}02\left(\frac{2B_1H^{\frac{3}{2}}}{3}+\frac{4H^{\frac{5}{2}}}{5}\right)=\mathbf{7{\cdot}8\ cusecs.}$$

41. *A notch of the form shown in the figure consists of a Vee notch for measuring small flows and a combined* V *and rectangular notch when the flow is great. Find an expression for the quantity passing under the latter conditions.*



$$b=\frac{B(H-h)}{H_1}$$

$$\left.\begin{aligned}Q&=Kbdh\sqrt{2gh}\\ Q&=K\frac{B(H-h)}{H_1}dh\sqrt{2gh}\end{aligned}\right\}\ \text{For the quantity passing the narrow strip in the V notch.}$$

$$\left.Q=\frac{KB\sqrt{2g}}{H_1}\int_{h=H_2}^{h=H}h^{\frac{1}{2}}(H-h)dh\right\}\ \text{For the quantity passing the triangular notch.}$$

$$Q=\text{etc.}\int_{H_2}^{H}Hh^{\frac{1}{2}}dh-\int_{H_2}^{H}h^{\frac{3}{2}}dh$$

$$Q=\text{etc.}\left[\frac{Hh^{\frac{3}{2}}}{\frac{3}{2}}\right]_{H_2}^{H}-\left[\frac{h^{\frac{5}{2}}}{\frac{5}{2}}\right]_{H_2}^{H}$$

$$Q=\text{etc.}\left[\tfrac{2}{3}H^{\frac{5}{2}}-\tfrac{2}{3}HH_2{}^{\frac{3}{2}}-\tfrac{2}{5}H^{\frac{5}{2}}+\tfrac{2}{5}H_2{}^{\frac{5}{2}}\right]$$

$$Q=\frac{KB\sqrt{2g}}{H_1}\left(\tfrac{4}{15}H^{\frac{5}{2}}-\tfrac{10}{15}HH_2{}^{\frac{3}{2}}+\tfrac{6}{15}H_2{}^{\frac{5}{2}}\right)$$

For the whole notch :

Q = The above expression + $K\frac{2}{3}B_0H_2\sqrt{2g}H_2^{\frac{1}{2}}$

$$Q=\frac{K\sqrt{2g}}{15}\left\{\left[\frac{B}{H_1}(4H^{\frac{5}{2}}-10HH_2^{\frac{3}{2}}+6H_2^{\frac{5}{2}})\right]+10B_0H_2^{\frac{3}{2}}\right\}$$

Let $H=3$ feet $\quad H_1=1{\cdot}5$ ft. $\quad H_2=1{\cdot}5$ ft.

$B=6$ ft. $\quad B_0=20$ ft. $\quad K=0{\cdot}6$

$$Q=\frac{0{\cdot}6\times 8{\cdot}02}{15}\left\{\left[\frac{6}{1{\cdot}5}(4\times 3^{\frac{5}{2}}-10\times 3\times 1{\cdot}5^{\frac{3}{2}}+6\times 1{\cdot}5^{\frac{5}{2}})\right]+10\times 20\times 1{\cdot}5^{\frac{3}{2}}\right\}$$

$$=0{\cdot}321\{[4(4\times 15{\cdot}6-30\times 1{\cdot}84+6\times 2{\cdot}76)]+200\times 1{\cdot}84\}$$

$$=\mathbf{148{\cdot}6\ cusecs.}$$

As a check, treat the notch as rectangular with a triangular orifice below, the velocity of flow being due to the head above the centre of gravity.

Rectangular notch :

$$Q_n=0{\cdot}6\times\tfrac{2}{3}\times 20\times 1{\cdot}5\times 8{\cdot}02\times 1{\cdot}225=117{\cdot}9$$

Triangular orifice :

$$Q_0=\frac{0{\cdot}6\times 6\times 1{\cdot}5}{2}\times 8{\cdot}02\sqrt{1{\cdot}5+0{\cdot}5}=30{\cdot}5$$

$$Q_n+Q_0=\mathbf{148{\cdot}5\ cusecs.}$$

42. *The area of the water surface of a tank varies as the square root of the head above an orifice in the bottom. The tank is filled to a height of 24″ above the orifice and the level is required to fall 1 inch per hour. Find the top area* (A) *of the water surface. The diameter of the orifice is* 0·1″. $K_d=0{\cdot}62$.

Then $A\,dh=0{\cdot}62\times 0{\cdot}785\times 0{\cdot}1^2\times 8{\cdot}02\sqrt{2}\,dt$

$$A=0{\cdot}0552\frac{dt}{dh}=\frac{0{\cdot}0552\times 3600}{\frac{1}{12}}=\mathbf{2{,}385\ sq.\ ins.}$$

If the horizontal section of the tank is circular it gives a diameter of 55 ins.

An arrangement of this kind was used by the ancient Egyptians for measuring time. The form of the vessel to give

equal differences of water-level for equal intervals of time was doubtless found by experiment.

43. *A tank of* 12 *sq. ft. horizontal cross-section has two orifices each* 2″ *diam. in one of its vertical sides. One is* 20 *ft. above the bottom of the tank and the other* 8 *ft. Find the time required to lower the water from* 30 *feet down to* 15 *ft.* $K_d = 0{\cdot}62$.

Let the initial head above the top orifice be H_a, and above the bottom orifice H_b. $H = H_b - H_a$. Final head above bottom orifice H_c.

When the water is above the top orifice we have :

$$dt = \frac{A}{K_d a\sqrt{2g}}\left(\frac{h_b^{\frac{1}{2}} - h_a^{\frac{1}{2}}}{h_b - h_a}\right)dh \qquad \text{where } h_a,\ h_b \text{ refer to any instant.}$$

$$T(\text{top}) = \frac{A}{K_d a\sqrt{2gH}}\left[\int_{H_c}^{H_b} h_b^{\frac{1}{2}}dh - \int_{0}^{H_a} h_a^{\frac{1}{2}}dh\right]$$

$$= \frac{A}{K_d a\sqrt{2gH}}\left(\tfrac{2}{3}H_b^{\frac{3}{2}} - H^{\frac{3}{2}} - H_a^{\frac{3}{2}}\right).$$

When the water is below the top orifice we have :

$$T(\text{bottom}) = \frac{A}{K_d a\sqrt{2g}}\int_{H}^{H} h^{-\frac{1}{2}}dh = \frac{2A(H^{\frac{1}{2}} - H_c^{\frac{1}{2}})}{K_d a\sqrt{2g}}$$

$$T = T(\text{top}) + T(\text{bottom})$$

$$T(\text{top}) = \frac{12}{0{\cdot}62 \times 0{\cdot}0218 \times 8{\cdot}02 \times 12}\tfrac{2}{3}(103{\cdot}2 - 41{\cdot}6 - 31{\cdot}6)$$

$$= 185 \text{ seconds}$$

$$T(\text{bottom}) = \frac{2 \times 12 \times 0{\cdot}82}{0{\cdot}62 \times 0{\cdot}0218 \times 8{\cdot}02} = 181 \text{ seconds}$$

$$T = 185 + 181 = \mathbf{366 \text{ seconds.}}$$

The above solution is not rigidly true, since it neglects the disturbance of the water as the surface passes the top orifice.

44. *Find the time required to lower the level of the water in a hemispherical vessel which is full of water initially down to a height of* 2·5′ *above an orifice in the bottom of the vessel. Diam. of vessel,* 10′ ; *diam. of orifice,* 2″ ; $K_d = 0{\cdot}62$.

(See p. 661, Vol. I.)

$$T=\frac{\pi}{K_d a\sqrt{2g}}\left(\frac{4}{5}R^{\frac{5}{2}}-\frac{2}{5}R^{\frac{5}{2}}-\frac{4R^{\frac{5}{2}}}{3\times 2^{\frac{3}{2}}}+\frac{2R^{\frac{5}{2}}}{5\times 2^{\frac{5}{2}}}\right)$$

$$T=\frac{\pi}{K_d a\sqrt{2g}}(0{\cdot}531R^{\frac{5}{2}})$$

$$T=\frac{3{\cdot}14\times 0{\cdot}531\times 5^{\frac{5}{2}}\times 144}{0{\cdot}62\times 3{\cdot}14\times 8{\cdot}02}=\mathbf{860\ secs.}$$

45. *Find the approximate time required to empty a swimming bath through a fullway sluice valve* 12 *ins. diameter with a well-rounded entrance. Length of bath,* 80 *ft.; breadth,* 30 *ft.; depth at one end,* 6 *ft.* 6 *ins.; at other end,* 3 *ft.* 6 *ins.; coefficient of discharge of valve,* 0·95.

(See p. 661, Vol. I.)

$$T=\frac{2\times 80\times 30(\sqrt{6{\cdot}5}-\sqrt{3})}{0{\cdot}95\times 0{\cdot}785\times 8{\cdot}02}+\frac{2\times 80\times 30\sqrt{3{\cdot}5}}{3\times 0{\cdot}95\times 0{\cdot}785\times 8{\cdot}02}$$

$$=\tfrac{4800}{6}(2{\cdot}55-1{\cdot}73+0{\cdot}62)=\mathbf{1{,}150\ secs.}$$

46. *A tapered tank with its apex below is of such a form that the area of the surface of the water* A *at any height* H *above an orifice at the bottom varies as* $H^{2\cdot3}$. *The surface is* 12 *sq. ft. at a height of* 5 *ft. above the orifice. Find the time required to lower the water from* 5 *feet to* 2 *feet. Diam. of orifice* 1″. $K_d=0{\cdot}93$.

Let $A=mH^{2\cdot3}$ Then $12=m\times 5^{2\cdot3}$ $m=0{\cdot}296$

$$A\,dh=K_d a\sqrt{2g}h\,dt \qquad mh^{2\cdot3}dh=K_d a\sqrt{2g}h^{0\cdot5}dt$$

$$mh^{1\cdot8}dh=K_d a\sqrt{2g}\,dt \qquad \int dt=\frac{m}{K_d a\sqrt{2g}}\int_{h=2}^{h=5}h^{1\cdot8}dh$$

$$T=\frac{m}{K_d a\sqrt{2g}}\left(\frac{5^{2\cdot8}}{2{\cdot}8}-\frac{2^{2\cdot8}}{2{\cdot}8}\right)$$

$$T=\frac{0{\cdot}296\times 144\times 83{\cdot}6}{0{\cdot}93\times 0{\cdot}785\times 8{\cdot}02\times 2{\cdot}8}=\mathbf{218\ secs.}$$

47. *A tank in the form of a wedge with the sharp edge downwards has a rectangular notch in one of the vertical ends. Find the time required to lower the water from* 2 *ft. to* 3 *ins. above the sill of the notch. Breadth of notch,* 2 *ft.* 6 *ins.; coefficient of discharge,* 0·6; *apex angle,* 30°; *upper water surface,* 10 *ft.* × 10 *ft.*

Let H = Height in feet of the water surface above apex.
z = Height in feet of the sill of the notch above apex.
b = Breadth in feet of a horizontal surface h ft. above sill of weir.
a = Area of the surface of the water h ft. above sill.

$\frac{5}{H} = \tan 15$ $\quad H = 18{\cdot}7$ ft. $\quad z = 16{\cdot}7$ ft.

$$b = 10\frac{(z+h)}{18{\cdot}7} \qquad a = \frac{100}{18{\cdot}7}(z+h) = 89{\cdot}3 + 5{\cdot}35h$$

$$a\,.\,dh = \tfrac{2}{3}\mathrm{K}Bh^{\frac{3}{2}}\sqrt{2g}\,dt \qquad dt = \frac{3(89{\cdot}3+5{\cdot}35h)h^{-\frac{3}{2}}dh}{2\mathrm{K}B\sqrt{2g}}$$

$$T = \frac{1}{8{\cdot}02}\int_{0{\cdot}25}^{2}(89{\cdot}3h^{-\frac{3}{2}} + 5{\cdot}35h^{-\frac{1}{2}})dh$$

$$= \frac{1}{8{\cdot}02}\left(\frac{89{\cdot}3\times 2^{-\frac{1}{2}}}{-\frac{1}{2}} + \frac{5{\cdot}35\times 2^{\frac{1}{2}}}{\frac{1}{2}} - \frac{89{\cdot}3\times 0{\cdot}25^{-\frac{1}{2}}}{-\frac{1}{2}} - \frac{5{\cdot}35\times 0{\cdot}25^{\frac{1}{2}}}{\frac{1}{2}}\right)$$

$$= \mathbf{30\ secs.}$$

48. *A wedge-shaped tank with the sharp edge uppermost and 12′ high is 10′ square at the base. Find the time required to lower the water from 8′ to 1′ through an orifice in the base 2″ diameter.* $\mathrm{K}_d = 0{\cdot}61$.

Let A = area of water surface h ft. above orifice.

$$\mathrm{A} = 100\frac{12-h}{12} = 8{\cdot}3(12-h) \qquad \mathrm{A}dh = \mathrm{K}_d a\sqrt{2g}h^{\frac{1}{2}}dt$$

$$T = \int_1^8 \frac{\mathrm{A}h^{-\frac{1}{2}}dh}{\mathrm{K}_d a\sqrt{2g}} = \int_1^8 \frac{8{\cdot}3(12-h)h^{-\frac{1}{2}}dh}{\mathrm{K}a\sqrt{2g}}$$

$$T = \frac{8{\cdot}3}{\mathrm{K}_d a\sqrt{2g}}\int_1^8 (12h^{-\frac{1}{2}} - h^{\frac{1}{2}})dh$$

$$T = \frac{8{\cdot}3}{0{\cdot}61\times 0{\cdot}0218\times 8{\cdot}02}\left(\frac{12\times 8^{\frac{1}{2}}}{\frac{1}{2}} - \frac{12\times 1}{\frac{1}{2}} - \frac{8^{\frac{3}{2}}}{\frac{3}{2}} + \frac{1}{\frac{3}{2}}\right)$$

$$= \mathbf{2{,}290\ secs.}$$

49. *A tank in the form of an inverted cone (apex downwards) is filled with water to a height of 40 ft. above the apex. Find the time required to lower the water to 12 ft. when it leaves by an orifice 1·5 in. diameter which may be assumed to*

be at the apex. The diameter of the tank is 32 ft. at a height of 40 ft. above the apex. Coefficient of discharge, 0·93.

Area of orifice $a=0{\cdot}0123$ sq. ft.

Area of tank surface $A=0{\cdot}785d^2=0{\cdot}785\dfrac{D^2h^2}{H^2}$

where d is the diam. at any height h above the orifice

D(32 ft.)	,,	,,	H (40′)	,,	,,
$A=0{\cdot}504h^2$			$Adh=K_da\sqrt{2g}h^{\frac{1}{2}}dt$		
$dt=\dfrac{Ah^{-\frac{1}{2}}dh}{K_da\sqrt{2g}}$			$\int dt=\dfrac{0{\cdot}504}{K_da\sqrt{2g}}\int_{12}^{40}h^{\frac{3}{2}}dh$		

$$T=\frac{0{\cdot}504(40^{\frac{5}{2}}-12^{\frac{5}{2}})}{0{\cdot}93\times0{\cdot}0123\times8{\cdot}02\times2{\cdot}5}=\textbf{21,100 secs.}$$

50. *The level of the water in a reservoir is lowered through a sluice* 2′ *diam. having a coefficient of discharge* $K_d=0{\cdot}8$. *Calculate the time required to lower the water from* 40′ *to* 25′.

Height above sluice in ft.	Area of water surface, sq. ft.	Height above sluice in ft.	Area of water surface, sq. ft.
40	2500	30	2000
35	2240	25	1600

This can be solved most readily by the method given on p. 663, Vol. I.

$$K_dA\sqrt{2g}=0{\cdot}8\times3{\cdot}14\times8{\cdot}02=20{\cdot}15$$

$$T=\left\{\frac{2500\times5}{2(20{\cdot}15\times6{\cdot}32)}+\frac{2240\times5}{20{\cdot}15\times5{\cdot}92}+\frac{2000\times5}{20{\cdot}15\times5{\cdot}48}+\frac{1600\times5}{2(20{\cdot}15\times5)}\right\}$$

$$=51{\cdot}4+93{\cdot}9+90{\cdot}6+39{\cdot}7=\textbf{276 secs.}\ \text{(approx.)}.$$

If the area of several horizontal sections of a hemispherical bowl be taken and the time of lowering the level be calculated by the above approximation and compared with the exact time (see p. 661) the two results will be found to agree quite closely.

51. *Two reservoirs with vertical walls are connected by means of a horizontal pipe having a bell-mouthed inlet end. Find the time required to bring the water to a common level in each. Area of reservoir* A, 4,000 *sq. ft.; area of reservoir* B, 2,500 *sq. ft.; level of water in* A *above datum,* 67 *ft.; level of water in* B *above datum,* 42 *ft.; diameter of connecting main,* 2 *ft.* 3 *ins.; length of connecting main,* 600 *ft.*

$$\frac{1}{K}=\frac{1}{5000}\left(1+\frac{1}{2D}\right)=\frac{1}{4090}$$ (See p. 682, Vol. I.)

Area of pipe section (a)=3·98 sq. ft.

$$h=\frac{600V^2}{4090\times 2\cdot 25}+\frac{V^2}{2g}=0\cdot 081\ V^2 \qquad V=\frac{\sqrt{h}}{0\cdot 284}$$

$$\frac{dh}{dt}=\frac{aV}{A_1}+\frac{aV}{A_2}=\frac{aV(A_1+A_2)}{A_1A_2}=\frac{3\cdot 98V\times 6500}{10{,}000{,}000}$$

$$dh=0\cdot 00259V\,dt$$ (See p. 664, Vol. I.)

$$dt=\frac{0\cdot 284\,dh}{0\cdot 00259h^{\frac{1}{2}}}=110h^{-\frac{1}{2}}dh$$

$$T=110\int_0^{25}h^{-\frac{1}{2}}dh=110\times 25^{\frac{1}{2}}\times 2=\mathbf{1100\ secs.}$$

If there were no pipe between, but a simple orifice having a value of K=0·62, then

$$dh=0\cdot 62\times 0\cdot 00259V\,dt=0\cdot 0016\sqrt{2gh}\,dt$$

$$dt=\frac{h^{-\frac{1}{2}}dh}{0\cdot 0016\times 8\cdot 02}=78h^{-\frac{1}{2}}dh$$

$$T=78\int_0^{25}h^{-\frac{1}{2}}dh=78\times 2\sqrt{25}=\mathbf{780\ secs.}$$

52. *Horizontal cross-sections of a reservoir at vertical intervals of* 5 *ft. are* 53,000, 48,000, *and* 46,000 *sq. ft. The reservoir may be emptied through a bell-mouthed pipe* 6 *ins. diam. and* 300 *yds. long, discharging freely at a level* 30 *ft. below the highest contour line. Find approximately how long it will take to lower the level of the reservoir from the highest to the lowest contour.* K *in D'Arcy's formula*=2000.

This must be solved by steps, which is only an approximation.

$$h=\frac{LV^2}{KD} \quad \text{and} \quad V=\sqrt{\frac{hKD}{L}}=\sqrt{\frac{h\times 2000}{2\times 900}}=1{\cdot}055\sqrt{h}$$

$H_1=30'$ $V_1=5{\cdot}78$ f.p.s.
$H_2=25'$ $V_2=5{\cdot}27$,,
$H_3=20'$ $V_3=4{\cdot}72$,,

Area of pipe $a=0{\cdot}196$ sq. ft. Quantity passing $=0{\cdot}196\times V$

We have $Q\,.\,\delta t=A\,.\,\delta h$

$$\frac{\delta t}{\delta h}=\frac{A_1}{Q_1}=\frac{53000}{0{\cdot}196\times 5{\cdot}78}=46{,}800$$

$$\frac{A_2}{Q_2}=\frac{48000}{0{\cdot}196\times 5{\cdot}27}=46{,}400$$

$$\frac{A_3}{Q_3}=\frac{46000}{0{\cdot}196\times 4{\cdot}72}=49{,}700$$

$$T=\frac{46800\times 5}{2}+46{,}400\times 5+\frac{49700\times 5}{2}$$

$$=\mathbf{473{,}000\ secs.=132\ hours}\ \text{(approx.)}.$$

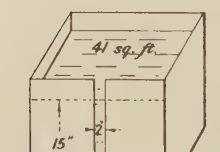
53. *The flow over a rectangular notch of breadth* B *feet and head* H *ft. in cusecs is given by* $Q=3{\cdot}1\,B^{1{\cdot}02}H^{1{\cdot}47}$. *Find the time required to lower the water in a tank, whose surface is of constant area* 41 *sq. ft., from* 1·25 *ft. to* 0·5 *ft. above the lip of the weir.* $B=0{\cdot}5$ *ft.*

$$A\,dh=3{\cdot}1B^{1{\cdot}02}H^{1{\cdot}47}dt$$

$$T=\frac{A}{3{\cdot}1B^{1{\cdot}02}}\int_{0{\cdot}5}^{1{\cdot}25}h^{-1{\cdot}47}dh$$

$$T=\frac{A}{3{\cdot}1\times 0{\cdot}5^{1{\cdot}02}\times 0{\cdot}47}\left(\frac{1}{0{\cdot}5^{0{\cdot}47}}-\frac{1}{1{\cdot}25^{0{\cdot}47}}\right)=\mathbf{28\ secs.}\ \text{(approx.)}.$$

54. *Find the time required to lower the water from* 15″ *to* 6″ *in the tank shown in the figure.* $K_d=0{\cdot}62$.



The quantity Q passing the notch when the head is h feet above the bottom is

$$Q=K_d\tfrac{2}{3}B\sqrt{2g}h^{\frac{3}{2}}$$

Let A be the area of the water surface.

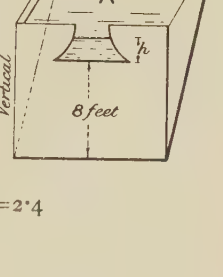
Then $$Q=\frac{A\,dh}{dt} \qquad dt=\frac{3Ah^{-\frac{3}{2}}dh}{2K_dB\sqrt{2g}}$$

$$T=\frac{3A}{2K_dB\sqrt{2g}}\int_{H_2}^{H_1}h^{-\frac{3}{2}}dh=\text{etc.}\left[\frac{h^{-\frac{1}{2}}}{-\frac{1}{2}}\right]_{H_2}^{H_1}=\text{etc.}\left[\frac{-2}{5^{\frac{1}{2}}}\right]_{H_2}^{H_1}$$

$$T=\frac{2\times3\times41\times12}{0\cdot62\times4\times8\cdot02}\left(\frac{1}{\sqrt{\frac{6}{12}}}-\frac{1}{\sqrt{\frac{15}{12}}}\right)=\mathbf{77\ secs.}$$

The value we have assumed for K_d is probably low, since the bottom of the approach chamber is at the same level as the sill where the contraction is suppressed; but there are no experimental data available to cover such a case.

55. *In the notch shown in the figure the width is so proportioned that the quantity of water passing varies directly as the head over the sill. Find the time required to lower the water from* 9″ (H_1) *to* 3″ (H_2) *over the sill.*



Vol. passing when $h=6''=1\cdot2$ *cusecs.*

Area of water surface A *at sill level*=244 *sq. ft.*

$$Q=mh \qquad m=\frac{1\cdot2}{0\cdot5}=2\cdot4$$

$$A=244\frac{(8+h)}{8}=30\cdot5(8+h)$$

$$Adh=mhdt$$

$$dt=\frac{Adh}{mh}=\frac{30\cdot5}{2\cdot4}(8+h)h^{-1}dh$$

$$T=12\cdot7\int_{H_2}^{H_1}(8h^{-1}+1)dh=12\cdot7(8\log_e\frac{H_1}{H_2}+H_1-H_2)$$

$$T=12\cdot7(8\log_e3+0\cdot75-0\cdot25)=\mathbf{118\ secs.}$$

56. *A Venturi water meter is employed for measuring the quantity of water delivered by a pump. Find the cusecs passing under the following conditions: Diameter of main,* 14 *ins.; diameter of throat,* 4·5 *ins.; mercury gauge (both limbs being full of water above the mercury),* 18·2 *ins.; coefficient of discharge,* 0·975.

Area of main (1) 153·93 sq.ins.; area of throat (2) 15·90 sq. ins.

$$\frac{V_1}{V_2}=\frac{15\cdot9}{153\cdot93}=0\cdot103$$

Head in feet of water $\frac{18{\cdot}2(13{\cdot}6-1)}{12}=19{\cdot}1$.

$$\frac{V_1^2}{2g}+\frac{p_1}{w}=\frac{V_2^2}{2g}+\frac{p_2}{w} \qquad \frac{V_1^2-V_2^2}{2g}=\frac{p_2-p_1}{w}$$

$$(0{\cdot}103^2-1)\frac{V_2^2}{2g}=-19{\cdot}1 \qquad V_2=35{\cdot}2 \text{ ft./sec.}$$

$$Q=\frac{35{\cdot}2\times 15{\cdot}9\times 0{\cdot}975}{144}=\mathbf{3{\cdot}78\ cusecs.}$$

57. *Find the quantity of water passing through a Venturi water meter having a throat ratio of* 16. *The diameter of the main is* 12″, *the difference of mercury head when both limbs are full of water above the mercury is* 14·2″ ; *the absolute pressure on the approach side of the meter is* 32 *lb. per sq. inch. At what velocity will the water break up in the throat ?*

$$\frac{V_1^2}{2g}+\frac{p_1}{w}=\frac{V_2^2}{2g}+\frac{p_2}{w} \qquad V_2^2=\frac{V_1^2}{256} \qquad \frac{V_1^2-V_2^2}{2g}=\frac{p_2-p_1}{w}$$

$$\frac{255V_1^2}{256\times 64{\cdot}4}=14{\cdot}2\,\frac{(13{\cdot}6-1)}{12}=14{\cdot}9 \text{ ft. water.}$$

$V_1=31$ f.p.s. through throat $\qquad Q=\frac{31}{16}\times\frac{113{\cdot}1}{144}=\mathbf{1{\cdot}52\ cusecs.}$

When breaking up occurs $p_1=0$

$$\frac{V_1^2-V_2^2}{2g}=\frac{p_2}{w}=\frac{32}{0{\cdot}434} \qquad V_1=\mathbf{69\ f.p.s.\ in\ throat.}$$

58. *A Venturi meter is placed with its axis vertical ; the difference of pressure in the two sections is measured by a mercury gauge which is full of water above the mercury. Find the quantity of water flowing per minute. Diam. of main,* 12″ ; *diam. of throat,* 2·5″ ; *mercury head,* 18·42″.

$$h_1+\frac{V_1^2}{2g}+\frac{p_1}{w}=h_2+\frac{V_2^2}{2g}+\frac{p_2}{w}, \quad h_1-h_2+\frac{V_1^2-V_2^2}{2g}=\frac{p_2-p_1}{w}$$

$$\frac{V_1^2-(23{\cdot}2V_1)^2}{64{\cdot}4}=\frac{18{\cdot}42(13{\cdot}6-1)}{12}$$

$$\frac{534V_1^2}{64{\cdot}4}=19{\cdot}4 \qquad V_1=1{\cdot}53 \text{ ft. per sec.}$$

$$Q=1{\cdot}53\times 0{\cdot}785\times 60\times 6{\cdot}24=\mathbf{448\ gals.\ p.\ min.}$$

59. *A horizontal Venturi meter has an enlarged waist. Find the quantity of water passing per second from the following data: Diam. of waist,* 30″; *diam. of main,* 12″; *mercury head (both limbs full of water),* 8·2″. K_d 0·97.

$$8\cdot2'' \text{ mercury} = \frac{8\cdot2 \times 12\cdot6}{12} = 8\cdot6 \text{ ft. water}$$

$$\frac{V_1}{V_2} = \frac{30^2}{12^2} = 6\cdot25 \qquad \frac{V_1^2}{2g} + \frac{p_1}{w} = \frac{V_2^2}{2g} + \frac{p_2}{w}$$

$$\frac{V_1^2 - \left(\frac{V_1}{6\cdot25}\right)^2}{2g} = \frac{p_2 - p_1}{w} = 8\cdot6 \qquad V_1 = 23\cdot8 \text{ f.p.s.}$$

$$Q = 0\cdot785 \times 0\cdot97 \times 23\cdot8 = \mathbf{18\cdot1 \text{ cusecs.}}$$

Contrary to what might be expected, the loss of head by friction and turbulence in a Venturi meter of this type is considerably greater than in a meter having a waist smaller than the body of the pipe.

60. *Water flows through a* 6″ *pipe in which there is a diaphragm having a hole* 2″ *diam. Find the pressure after passing the diaphragm. Initial pressure,* 100 *lb. sq. in.; velocity in pipe,* 6 *f.p.s.; K_c for hole in diaphragm,* 0·64.

Velocity at vena contracta $\frac{36}{4 \times 0\cdot64} = 14\cdot1 \times$ vely. in pipe.

$$\frac{p_1}{w} + \frac{V_1^2}{2g} = \frac{p_2}{w} + \frac{V_2^2}{2g} + \frac{V_2^2}{2g}(14\cdot1 - 1)^2$$

$$p_2 = 100 - \frac{0\cdot434 \times 36 \times 13\cdot1^2}{64\cdot4} = \mathbf{58\cdot3 \text{ lb. sq. in.}}$$

If the hole in the diaphragm is very small the water breaks up and p_2 becomes negative, which indicates that a vacuum is formed and the pressure after passing the diaphragm will be in the neighbourhood of 14·7 lb. sq. in. below atmosphere.

The diaphragm is exactly the same in theory as a Venturi meter with a throat ratio of 1 to 14·1. The pressure close to the diaphragm will be that corresponding to the Venturi head of the diaphragm which is much less than p_2. The full pressure p_2 will be found some 2 or 3 feet beyond the diaphragm.

61. *A piece of glass tube is drawn down locally to a small waist and water under pressure is forced through the tube. When the velocity is low, the flow is quite normal, but when the velocity is increased the water breaks up and boils vigorously. If the tube were* 0·75″ *diam., and the diam. of the waist* 0·1″, *calculate the velocity of flow in the waist at which the stream will break up. The tube is horizontal and the delivery (from the* 0·75″ *portion) is into the atmosphere. Height of water barometer,* 34′.

Ratio of sections $7{\cdot}5^2=56{\cdot}25$

$$\frac{p_1}{w}+\frac{V_1^2}{2g}=\frac{p_2}{w}+\frac{V_2^2}{2g} \qquad 34+\frac{V_1^2}{2g}=0+\frac{V_2^2}{2g}$$

$$34\times 64{\cdot}4=V_2^2-V_1^2=56{\cdot}25^2V_1^2-V_1^2=3163\ V_1^2$$

$V_1=0{\cdot}83$ f.p.s. in large section $V_2=$ **46·8 f.p.s.** in waist.

In the above $\frac{p_2}{w}$ is taken as zero—which is not rigidly true. It will always be slightly above zero, depending upon the temperature of the water, and to what extent it contains air in solution.

62. *If the delivery tube of an injector diverges gradually from* $\frac{3}{8}$″ *to* $\frac{3}{4}$″ *diam., and the pressure at the throat is just sufficient to overcome the friction in the delivery pipe, calculate the minimum rate of delivery into a boiler working at* 80 *lb. pressure.*

Suffix (1) denotes conditions at the throat.
„ (2) „ „ „ entrance to the boiler.

$$\frac{p_1}{w}+\frac{v_1^2}{2g}=\frac{p_2}{w}+\frac{v_2^2}{2g}+h_x$$

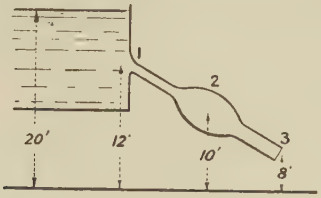
Given that $\frac{p_1}{w}=h_x$ and $v_1=4v_2$

Hence $\frac{p_2}{w}=\frac{1}{2g}(v_1^2-v_2^2)$

$\frac{80\times 144}{62{\cdot}5}=\frac{v_2^2}{2g}(16-1)$ for minimum delivery, $v_2=28{\cdot}1$ f.p.s.

Minimum delivery $=28{\cdot}1\times\frac{\pi}{4}\times\left(\frac{3}{48}\right)^2$ cusecs $=$ **32·3 g.p.m.**

63. *In the pipe of variable section,* $D_1=2''$, $D_2=8''$, $D_3=3''$. *Find* V_1, *and* Q. *Water barometer* 34 *ft.*



$$20+34=\frac{V_1^2}{2g}+\frac{p_1}{w}+h_1=\frac{V_3^2}{2g}+\frac{p_3}{w}+h_3 \text{ (neglecting losses)}$$

$$54=\frac{V_3^2}{2g}+34+8 \qquad V_3=27{\cdot}8 \text{ ft./sec.}$$

Then $V_1=\frac{9}{4}V_3$ $\qquad V_1=62{\cdot}5$ ft./sec. if pipe runs full.

We also have $54=\frac{62{\cdot}5^2}{64{\cdot}4}+12+\frac{p_1}{w}$

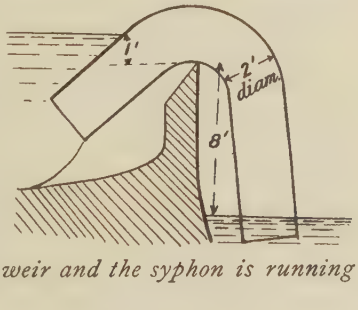
$\frac{p_1}{w}=-18{\cdot}8$ ft., *i.e.* the water breaks up in section 1.

Hence $\qquad 54=\frac{V_1^2}{2g}+34+10 \qquad V_1=$ **25·4 ft./sec.**

This is the max. vel. which can occur in section 1 under the given conditions, which is less than the value found above, hence the pipe does not run full, and p_2 and p_3 are atmospheric.

$$Q=\frac{3{\cdot}14\times 25{\cdot}4}{144}=\textbf{0·554 cusec.}$$

64. *A syphon spillway consists of a pipe* 2′ *diameter passing over the sill of a weir which is* 8′ *above the level of the tail race. If three-quarters of the available head is wasted in friction and turbulence in the pipe, find the increase in discharge obtained by use of the syphon when the water-level in the headrace is* 1 *ft. above the sill of the weir and the syphon is running full.*



The syphon may be regarded as occupying 2′ of the available length of the weir. The discharge under a head of 1 ft. is

$$Q = 0{\cdot}63 \times \tfrac{2}{3} \times 2 \times 1\sqrt{64{\cdot}4 \times 1} = 6{\cdot}7 \text{ cusecs}$$

The discharge through the syphon is

$$AV = 3{\cdot}14\sqrt{2g \times \tfrac{2}{4}} = 37{\cdot}7 \text{ cusecs}$$

Increase in discharge $= 37{\cdot}7 - 6{\cdot}7 =$ **31 cusecs.**

65. *Find the loss of head and the horse-power required to force water through the tubes of a surface condenser. The water passes through half the number of tubes into a chamber and from thence returns through the rest of the tubes. The ends of the tubes in all cases are flush with the tube-plates. Number of tubes*, 600 ; *length*, 8′ ; *internal diameter of tubes*, 0·75″ ; *quantity of water passing per minute*, 1500 *gals.* K *in D'Arcy's formula* $= 1370$.

The ratio of the area of the chamber to the area of the cross-section of a tube is practically infinite, hence the loss due to the sudden enlargement is $\frac{V_2^2}{2g}$, where V_2 is the velocity of flow through the tubes.

Area of each tube 0·441 sq. in.

Total area available for flow $= \dfrac{0{\cdot}441 \times 300}{144} = 0{\cdot}92$ sq. ft.

Velocity through tubes $(V_2) = \dfrac{1500}{6{\cdot}24 \times 60 \times 0{\cdot}92} = 4{\cdot}35$ f.p.s.

Loss of head in each set of tubes

= loss at entry + friction + loss at sudden enlargement

$$= \frac{0{\cdot}3V_2^2}{2g} + \frac{LV_2^2}{KD} + \frac{V_2^2}{2g}$$

$$= V_2^2\left(\frac{1{\cdot}3}{2g} + \frac{L}{KD}\right) = 4{\cdot}35^2\left(0{\cdot}02 + \frac{8 \times 12}{1370 \times 0{\cdot}75}\right) = 2{\cdot}14'$$

Loss of head in both sets of tubes $=$ **4·28′**.

H.P. absorbed $= 0{\cdot}1135 \times \dfrac{1500}{6{\cdot}24 \times 60} \times 4{\cdot}28 =$ **1·94.**

(See p. 703, Vol. I.)

66. *A pipe* 6″ *diameter suddenly enlarges to* 18″ *diameter ; it then suddenly contracts to* 7″ *diameter. Find the pressure*

in the 7" pipe when the velocity of flow in it is 20 f.p.s. The pressure is 80 lb. sq. in. in the 6" portion.

$$\frac{V_6}{V_7}=\frac{7^2}{6^2} \qquad \therefore \quad V_6=27{\cdot}2 \text{ f.p.s.}$$

$$\frac{V_{18}}{V_7}=\frac{7^2}{18^2} \qquad \therefore \quad V_{18}=3{\cdot}02 \text{ f.p.s.}$$

By Bernoulli's law we have

$$\frac{27{\cdot}2^2}{64{\cdot}4}+\frac{80}{0{\cdot}434}=\frac{20^2}{64{\cdot}4}+\frac{p}{0{\cdot}434}+\frac{(27{\cdot}2-3{\cdot}02)^2}{64{\cdot}4}+\frac{\left(\frac{20}{0{\cdot}6}-20\right)^2}{64{\cdot}4}$$

$$p=\mathbf{73 \text{ lb. sq. inch.}}$$

67. *Find the pressure in the outlet pipe* (3) *when* $V_1=30$ *f.p.s., and the pressure* $p_1=80$ *lb. p. sq. in.*

Loss at sudden enlargement from 1 to 2

$$\frac{(V_1-V_2)^2}{2g}=\frac{\left(V_1-\frac{V_1}{7{\cdot}1}\right)^2}{2g}=\frac{(0{\cdot}86V_1)^2}{2g}$$

Loss at entry in section (3) $K_c=0{\cdot}5$

$$\frac{(2V_3-V_3)^2}{2g}=\frac{V_3^2}{2g}=\frac{81V_1^2}{16\times 2g}$$

$$\frac{p_1}{w}+\frac{V_1^2}{2g}=\frac{p_3}{w}+\frac{V_3^2}{2g}+\frac{(0{\cdot}86V_1)^2}{2g}+\frac{81V_1^2}{32g}$$

$$\frac{80}{0{\cdot}434}=\frac{p_3}{0{\cdot}434}+\frac{2\times 81V_1^2}{32g}+\frac{0{\cdot}74V_1^2}{2g}-\frac{V_1^2}{2g}$$

$$p_3=\mathbf{20{\cdot}5 \text{ lb. sq. in.}}$$

68. *Water flows in a 2' pipe round a bend of 10' mean radius at a mean speed of 20 f.p.s. Find the difference of pressure over any given section of the pipe due to centrifugal force.*

Take an elemental radial prism of water 1 sq. inch in section.

$$C=\frac{WV^2}{gR}=\frac{0{\cdot}464\times 2(20^2)}{32{\cdot}2\times 10}=\mathbf{1{\cdot}08 \text{ lb. sq. in.}}$$

or (see p. 666, Vol. I.)

$$\text{Diff. of press.} = w\left(\frac{V_e^2 - V_i^2}{2g}\right) = 0{\cdot}434\left(\frac{22^2 - 18^2}{64{\cdot}4}\right)$$

$$= \mathbf{1{\cdot}08 \text{ lb. sq. inch.}}$$

69. For calculating the flow of water in pipes and open channels, also over notches, etc., readers should refer to Barnes' "Hydraulic Flow Reviewed," which is probably the most complete and up-to-date work on the subject, but it is doubtful if data are sufficiently accurate to warrant any but the more simple expressions.

Water flows along a clean cast-iron channel, the section of the stream being 14·28 *sq. ft. and the wetted perimeter* 12·68 *sq. feet. Find the fall per mile required to give a velocity of flow of* 3 *f.p.s. Compare results obtained from D'Arcy, Barnes, Thrupp.*

$$\text{Hydraulic mean depth} = \frac{14{\cdot}28}{12{\cdot}68} = 1{\cdot}13 = R \qquad D = 4 \times 1{\cdot}13.$$

D'Arcy. $\dfrac{1}{K} = \dfrac{1}{5000}\left(1 + \dfrac{1}{2 \times 4 \times 1{\cdot}13}\right) = \dfrac{1}{4500}$

$$h = \frac{LV^2}{KD} \text{ or } \frac{LV^2}{4KR} = \frac{5280 \times 9}{4 \times 4500 \times 1{\cdot}13} = \mathbf{2{\cdot}34'}$$

Thrupp. $V = \dfrac{R^{0{\cdot}67}}{0{\cdot}00535 \times \sqrt[1{\cdot}85]{S}}$ $\qquad 3 = \dfrac{1{\cdot}13^{0{\cdot}67}}{0{\cdot}00535 \times \sqrt[1{\cdot}85]{S}}$

$$S = 2180 = \frac{L}{h} \qquad h = \frac{5280}{2180} = \mathbf{2{\cdot}42'}$$

Barnes. $V = 136{\cdot}6 \times 1{\cdot}13^{0{\cdot}6} \times \dfrac{h^{0{\cdot}512}}{5280^{0{\cdot}512}} = 3$

$h = \mathbf{2{\cdot}63'}$. For a coated pipe $h = \mathbf{2{\cdot}1'}$.

70. *It is found by experiment that when a plank is towed through still water at* 1 *f.p.s., the resistance is* 0·004 *lb. per sq. foot of wetted surface, and further that it increases as the square of the speed. Utilising this data, find the quantity of water that will flow through a wooden stave pipe,* 4′ *diam., when the pipe falls* 11 *feet per mile.*

(See p. 681, Vol. I.)

$$(P_1 - P_2)\left(\frac{\pi D^2}{4}\right) = \pi D l f V^2 \qquad P = \frac{4fLV^2}{D}$$

$$h=\frac{4f}{W_w}\cdot\frac{LV^2}{D}=\frac{4\times 0{\cdot}004}{62{\cdot}4}\cdot\frac{LV^2}{D}$$

$$h=\frac{LV^2}{3900D} \qquad V=\sqrt{\frac{3900Dh}{L}}$$

$$V=\sqrt{\frac{3900\times 4\times 11}{5280}}=5{\cdot}7 \text{ f.p.s.}$$

$$Q=0{\cdot}785\times 4^2\times 5{\cdot}7=\mathbf{71{\cdot}6 \text{ cusecs.}}$$

71. *Compare the discharge of* (i) *two* 8″ *pipes with that of* (ii) *one* 12″ *pipe, the roughness, length, and slope being the same in both cases.*

$$Q=\frac{\pi D^2}{4}\sqrt{\frac{KDh}{L}}.$$

If K is constant $Q=$ const. $\times D^{\frac{5}{2}}$ per pipe.

(i) $Q=2\times 8^{\frac{5}{2}}=362$ (ii) $Q=12^{\frac{5}{2}}=500$

Ratio **1 to 1·38.**

If K varies (see p. 682, Vol. I.)

$$\frac{\text{(i)}}{\text{(ii)}}=\frac{2D_1^{\frac{5}{2}}\sqrt{K_1}}{D^{\frac{5}{2}}\sqrt{K_2}}=\frac{\sqrt{2859}}{1{\cdot}38\sqrt{3333}}=\frac{\mathbf{1}}{\mathbf{1{\cdot}49}}.$$

A pipe discharges 250 *g.p.m., and the head lost in friction is* 3′. *Find the loss of head when the discharge is increased to* 300 *gals. Find the work done in overcoming the friction in the two cases.*

$$\frac{h_1}{h_2}=\frac{Q_1^2}{Q_2^2}=\frac{300^2}{250^2} \qquad h_1=\frac{300^2}{250^2}\times 3=\mathbf{4{\cdot}32 \text{ ft.}}$$

Work done $=2500\times 3$ $=$**7,500** ft. lb. per min.

„ $=3000\times 4{\cdot}32=$**12,960** „ „ „

72. *Compare the loss of head on two pipes* 5 *inches and* 6 *inches diameter of the same length* (i) *with same velocity of flow,* (ii) *with same quantity passing.*

$K_5=2670$ $K_6=2740$, in D'Arcy's formula.

(i) $\dfrac{\text{Loss in } 5''}{\text{Loss in } 6''}=\dfrac{LV^2\times 2740\times 6}{2670\times 5\times LV^2}=\dfrac{6\times 2740}{5\times 2670}=\mathbf{1{\cdot}23}$

(ii) $V_6=\dfrac{25V_5}{36}$ $\therefore \dfrac{\text{Loss in } 5''}{\text{Loss in } 6''}=\dfrac{LV_5^2\times 6\times 2740\times 36^2}{5\times 2670\times LV_5^2\times 25^2}=\mathbf{2{\cdot}56.}$

73. *Find the flow through a clean cast-iron pipe* 2′ *diam. by D'Arcy, Thrupp, and Barnes.*

$h = 100'$ $\quad L = 1$ *mile.*

D'Arcy (modified). $V = \sqrt{\dfrac{KDh}{L}}$ $\quad \dfrac{1}{K} = \dfrac{1}{5000}\left(1 + \dfrac{1}{2D}\right) = \dfrac{1}{4000}$

$$V = \sqrt{\frac{4000 \times 2 \times 100}{5280}} = 12{\cdot}3 \text{ f.p.s.}$$

$$Q = 12{\cdot}3 \times 3{\cdot}14 = \mathbf{38{\cdot}7 \text{ cusecs.}}$$

Thrupp. $V = 187 \times 0{\cdot}5^{0{\cdot}67}\left(\dfrac{100}{5280}\right)^{0{\cdot}54}$

$V = 13{\cdot}8$ f.p.s. $\quad Q = \mathbf{43{\cdot}3 \text{ cusecs.}}$

Barnes. $V = 174{\cdot}1 \times 0{\cdot}5^{0{\cdot}769}\left(\dfrac{100}{5280}\right)^{0{\cdot}529}$

$= 12{\cdot}5$ f.p.s. $\quad Q = \mathbf{39{\cdot}4 \text{ cusecs.}}$

74. *A* $1\frac{1}{2}''$ *pipe delivers* 20 *g.p.m. at full bore. A pipe is to be installed drawing from the same supply by the same route, to deliver* 300 *g.p.m. Determine a suitable diameter for the pipe.*

Delivery, head, and diameter are connected by the expression :

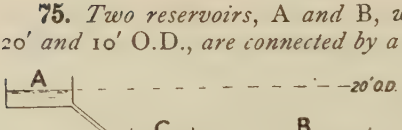
$$h_x = f \cdot \frac{4l}{d} \cdot \frac{v^2}{2g} = f \cdot \frac{4l}{d} \cdot \frac{8Q^2}{\pi^2 d^4 g} = K \cdot \frac{flQ^2}{d^5}$$

For the two pipes h_x, l are the same. Assume f is the same.

Then $\dfrac{Q_1^2}{d_1^5} = \dfrac{Q_2^2}{d_2^5}$

$$\therefore \; d_2 = d_1\left(\frac{Q_2}{Q_1}\right)^{\frac{2}{5}} = 1{\cdot}5 \times \left(\frac{300}{20}\right)^{\frac{2}{5}} = \mathbf{4{\cdot}43''}.$$

75. *Two reservoirs,* A *and* B, *whose water surfaces are* 20′ *and* 10′ O.D., *are connected by a* 12″ *pipe line* 2000′ *long.*



A third reservoir, C, *halfway between* A *and* B *is placed so that water passes through* C *on its way from* A *to* B. *The water level in* C *is to be* 12′ O.D., *and the flow is to remain unaltered. Find the diameter of pipes necessary from* A *to* C *and from* C *to* B.

$$h=\frac{K_0LQ^2}{d^5}=\frac{LK}{d^5}$$

$$10=\frac{2000K}{12^5} \quad 8=\frac{1000K}{{d_1}^5} \quad \therefore \frac{d_1}{12}=\left(\frac{5}{8}\right)^{\frac{1}{5}}=0{\cdot}91' \quad d_1=\mathbf{10{\cdot}9''}$$

$$10=\frac{2000K'}{12^5} \quad 2=\frac{1000K'}{{d_2}^5} \quad \therefore \frac{d_2}{12}=\left(\frac{5}{2}\right)^{\frac{1}{5}}=1{\cdot}2' \quad d_2=\mathbf{14{\cdot}4''}.$$

76. *The sprinkler installation in a mill building* 60′ *high is served from a reservoir through a* 3″ *pipe* 2 *miles long. The reservoir is* 150′ *above the level of the mill floor. If a minimum pressure of* 20 *lb. per sq. in. must be maintained at all parts of the building, find the maximum discharge for which the installation may be designed.* $f=0{\cdot}01$.

Max. loss of head permissible $=150-60-46{\cdot}2$ $h=43{\cdot}8'$

$$h=4\times0{\cdot}01\times\frac{2\times5280}{0{\cdot}25}\times\frac{v^2}{2g}$$

$$v^2=\frac{64{\cdot}4\times43{\cdot}8\times0{\cdot}25}{4\times0{\cdot}01\times2\times5280}=1{\cdot}67$$

$v=1{\cdot}29$ ft. per sec. $\quad Q=1{\cdot}29\times\frac{\pi}{4}\times\left(\frac{1}{4}\right)^2$ cusecs $=\mathbf{23{\cdot}7\ g.p.m.}$

77. *Estimate the H.P. required to pump* 300 *g.p.m. through a* 6″ *pipe one mile long to an open tank whose level is* 80′ *above the point of supply. What would be the H.P. if the pipe were* 4″ *in diameter?* $f=0{\cdot}01$.

$$Q=\frac{300}{60\times6{\cdot}25}\text{ cusecs}$$

$$v=\frac{Q}{\frac{\pi}{4}d^2}=\frac{300}{60\times6{\cdot}25}\times\frac{4\times4}{\pi}=4{\cdot}1\text{ f.p.s.}$$

Loss of head in pipe $=\frac{1}{100}\times5280\times8\times\frac{4{\cdot}1^2}{64{\cdot}4}=110'$

Head at pump $=80+110=190'$

Water H.P. developed by pump $=\frac{3000\times190}{33000}=\mathbf{17{\cdot}3}$

If the pipe had been 4″ diameter,

Loss of head $=110\times\left(\frac{6}{4}\right)^5=760$ Total head $=80+760=840$

Water horse-power developed by pump $=\mathbf{76{\cdot}4}$.

78. *Calculate the quantity of water which will flow per sec. through a long clean cast-iron pipe* 18″ *diam. The outlet end of the pipe is partially closed by a valve. Length of pipe,* 4000′ ; *area of valve opening,* 52 *sq. in. ; coefficient of discharge* 0·84 ; *head of water above valve,* 86′ ; *coefficient of friction,* 0·0065.

Area of 18″ pipe = 254·5 sq. ins.
Area of stream through valve = $52 \times 0{\cdot}84$.

$$\text{Ratio} = \frac{254{\cdot}5}{52 \times 0{\cdot}84} = 5{\cdot}84$$

$$86 = \frac{4 \times 0{\cdot}0065 \times 4000 V^2}{62{\cdot}4 \times 1{\cdot}5} + \frac{5{\cdot}84^2 V^2}{64{\cdot}4} = 1{\cdot}64 V^2$$

$V = 7{\cdot}3$ f.p.s. $\qquad Q = \mathbf{12{\cdot}8}$ **cusecs.**

By D'Arcy's formula, with the modified value of K.

$$86 = \frac{4000 V^2}{3750 \times 1{\cdot}5} + \frac{5{\cdot}84^2 V^2}{64{\cdot}4} = 1{\cdot}24 V^2$$

$V = 8{\cdot}35$ f.p.s. $\qquad Q = \mathbf{14{\cdot}7}$ **cusecs.**

The latter value is probably more reliable. The coefficient of friction given is evidently too high.

79. *Find the quantity of water which will flow through a clean cast-iron pipe having a bellmouth entry. Length,* 2,000′ ; *diam.,* 18″ ; *head,* 80′. (i) *By Barnes' formula,* viz. $V = 174{\cdot}1 m^{0{\cdot}769} i^{0{\cdot}529}$. (ii) *By D'Arcy's formula.*

(i) In the first place neglect the head required to impart kinetic energy to the water.

$$V = 174{\cdot}1 \times 0{\cdot}375^{0{\cdot}769} \times \left(\tfrac{80}{2000}\right)^{0{\cdot}529}$$
$$= 174{\cdot}1 \times 0{\cdot}47 \times 0{\cdot}182 = 14{\cdot}9 \text{ f.p.s.}$$

The above expression may be written :

$$V = 174{\cdot}1 \times m \times \frac{h^{0{\cdot}529}}{L^{0{\cdot}529}} \qquad h = \frac{L V^{1{\cdot}89}}{174{\cdot}1^{1{\cdot}89} m^{1{\cdot}454}}$$

$$H = \frac{L V^{1{\cdot}89}}{17400 \times m^{1{\cdot}454}} + \frac{V^2}{64{\cdot}4} \text{ allowing for kinetic energy}$$

$$80 = \frac{2000 V^{1{\cdot}89}}{17400 \times 0{\cdot}375^{1{\cdot}454}} + \frac{V^2}{64{\cdot}4}$$

Solve by trial and error

$V = 14{\cdot}7$ f.p.s. $\qquad Q = \mathbf{26}$ **cusecs.**

Alternative solution :

Having got V = 14·9 f.p.s. as a first approximation,

$$\text{Kinetic head} = \frac{14{\cdot}9^2}{64{\cdot}4} = 3{\cdot}45'$$

$$\text{Hence a closer value for } i = \frac{80 - 3{\cdot}45}{2000}$$

$$\text{and} \quad \frac{\delta V}{V} = 0{\cdot}529\,\frac{\delta i}{i} = 0{\cdot}529\,\frac{3{\cdot}45}{80} = 2{\cdot}3 \text{ per cent.}$$

$$\text{Hence corrected } V = 14{\cdot}9\left(\frac{100 - 2{\cdot}3}{100}\right) = 14{\cdot}6 \text{ f.p.s.}$$

(ii) K in D'Arcy's formula = 3750

$$V = \sqrt{\frac{H}{\frac{L}{KD} + \frac{1}{2g}}} = \sqrt{\frac{80}{\frac{2000}{5625} + \frac{1}{64{\cdot}4}}} = 14{\cdot}7 \text{ f.p.s.}$$

$$Q = \mathbf{26 \text{ cusecs.}}$$

80. *A* 6″ *pipe discharges* Q *cusecs. Find how many* 2″ *pipes would be required to pass the same quantity of water, the length, slope, and coefficient of friction being the same in all the pipes.*

$$Q = \frac{\pi D^2}{4}\sqrt{\frac{KDh}{L}} \qquad \text{(See p. 682, Vol. I.)}$$

Then $$\frac{\pi D^2}{4}\sqrt{\frac{KDh}{L}} = \frac{n\pi d^2}{4}\sqrt{\frac{Kdh}{L}}$$

$D^{\frac{5}{2}} = nd^{\frac{5}{2}}$ $\quad 6^{\frac{5}{2}} = n2^{\frac{5}{2}}$ $\quad n = 15{\cdot}6$, or, say, **16 pipes** 2″ diam.

81. *In a heating system a radiator consists of coils of* 1″ *diam. pipe, the total length of piping being* 72′. *The main pipe from which hot water is supplied to the radiator is* 3″ *diam., and the connection to the radiator is at a point* A. *From* A *the main pipe is carried forward for some distance and then returned so that the connection for discharge from the radiator is at a point* B *almost below* A. *The length between the points* A *and* B *measured along the main pipe is* 144′. *If* 0·15 *cusec flow along the main pipe before the point* A *is reached, determine the quantity which will circulate round the radiator per second.*

(Edinburgh University.)

The loss of head in the coils is the same as in the main.

$$h_1=\frac{L_1V_1^2}{K_1D_1}=\frac{L_3V_3^2}{K_3D_3} \qquad \frac{V_1^2}{V_3^2}=\frac{K_1D_1L_3}{K_3D_3L_1}$$

$$K_1=1200,\quad K_3=1800,\quad V_1=V_3\sqrt{\frac{1200\times1\times144}{1800\times3\times72}}=0\cdot67V_3$$

$$V_1A_1+V_3A_3=0\cdot15 \quad \text{also} \quad V_1\left(A_1+\frac{A_3}{0\cdot67}\right)=0\cdot15$$

$$V_1\left(0\cdot00545+\frac{0\cdot049}{0\cdot67}\right)=0\cdot079V_1=0\cdot15$$

$V_1=1\cdot9$ f.p.s. $\quad Q=1\cdot9\times0\cdot00545=$ **0·014 cusec.**

82. The Chézy formula for the flow of water, viz. $V=C\sqrt{m}\sqrt{i}$, where m is the hydraulic mean depth and i is the inclination $\frac{h}{L}$, can be derived from the D'Arcy expression thus:

$$h=\frac{LV^2}{KD} \qquad V=\sqrt{\frac{KDh}{L}}$$

$$V=\sqrt{K}\sqrt{4m}\sqrt{i} \qquad V=2\sqrt{\frac{3200}{1+\frac{1}{12D}}}\sqrt{m}\sqrt{i}$$

For pipes over 8″ diam. the following value for K better fits experiments:

$$V=2\sqrt{\frac{5000}{1+\frac{1}{2D}}}\sqrt{m}\sqrt{i} \qquad V=C\sqrt{m}\sqrt{i}$$

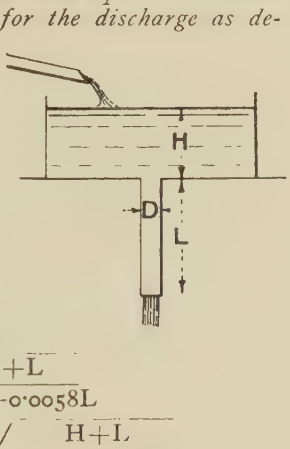
Diam. ins.	C	Diam. ins.	C	Diam. ins.	C
½ . .	65	4 . .	101	12 . .	116
1 . .	80	6 . .	103	18 . .	123
2 . .	92	8 . .	107	24 . .	126
3 . .	98	10 . .	112	36 . .	129

83. *The loss of head due to friction in a pipe is measured by an inverted oil gauge with water in contact with the oil in both limbs. Find f in the expression* $h=\frac{4fLV^2}{2gD}$. *Diam. of pipe,* 8″; *length of pipe,* 25′; *discharge,* 1·2 *cusecs; specific gravity of oil,* 0·88; *difference in level of oil in gauge,* 18″.

Equivalent loss of head h in water $=\frac{18}{12}(1-0{\cdot}88)=0{\cdot}18'$

$$Q=AV \quad V=\frac{1{\cdot}2\times144}{50{\cdot}26}=3{\cdot}44 \text{ f.p.s.} \quad \frac{2gDh}{4LV^2}=f=\mathbf{0{\cdot}0065.}$$

84. *A vertical pipe of diameter* D *is screwed flush into the bottom of a tank in which the head is kept at a constant height* H. *Obtain an expression for the discharge as depending upon* H, D, *and* L. *Find the discharge when* D=0·1′, H=9′, *and* L=1′, 3′, 6′, 100′, 1,000′, 10,000′. K=1720.



Total head=loss of head at entry + kinetic energy + friction head.

$$H+L=\frac{0{\cdot}35V^2}{2g}+\frac{V^2}{2g}+\frac{LV^2}{KD}$$

$$=V^2\left(\frac{1{\cdot}35}{64{\cdot}4}+\frac{L}{1720\times0{\cdot}1}\right)$$

$$V=\sqrt{\frac{H+L}{0{\cdot}021+0{\cdot}0058L}}$$

$$Q=0{\cdot}785D^2\sqrt{\frac{H+L}{0{\cdot}021+0{\cdot}0058L}}$$

When L=	1′	Q=**0·15** cusec
=	3′	=**0·14** ,,
=	6′	=**0·13** ,,
=	100′	=**0·11** ,,
=	1,000′	=**0·10** ,,
=	10,000′	=**0·10** ,,

85. *When a screw-down valve on the outlet end of a long pipe is opened the first turn of the spindle allows a much larger percentage of the full-way quantity of water to pass than does any subsequent turn of the spindle. In a given instance it takes* 10 *turns to open fully a valve and each turn increases the area of the stream by an equal amount. Length of pipe,* 10,000′. *Diameter,* 2′. *Head of water above valve,* 100′. K *in D'Arcy's formula*=3,200. *Find the quantity flowing when the valve is open one turn and when fully open.*

$$H=\frac{LV^2}{KD}+\frac{V^2}{2g}$$

Full open: $100=\left(\frac{10000}{3200\times 2}+\frac{1}{64}\right)V^2$

$V=8$ f.p.s. nrly. $\quad Q=8\times 3\cdot 14=$ **25 cusecs**

One turn open: $100=\left(\frac{10000}{3200\times 2}+\frac{100}{64}\right)V^2$

$V=5\cdot 7$ f.p.s. $\quad Q=5\cdot 7\times 3\cdot 14=$ **18 cusecs.**

Ratio $\frac{5\cdot 7}{8}=0\cdot 71$, which will vary slightly with the coefficient of friction assumed.

86. *Water issues vertically from a bend on the end of a pipe* 1·5″ *diam. Head above outlet,* 30′. *Height of jet above outlet,* 6′. *Find the height of the jet when it issues from a* $\frac{1}{2}$″ *nozzle on the end of the pipe.*

No nozzle:— $V=\sqrt{2gH}=8\cdot 02\sqrt{6}=19\cdot 6$ ft./sec.

Friction in pipe $=30-6=24$ ft.

With nozzle $V_1=$ Vely. in pipe. $V_n=$ Vely. thro. nozzle $=9V_1$

$$\text{Friction}=24\left(\frac{V_1}{19\cdot 6}\right)^2=0\cdot 062V_1^2$$

$$30=0\cdot 062V_1^2+\frac{(9V_1)^2}{2g}=1\cdot 32V_1^2 \qquad V_1^2=\frac{30}{1\cdot 32}$$

$$H=\frac{(9V_1)^2}{2g}=\frac{9^2\times 30}{64\cdot 4\times 1\cdot 32}=\mathbf{28\cdot 7'}.$$

87. *A pipe is provided with a vertical nozzle at the outlet end, the resistance of which is negligible. Find the ratio of the height of jet to head of water. Length,* 800′; *static head,* 150′; *diameter of pipe,* 4″; *diameter of nozzle,* $1\frac{1}{2}$″; *ratio of areas,* 7·15. *Coefficient of friction* f=0·008.

$$H=\frac{4fLV^2}{W_wD}+\frac{V_1^2}{2g} \qquad W_w=62\cdot 4 \text{ lb.}$$

Some writers use the term "coefficient of friction" in a different sense, and write $2g$ instead of W_w.

$$150=\frac{4\times 0\cdot 008\times 800\times V_1^2}{62\cdot 4\times 0\cdot 33\times 7\cdot 11^2}+\frac{V_1^2}{64\cdot 4}$$

$$150=\frac{V_1^2}{40\cdot 7}+\frac{V_1^2}{64\cdot 4}=0\cdot 04V_1^2 \qquad V_1^2=3740$$

$$h=\frac{3740}{2g}=58' \qquad \text{Ratio}=\frac{58}{150}=\mathbf{0\cdot 39}.$$

88. *Find the loss of head due to friction in a tapered pipe* 1 *mile in length, the diameter at one end being* 1·1′ *and* 2′ *at the other end. The velocity in the small end is* 10 *f.p.s. D'Arcy's* K=4,000 *throughout the whole length.*

Let the velocity at the small end be V and the diam. D; let the velocity at the large end be V_1 and the diam. D_1; let the velocity at an intermediate section distant l from the small end be v and the diam. d.

Let the length of the pipe be L.

Then $$l=L\frac{d-D}{D_1-D} \qquad \delta l=\frac{L}{D_1-D}\delta d$$

$$vd^2=VD^2=V_1D_1^2 \qquad v=\frac{VD^2}{d^2}$$

h=loss of head due to friction $$\delta h=\frac{\delta l v^2}{Kd}$$

$$K\int dh=\int_D^{D_1}\frac{LV^2D^4}{D_1-D}d^{-5}dd$$

$$Kh=\frac{LV^2D^4}{D_1-D}\left(\frac{D_1^{-4}-D^{-4}}{-4}\right)=\frac{LV^2D^4}{4(D_1-D)}\left(\frac{1}{D^4}-\frac{1}{D_1^4}\right)$$

$$h=\frac{LV^2D^4}{4K(D_1-D)}\left(\frac{1}{D^4}-\frac{1}{D_1^4}\right)$$

$$h=\frac{5280\times100\times1\cdot46}{4\times4000\times0\cdot9}\left(\frac{1}{1\cdot46}-\frac{1}{16}\right)=\mathbf{33\cdot4'}.$$

Taking Thrupp's expression for a new cast-iron parallel pipe, viz.

$$h=\frac{LV^{1\cdot85}}{3200D^{1\cdot24}}$$

$$\delta h=\frac{L}{D_1-D}\delta d\times\frac{V^{1\cdot85}D^{3\cdot7}}{d^{3\cdot7}}\times\frac{1}{3200d^{1\cdot24}}$$

$$h=\frac{LV^{1\cdot85}D^{3\cdot7}}{3200(D_1-D)}\int_D^{D_1}d^{-4\cdot94}dd$$

$$h=\frac{5280\times10^{1\cdot85}\times1\cdot1^{3\cdot7}}{3200(0\cdot9)}\times\frac{1}{3\cdot94}\left(\frac{1}{1\cdot1^{3\cdot94}}-\frac{1}{2^{3\cdot94}}\right)=\mathbf{29'}.$$

The latter value is probably nearer the truth than the former, which assumes a constant value for K although the diameter varies.

89. *Pipe,* 200′ *in length. Diameter,* 0·6″=$\frac{1}{20}$′. *Find cusecs for heads of* $\frac{1}{2}$″, 1″, 6″, 12″. *Take* C=83,000 *in*

continuous flow formula and $DV_c=\frac{1}{45}$, *where* V_c=*critical velocity.*

$$V_c=\tfrac{20}{45}=0{\cdot}44 \text{ f.p.s.}$$

If the flow be continuous

$$h=\frac{LV}{CD^2} \qquad V=\frac{CD^2h}{L}$$

$$(\tfrac{1}{2}'')\ V=\frac{83000(\frac{1}{20})^2\frac{1}{24}}{200}=0{\cdot}043 \text{ f.p.s.}$$

$\therefore$ the flow is continuous because it is below the critical velocity.

$$Q=\frac{0{\cdot}043\times0{\cdot}785}{20^2}=\mathbf{0{\cdot}000084\ cusec}$$

(1″) V=0·086 f.p.s. Flow continuous

Q=0·00017 cusec.

(6″) If continuous V=0·516 f.p.s., but this is beyond the critical velocity, therefore the flow is sinuous then

$$V=\sqrt{\frac{KDh}{L}}=\sqrt{\frac{1200}{20\times2\times200}}=0{\cdot}387 \text{ f.p.s.}$$

Q=0·00076 cusec

(12″) V=0·545 f.p.s. **Q=0·0011 cusec.**

90. *Water flows from a tank through a flush-ended pipe* 6″ *diameter which suddenly enlarges to* 12″ *diam. Both pipes are* 100′ *long. The outlet of the* 12″ *pipe is* 20′ *below the surface of the water in the tank. Find the discharge and sketch the hydraulic gradient.*

Take $K_1=2000$ $K_2=2200$ D'Arcy's formula

Loss of head at entry $=\frac{0{\cdot}3V_1^2}{2g}$ (See p. 673, Vol. I.)

Loss of head at sudden enlargement

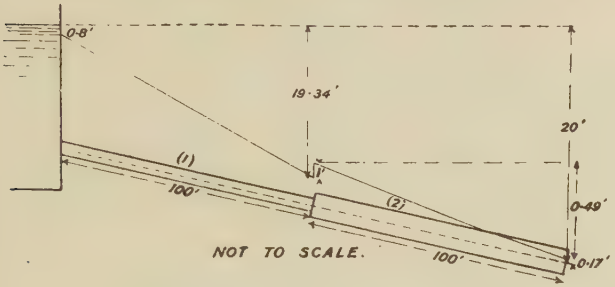
$$=\frac{(V_1-V_2)^2}{2g}=\frac{\left(V_1-\frac{V_1}{4}\right)^2}{2g}=\frac{9}{16}\cdot\frac{V_1^2}{2g}$$

Gain of pressure head due to enlargement $\Big\}=\frac{p_2-p_1}{w}=\frac{V_1^2-V_2^2}{2g}$

$$\frac{p_2-p_1}{w}=\frac{V_1^2-\left(\frac{V_1}{4}\right)^2}{2g}=\frac{15}{16}\cdot\frac{V_1^2}{2g}$$

Head = Loss of head at entry + Friction (1) + Sudden enlargement + Friction (2) + Kinetic energy at discharge

$$20 = 0{\cdot}3\frac{V_1^2}{2g} + \frac{100V_1^2}{2000\times 0{\cdot}5} + \frac{9V_1^2}{16\times 2g} + \frac{100V_2^2}{2200\times 1} + \frac{V_2^2}{2g}$$



$$20 = \frac{0{\cdot}3V_1^2}{2g} + \frac{V_1^2}{10} + \frac{0{\cdot}56V_1^2}{2g} + \frac{V_1^2}{35^2} + \frac{0{\cdot}062V_1^2}{2g} = 0{\cdot}1173V_1^2$$

$V_1 = \sqrt{170{\cdot}6} = 13{\cdot}1$ f.p.s. $\quad Q = 13{\cdot}1\times 0{\cdot}196 = \mathbf{2{\cdot}6\ cusecs}$

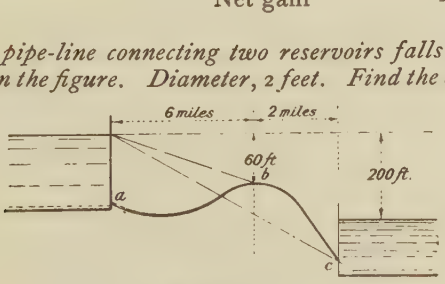
Loss of head at entry	$=170{\cdot}6\times 0{\cdot}0047$	$= 0{\cdot}80$ feet
,, by friction (1)	$=170{\cdot}6\times 0{\cdot}1$	$= 17{\cdot}06$,,
,, by sudden enlargement	$=170{\cdot}6\times 0{\cdot}0087$	$= 1{\cdot}48$,,
,, by friction (2)	$=170{\cdot}6\times 0{\cdot}0029$	$= 0{\cdot}49$,,
,, by kinetic energy at discharge	$=170{\cdot}6\times 0{\cdot}00096$	$= 0{\cdot}17$,,
Total loss		$= 20{\cdot}00$,,

Gain of pressure head at enlargement $= \frac{15\times 170{\cdot}6}{16\times 64{\cdot}4} = 2{\cdot}48$,,

Loss due to shock $= 1{\cdot}48$,,

Net gain $= 1{\cdot}00$,,

91. *A pipe-line connecting two reservoirs falls and rises as shown in the figure. Diameter, 2 feet. Find the discharge.*



K = 4000

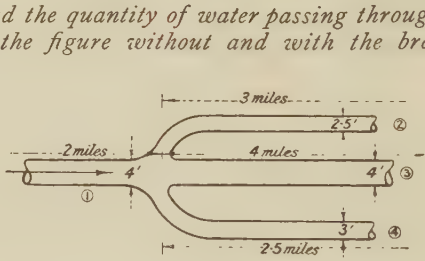
Since syphoning cannot occur the least hydraulic gradient is 60′ in 6 miles.

$$60=\frac{6\times 5280V^2}{4000\times 2} \qquad V=3{\cdot}89 \text{ f.p.s.}$$

$$Q=AV=3{\cdot}14\times 3{\cdot}89=\mathbf{12{\cdot}2 \text{ cusecs.}}$$

From *b* to *c* the pipe will not run full; an air inlet valve must be placed at *b* in order to prevent the water breaking up at this section.

92. *Find the quantity of water passing through the pipes shown in the figure without and with the branch pipes. Head, 50′.*



In D'Arcy's formula $K_1=K_3=4440$, $K_2=4170$, $K_4=4290$.

Without branches—

$$V=\sqrt{\frac{K_1D_1h}{L}}=\sqrt{\frac{4440\times 4\times 50}{6\times 5280}}=5{\cdot}3 \text{ f.p.s.}$$

$$Q=0{\cdot}785\times 4^2\times 5{\cdot}3=\mathbf{66{\cdot}6 \text{ cusecs}}$$

With branches—

$$D_1^2V_1=D_2^2V_2+D_3^2V_3+D_4^2V_4$$

Let $V_3=mV_2$, $V_4=pV_2$

$$\left(\frac{D_1^2}{D_2^2+mD_3^2+pD_4^2}\right)V_1=V_2=nV_1=0{\cdot}453V_1$$

$$V_3=V_2\sqrt{\frac{L_2K_3D_3}{L_3K_2D_2}}=mV_2=1{\cdot}13V_2=0{\cdot}512V_1$$

$$V_4=V_2\sqrt{\frac{L_2K_4D_4}{L_4K_2D_2}}=pV_2=1{\cdot}22V_2=0{\cdot}553V_1$$

$$V_1=\sqrt{\frac{50}{\frac{L_1}{K_1D_1}+\frac{L_2n^2}{K_2D_2}}}=7{\cdot}42 \text{ f.p.s.}$$

$$Q=0{\cdot}785\times 4^2\times 7{\cdot}42=\mathbf{93{\cdot}3 \text{ cusecs.}}$$

The quantity delivered by each branch can be arrived at thus—

(2)	$0{\cdot}785\times2{\cdot}5^2\times0{\cdot}453\times7{\cdot}42$	$=\mathbf{16{\cdot}5}$ cusecs
(3)	$0{\cdot}785\times4^2\times0{\cdot}512\times7{\cdot}42$	$=\mathbf{47{\cdot}7}$,,
(4)	$0{\cdot}785\times3^2\times0{\cdot}553\times7{\cdot}42$	$=\mathbf{29{\cdot}0}$,,
	Total	$\mathbf{93{\cdot}2}$,,

The head corresponding to the kinetic energy of the water on leaving is neglected in the above treatment, since it is very small compared with the friction loss.

93. *A pipe 4·2 miles in length and 3′ diam. conveys water from a reservoir to one 150′ below. A branch pipe is laid from the lower reservoir to join the above-mentioned main at a point 2·8 miles distant, the diameter being 2′ 6″. Find the increase in the flow due to the branch pipe, neglecting the kinetic energy at discharge in both cases.*

Without branch pipe $K_1=4300$

$$V_1=\sqrt{\frac{K_1D_1h}{L_1}}=\sqrt{\frac{4300\times3\times150}{4{\cdot}2\times5280}}=9{\cdot}34 \text{ f.p.s.}$$

$Q=7{\cdot}06\times9{\cdot}34=65{\cdot}9$ cusecs.

With branch pipe

$$\frac{L_2V_2^2}{K_2D_2}=\frac{L_3V_3^2}{K_3D_3} \qquad K_2=4300 \qquad K_3=4170 \qquad L_2=L_3$$

$$\frac{V_2^2}{4300\times3}=\frac{V_3^2}{4170\times2{\cdot}5} \qquad V_3=0{\cdot}9V_2$$

$$Q=A_1V_1=A_2V_2+A_3V_3 \qquad 9V_1=9V_2+6{\cdot}25V_3$$

$$9V_1=9V_2+6{\cdot}25\times0{\cdot}9V_2 \qquad V_2=0{\cdot}616V_1$$

$$150=\frac{L_1V_1^2}{K_1D_1}+\frac{L_2V_2^2}{K_2D_2}=\frac{5280\times1{\cdot}4\times V_1^2}{4300\times3}+\frac{2{\cdot}8\times5280V_2^2}{4300\times3}$$

Substituting the value of V_2

$V_1=12{\cdot}2$ ft.-sec. $\qquad Q=7{\cdot}06\times12{\cdot}2=86$ cusecs

Increase due to branch=**20 cusecs.**

NOTE.—The head lost in friction in pipe 2 is equal to that lost in 3. Hence

$$H=h_1+h_2.$$

94. *A pipe 4 feet diameter is reduced to 3′ diam. (gradually) at a section 3000′ from the inlet, the 3′ pipe is 8000′ long. At the section where the reduction in diameter occurs a pipe 2′ diameter branches off, its length is 10,000′. Find the quantity of water passing. Head, 25′.* $K_1=4440$, $K_2=4280$ for 3-*ft. pipe*, $K_3=4000$ *for 2-ft pipe.*

$$\frac{L_2V_2^2}{K_2D_2}=\frac{L_3V_3^2}{K_3D_3}$$

$$V_3=\sqrt{\frac{L_2V_2^2K_3D_3}{L_3K_2D_2}}=V_2\sqrt{\frac{L_2K_3D_3}{L_3K_2D_2}}=mV_2$$

$$Q=A_1V_1=A_2V_2+A_3V_3$$

$$D_1^2V_1=D_2^2V_2+D_3^2V_3=V_2(D_2^2+mD_3^2)$$

$$V_2=\frac{D_1^2V_1}{D_2^2+mD_3^2}=nV_1$$

$$h=\frac{L_1V_1^2}{K_1D_1}+\frac{L_2V_2^2}{K_2D_2}=V_1^2\left(\frac{L_1}{K_1D_1}+\frac{L_2n^2}{K_2D_2}\right)$$

$$V_1=\sqrt{\frac{h}{\frac{L_1}{K_1D_1}+\frac{L_2n^2}{K_2D_2}}}$$

Substituting the numerical values

$$V_3=V_2\sqrt{\frac{8000\times4000\times2}{10000\times4280\times3}}=0{\cdot}71V_2,\qquad m=0{\cdot}71$$

$$V_2=\frac{4^2V_1}{3^2+0{\cdot}71\times2^2}=1{\cdot}35V_1,\qquad n=1{\cdot}35$$

$$V_1=\sqrt{\frac{25}{\frac{3000}{4440\times4}+\frac{8000\times1{\cdot}35^2}{4280\times3}}}=4{\cdot}37\text{ f.p.s.}$$

$$Q=0{\cdot}785\times4^2\times4{\cdot}37=\mathbf{55\ cusecs.}$$

The kinetic energy of the water on leaving is neglected in the above, since it is a very small proportion of the head lost in friction.

95. *Water flows down a steep slope channel in which both the roughness of the surface and the hydraulic mean depth vary. Find the velocity after it has fallen 40. Initial velocity, 8 f.p.s.* $\Theta=30°$.

Fall (feet).	Hy. mean depth (feet).	K in pipe formula.
0	3·00	2200
10	2·92	2100
20	2·81	1950
30	2·70	1860

(See p. 687, Vol. I.)

$$V_1=\sqrt{\frac{(u^2+2gH)K_cR}{K_cR+2gl}} \qquad K_c=4K \quad l=20'$$

(10 ft.) $V_1=\sqrt{\frac{(64+644)8800\times3}{8800\times3+1288}}=26$ f.p.s.

(20 ft.) $V_2=\sqrt{\frac{(675+644)8400\times2\cdot92}{8400\times2\cdot92+1288}}=35\cdot4$ f.p.s.

(30 ft.) $V_3=\sqrt{\frac{(1253+644)7800\times2\cdot81}{7800\times2\cdot81+1288}}=42\cdot3$ f.p.s.

(40 ft.) $V_4=\sqrt{\frac{(1792+644)7440\times2\cdot70}{7440\times2\cdot70+1288}}=$ **47·8 f.p.s.**

96. *Water flows down a new cast-iron open channel* 1 *mile in length of* V *section, apex angle* 60°. *Find the fall required to give a flow of* 4 *feet per second. Depth of water,* 3′.

Breadth of water surface $=2\times3\tan30°=3\cdot46'$

Area of water section $=\frac{3\times3\cdot46}{2}=5\cdot19$ sq. ft.

Hydraulic mean depth $=\frac{5\cdot19}{6\cdot92}=0\cdot75'$

Equivalent diameter $=3'$ $\qquad \frac{1}{K}=\frac{1}{5000}\left(1+\frac{1}{2\times3}\right)=\frac{1}{4280}$

$\frac{h}{L}=\frac{4^2}{4280\times3}=\frac{1}{804}$ or $\frac{5280}{804}=$ **6·6′ per mile**

$Q=4\times5\cdot19=20\cdot8$ cusecs.

97. *Water flowing along a horizontal open* 90° *Vee channel is* 8″ *deep at a certain section and* 7·8″ *at a section* 40′ *away, and* 7·58″ *at a section* 80′ *from the first-mentioned section. The quantity flowing,* Q=0·4 *cusec. Find the coefficient of friction on the two sections.*

Let D_1=depth of water at section 1,
A_1=area of waterway at section 1.

Then $A_1=D_1^2$ Wetted surface $S_1=2\sqrt{2}D_1$

Hydraulic mean depth $m_1=\frac{D_1}{2\sqrt{2}}$

Mean depth of water in section 1–2=7·9″=0·66′

Mean velocity$=\frac{0\cdot4}{0\cdot66^2}=0\cdot92$ f.p.s.

Loss of head in first section

$$=\frac{fLV^2}{2gm}=\frac{f\times40\times0\cdot92^2\times2\times1\cdot41}{64\cdot4\times0\cdot66}=\frac{0\cdot2}{12}$$

$f=$**0·0074** on first section

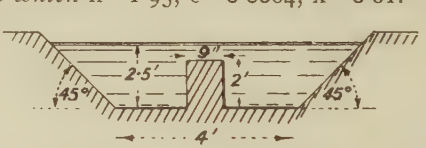
Mean depth of water in section 2–3=7·69″=0·64′

Mean velocity „ „ „ $=\frac{0\cdot4}{0\cdot64^2}=0\cdot97$ f.p.s.

Loss of head in second section $=\frac{f\times40\times0\cdot97^2\times2\times1\cdot41}{64\cdot4\times0\cdot64}=\frac{0\cdot22}{12}$

$f=$**0·0071** on second section.

98. *Find the necessary slope of a channel of the section shown to give a velocity of flow of* 4 *f.p.s. by Thrupp's formula, in which* n=1·95, c=0·0064, x=0·61.



Area of water section$=(4+2\cdot5)2\cdot5-\frac{2\times3}{4}=14\cdot75$ sq. ft.

Wetted perimeter=1·41×2·5×2+3·25+4+0·75=15·05 ft.

Hy. mean depth $R=\frac{14\cdot75}{15\cdot05}=0\cdot98$

$$V=\frac{R^x}{C\sqrt[n]{S}}=\frac{0\cdot98^{0\cdot61}}{0\cdot0064\times\sqrt[1\cdot95]{S}}$$

$$S=\left(\frac{0\cdot98^{0\cdot61}}{0\cdot0064\times4}\right)^{1\cdot95}=\mathbf{1239}=\frac{L}{h}.$$

99. *The wetted surface of a ship is* 15,000 *sq. feet. Find the skin resistance when travelling at* 8 *knots, also the horse-power required to overcome skin resistance. Coefficient of friction* 0·004.

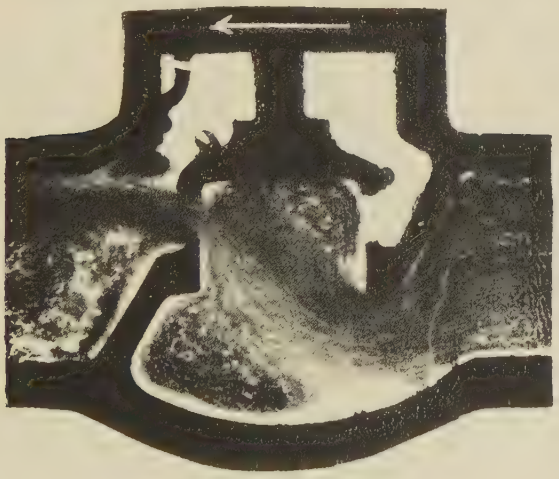
NOTE.—1 knot=1 nautical mile (6080 feet) per hour.

We have $R=sfV^2$ (see p. 679, Vol. I.)

$$R=\frac{15000\times 0\cdot 004(6080\times 8)^2}{3600^2}=10{,}960 \text{ lb.}$$

$$H.P.=\frac{RV}{33000\times 60}=\frac{10960\times 6080\times 8}{33000\times 60}=\mathbf{270.}$$

100. The illustration is a reproduction of a photograph taken by the author for Messrs. Hopkinson, Huddersfield, to show the manner in which water flows through an ordinary wheel valve. The water is nearly stationary above the valve and it eddies very violently below, with a consequent loss of pressure and scoring of the valve seat.



CHAPTER XIX

HYDRAULIC MOTORS

1. *Find the hydraulic efficiency, the horse-power, and the width of an overshot water-wheel. Fall* 30 *feet, quantity of water* 40 *cusecs, sluice opening* 3″ *deep. Peripheral velocity of wheel* 8 *f.p.s. The water leaves the buckets after passing through an angle of* 150°.

Vely. of water entering wheel = 16 f.p.s. for best efficiency

Velocity head at sluice $h = \frac{V^2}{2g} = 4'$

(In practice a smaller head than this would be adopted.)

Losses :	Half head at sluice	$= 2'$
	Shroud (about)	$= 1'$
	Water leaving before reaching tail race 0·07D, about	$= 1{\cdot}8'$
	Clearance	$= 0{\cdot}5'$
	Total	$5{\cdot}3$

Diameter of wheel $D = 30 - 4 - 0{\cdot}5 = 25{\cdot}5'$

Hydraulic efficiency $= \frac{30 - 5{\cdot}3}{30} =$ **82 per cent.**

Taking the mean depth of the water as 0·25 foot in the buckets.

Width of wheel $\frac{40}{16 \times 0{\cdot}25} =$ **10′**

H.P. $= 0{\cdot}1135 \times 40 \times 30 \times 0{\cdot}82 =$ **112.**

The horse-power available would be about 100. The above treatment is only a rough approximation but it fairly represents what would be obtained in practice.

2. *The radial vanes of an undershot water-wheel are closely spaced and are acted upon by water moving with a velocity of 45 f.p.s. If the quantity of water which reaches the wheel is 2 cusecs, and if the vanes are moving with a velocity of 22·5 f.p.s., find: (a) The pressure of water on the vanes; (b) the theoretical efficiency; (c) the horse-power developed.*

Weight of water delivered per second $=2\times62\cdot4=124\cdot8$ lb.
Striking velocity $45-22\cdot5=22\cdot5$ f.p.s.

(*a*) Pressure on vanes $\frac{124\cdot8\times22\cdot5}{32\cdot2}=$ **87·2 lb.**

Kinetic energy of water $=\frac{124\cdot8\times45^2}{2\times32\cdot2}=3{,}920$ ft.-lb.

Work done on vanes per second $=87\cdot2\times22\cdot5=1{,}960$ ft.-lb.

(*b*) Efficiency $=\frac{1960}{3920}=$ **50 per cent.**

(*c*) Horse-power $=\frac{1960}{550}=$ **3·56.**

3. *High-pressure water is supplied at 1/- per thousand gallons at a pressure of 700 lb. per sq. inch. If the efficiency of the hydraulic engine is 82 per cent., what is the cost per useful horse-power hour?*

Let 1,000 gals. be supplied per hour.

H.P. $=0\cdot1135QH\eta$ (see p. 703, Vol. I.)

$=0\cdot1135\times\frac{1000}{6\cdot24\times3600}\times700\times2\cdot31\times0\cdot82=6\cdot7$ for 1/-

or $\frac{12}{6\cdot7}=$ **1·8 pence** per H.P. hour.

4. Pressure in a Pipe due to Gradual Stoppage of Water

The question of sudden stoppage is treated on p. 699, Vol. I. Many theories have been advanced for gradual stoppage, but they differ widely and some are certainly very seriously in error. Readers should refer to Gibson's "Hydraulics and its Applications."

The few experimental results which are on record are very erratic, probably on account of the difficulty of determining the exact manner in which the stoppage takes place.

The change of momentum theory is often met with, viz.

Change of momentum of the flowing water = Mean retarding force due to stoppage × Time of closing valve

or $$\frac{w\mathrm{LV}}{32{\cdot}2} = p\mathrm{T}$$

$$p_{(\text{mean.})} = \frac{w\mathrm{LV}}{32{\cdot}2\mathrm{T}} = \frac{\mathrm{LV}}{74\mathrm{T}} \quad . \quad . \quad . \quad . \quad \text{(i)}$$

But there is much experimental evidence to show that the maximum pressure is greater than the mean; it is often assumed that it is twice, if so we have:

$$p_{\max.} = \frac{\mathrm{LV}}{37\mathrm{T}} \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

But this expression is obviously wrong for small values of T. If T be zero, *i.e.* sudden closing, $p_{\max.}$ becomes infinitely great, whereas we know (see p. 702, Vol. I.) that it cannot exceed 63·5V. The reason why the momentum formula gives erroneous results is because the elasticity of the water and pipe is neglected.

When a valve is suddenly closed a pressure wave traverses the pipe towards the valve, and is reflected back to the open end.

The time taken by the wave in making the return trip is $\mathrm{T}_r = \frac{2\mathrm{L}}{4720}$ (see p. 701, Vol. I.), hence if the time occupied in closing a valve is less than T_r the pressure is the same as for sudden closing, and if greater than T_r the pressure is assumed to be given by (ii).

In another theory it is assumed that the pressure set up by gradual closing is less than with sudden closing in the ratio:

$$\frac{\text{Pressure with gradual closing}}{\text{Pressure with sudden closing}} = \frac{\mathrm{P}_g}{p} = \frac{\mathrm{T}r}{\text{Time of closing } (\mathrm{T}_g)} \quad \text{(iii)}$$

which gives the same results as (ii), or

$$\mathrm{P}_g = 63{\cdot}5\mathrm{V}\frac{\mathrm{T}_r}{\mathrm{T}_g} \quad \text{(iv) neglecting the elasticity of the pipe.}$$

Owing to the uncertainty of the reasoning, one is not justified in going to the refinement of allowing for the elasticity of the pipe.

EXAMPLE.—*Pipe* 10,000 *feet in length. Initial velocity*, V = 6 *ft./sec. Static pressure*, 500 *lb. sq. in. Find the*

instantaneous pressure when a valve is closed : suddenly, and in 1, 2, 4, 8, 20 *seconds. Neglecting the elasticity of the pipe in all cases.*

Sudden closing $P=63{\cdot}5\times6=380$ lb. sq. in.
Total pressure **880 lb. sq. in.**

$$T_t=\frac{20000}{4720}=4{\cdot}2 \text{ seconds}$$

P_g=**880 lb. sq. in.** for 1, 2, 4 seconds

$$P_g=380\times\frac{4{\cdot}2}{8}=200 \text{ lb. sq. in. for 8 seconds}$$

or **700 lb. sq. in.** total

$$P_g=380\times\frac{4{\cdot}2}{20}=80 \text{ lb. sq. in. for 20 secs.}$$

or **580 lb. sq. in.** total.

By the momentum method (ii) we get

$$p_{max.}=\frac{10000\times6}{37\times8}=200 \text{ lb. sq. in. for 8 secs.}$$

$$p_{max.}=\frac{10000\times6}{37\times20}=80 \text{ lb. sq. in. for 20 secs.}$$

Although the two methods give the same results it cannot be pretended that either of the gradual closing theories is based on sound reasoning. As far as the author is aware, no reliable data upon the question of gradual closing has been published ; the results of the few experiments on record are marred by leaky joints, the pipes not being of uniform diameter throughout and straight, valves not being "full way," lack of data as to the exact manner in which the valves were closed, and the coefficients of discharge of the valves at various openings. In one instance a steam engine indicator was used for recording the pressure ; but it rendered the experiment valueless because the movement of the piston was equivalent to an air vessel on the pipe.

The question is well worthy of the attention of research workers ; but those who undertake it should recognize that it is by no means an easy problem, and will involve a considerable amount of expense to carry through.

5. *Find the efficiency of transmission of power through a pipe-line to a hydro-electric power station if a maximum overload of* 25 *per cent. is to be allowed for.*

If H is the gross head, and h the head lost in the pipe-line, the horse-power supplied at the station :

$$P=C_1(H-h)v=C_2(H-h)h^{\frac{1}{2}}$$

If η is the efficiency of transmission :

$$P=C_2H^{\frac{3}{2}}\eta\sqrt{1-\eta}$$

If a maximum overload of 25 per cent. is permitted :

$$\tfrac{5}{4}P=C_2H^{\frac{3}{2}}\tfrac{2}{3}\times\sqrt{1-\tfrac{2}{3}}$$

$$\tfrac{4}{5}=\frac{\eta\sqrt{1-\eta}}{\frac{2}{3}\sqrt{\frac{1}{3}}} \qquad \eta\sqrt{1-\eta}=0{\cdot}308 \qquad \eta=\mathbf{0{\cdot}876.}$$

If the efficiency of transmission at rated load is less than this the power station cannot develop 25 per cent. overload with the pipe-line considered.

6. *Water is delivered by a hydraulic pipe-line* 200 *yards long at a pressure of* 1,000 *lb. per sq. in. If the H.P. is* 100 *and the pipe is* 4 *ins. in diameter, and* $f=0{\cdot}0075$, *find what H.P. is lost by frictional resistance in the pipe.*

$$100\times33{,}000=1000\times144\times60Q$$

$$Q=0{\cdot}382 \text{ cusecs.}$$

$$\text{Area of pipe}=0{\cdot}087 \text{ sq. ft.}$$

$$\text{Vely. of flow}=\frac{0{\cdot}382}{0{\cdot}087}=4{\cdot}38 \text{ ft.-sec.}$$

$$h=\frac{4fLV^2}{2gD}=\frac{4\times0{\cdot}0075\times600\times4{\cdot}38^2}{64{\cdot}4\times0{\cdot}33}=16{\cdot}3'$$

$$\text{H.P. lost in friction}=\frac{16{\cdot}3\times0{\cdot}382\times62{\cdot}4}{550}=\mathbf{0{\cdot}71.}$$

7. *A turbine of* 70 *per cent. efficiency working under a head of* 30 *feet and giving* 50 *H.P., draws water from a screening chamber separated from the main source of supply by* 3 *gates each* 2′ *wide and having* $\frac{3}{8}$″ *sq. mesh with* $\frac{1}{8}$″ *wire. If the water-level outside is* 2′ 6″ *above the bottom of the screens, estimate the fall in level through them when they are* (i) *free*, (ii) *half choked with weeds.*

$$\text{Water supplied per sec.}=\frac{100\times50\times550}{70\times30}=21 \text{ cusecs.}$$

Water passing each screen=7 cusecs.
Take K_d for screens as 0·8.

Available area $=\left(\frac{0\cdot25}{0\cdot375}\right)^2=\frac{4}{9}$ of gross area.

Effective coefficient of discharge $=\frac{0\cdot8\times4}{9}=0\cdot355$.

Let level inside screen chamber be H feet below the outside.

Discharge through one screen

$$=\int_0^H 0\cdot355\times2\sqrt{2ghdh}+0\cdot355\times2\sqrt{2g}(2\cdot5-H)H^{\frac{1}{2}}$$

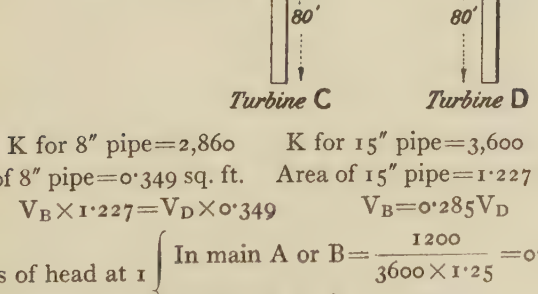
$$7=5\cdot7[\tfrac{2}{5}H^{\frac{5}{2}}+\tfrac{5}{2}H^{\frac{1}{2}}-H^{\frac{3}{2}}]$$

$$1\cdot23=\tfrac{5}{2}H^{\frac{1}{2}}-\tfrac{1}{3}H^{\frac{3}{2}} \qquad H=\mathbf{0\cdot26'} \text{ (by trial and error)}$$

When the screens are half choked with weeds the corresponding equation is :

$$2\cdot46=\tfrac{5}{2}H^{\frac{1}{2}}-\tfrac{1}{3}H^{\frac{3}{2}} \qquad H=\mathbf{1\cdot5'}.$$

8. *Two turbines* C *and* D *are supplied with water from a reservoir* 200′ *above, but owing to friction in the supply pipes, the head of water at each turbine is only* 100′. *Find the horse-power supplied to each turbine.*



K for 8″ pipe $=2{,}860$ K for 15″ pipe $=3{,}600$

Area of 8″ pipe $=0\cdot349$ sq. ft. Area of 15″ pipe $=1\cdot227$ sq. ft.

$$V_B\times1\cdot227=V_D\times0\cdot349 \qquad V_B=0\cdot285V_D$$

Loss of head at 1 foot per second:

In main A or B $=\frac{1200}{3600\times1\cdot25}=0\cdot267'$

In branch C or D $=\frac{80}{2860\times0\cdot67}=0\cdot042'$

$$H_1 - 100 = 0{\cdot}042V_C^2 = 0{\cdot}042V_D^2 + 0{\cdot}267V_B^2 = 0{\cdot}064V_D^2$$

Substituting the value of V_B in terms of V_D

$$V_C = 1{\cdot}23V_D$$

$$V_A = 0{\cdot}285V_C + 0{\cdot}285V_D = 0{\cdot}285V_C + \frac{0{\cdot}285}{1{\cdot}231}V_C = 0{\cdot}517V_C$$

$$100 = 0{\cdot}042V_C^2 + 0{\cdot}267V_A^2 = \frac{0{\cdot}042V_A^2}{0{\cdot}517^2} + 0{\cdot}267V_A^2 = 0{\cdot}424V_A^2$$

$V_A = 15{\cdot}36$ f.p.s. $\quad Q_A = 18{\cdot}8$ cusecs.

$V_C = 29{\cdot}7$,, $\quad Q_C = 10{\cdot}4$,,

$V_D = 24{\cdot}1$,, $\quad Q_D = 8{\cdot}4$,,

$$\text{H.P.}_{(C)} = \frac{10{\cdot}4 \times 62{\cdot}4 \times 100}{550} = \mathbf{118}$$

$$\text{H.P.}_{(D)} = \frac{8{\cdot}4 \times 62{\cdot}4 \times 100}{550} = \mathbf{95{\cdot}3}.$$

9. *Find the maximum horse-power which can be transmitted through a pipe 2 miles in length and 30″ diam., the outlet end being 800′ below the water supply surface. If two similar nozzles were fitted to the outlet end of the pipe, what should be their diameter for the maximum kinetic energy of the jets?*

The maximum power is transmitted when one-third of the head is expended in friction (see p. 704, Vol. I.).

Hence $\frac{800 \times 2}{3} = 533$ ft. is available for power transmission.

Then using D'Arcy's expression for the loss of head due to friction in which K=4,170

$$h = \frac{800}{4} = \frac{2 \times 5280V^2}{4170 \times 2{\cdot}5} \qquad V = 16{\cdot}2 \text{ f.p.s.}$$

$$Q = 0{\cdot}785 \times 2{\cdot}5^2 \times 16{\cdot}2 = 79 \text{ cusecs}$$

$$\text{H.P.} = 0{\cdot}1135 \times 79 \times 533 = \mathbf{4{,}800} \text{ (p. 703, Vol. I.)}$$

(See p. 719, Vol. I.)

$$n = \frac{\text{nozzle area}}{\text{pipe area}} = \sqrt{\frac{K}{4g}}\sqrt{\frac{D}{L}}$$

$$n = \sqrt{\frac{4170}{4 \times 32{\cdot}2}}\sqrt{\frac{2{\cdot}5}{10560}} = \frac{1}{11{\cdot}4}$$

Area of pipe=707 sq. ins.

Area of each nozzle $= \frac{707}{2 \times 11 \cdot 4} = 31$ sq. ins.

Diam. ,, ,, $= \mathbf{6 \cdot 29''}$.

10. *Find the maximum horse-power which can be transmitted through a 6" main 2 miles long, in which there are 3 bends (4 diameters), 2 elbows (40 diams.). Head, 300'. Use Thrupp's formula.*

$$V = \frac{R^x}{C \sqrt[n]{S}} = \frac{R^{0 \cdot 67}}{0 \cdot 004 \sqrt[1 \cdot 75]{S}}$$

3 bends $= 3(4 \times \frac{1}{2}) = 6'$. 2 elbows $= 2(40 \times \frac{1}{2}) = 40'$

Equivalent length $= 10{,}560 + 46 = 10{,}606'$

Power transmitted $P = AV(H - KV^n) \times$ constant

$$\frac{dP}{dV} = AH - 2 \cdot 75KAV^{1 \cdot 75} = 0 \text{ for maximum power}$$

$$H = 2 \cdot 75KV^{1 \cdot 75}$$

$$\frac{H}{2 \cdot 75} = KV^{1 \cdot 75} = h, \textit{ i.e.} \text{ loss of head due to friction}$$

$$h = \frac{300}{2 \cdot 75} = 109 \text{ feet for max. power}$$

$$V = \frac{0 \cdot 125^{0 \cdot 67}}{0 \cdot 004 \sqrt[1 \cdot 75]{\frac{10606}{109}}} = 4 \cdot 54 \text{ f.p.s.}$$

$$\text{H.P.} = \frac{0 \cdot 785 \times 0 \cdot 5^2 \times 4 \cdot 54 \times 62 \cdot 4(300 - 109)}{550} = \mathbf{19 \cdot 3}.$$

From the above reasoning we find that if the friction in the pipe varies as V^n the maximum horse-power is transmitted when $\frac{1}{n+1}$ of the total head is absorbed in friction.

11. *Water issues from a nozzle on the end of a pipe of length* L *feet. The velocity in the pipe is* V *f.p.s. The jet impinges normally on a flat plate. Show that the pressure* P *is a maximum when one-half the head* H *is absorbed in pipe friction. Find the ratio of nozzle to pipe area* n *to give the maximum pressure.*

$$H = \frac{V^2}{2gn^2} + \frac{LV^2}{KD}$$

$$V = n\sqrt{2g\left(H - \frac{LV^2}{KD}\right)} \qquad n = \frac{V}{\sqrt{2g\left(H - \frac{LV^2}{KD}\right)}}$$

$$P = \frac{WV}{n} = \frac{62{\cdot}4AV^2}{ng}$$

$$P = \frac{62{\cdot}4AV}{g}\sqrt{2g\left(H - \frac{LV^2}{KD}\right)} = \frac{62{\cdot}4A\sqrt{2g}}{g}\sqrt{HV^2 - \frac{LV^4}{KD}}$$

This is a maximum when the quantity under the root is a maximum.

$$\frac{dP}{dV} \propto 2HV - \frac{4LV^3}{KD} = 2HV - 4V\left(\frac{LV^2}{KD}\right)$$

P is a maximum when $2H = 4h$ or $h = \frac{H}{2}$

We have shown in Q. 13 that $h = \frac{H}{3}$ for maximum kinetic energy.

We also have $H - h = \frac{V^2}{2gn^2} = 2h - h = h$

$$\frac{V^2}{2gn^2} = \frac{LV^2}{KD} \qquad n = \sqrt{\frac{KD}{2gL}}$$ for maximum momentum

and $n = \sqrt{\frac{KD}{4gL}}$ for maximum kinetic energy.

12. *Find the greatest pressure which can be obtained from a jet of water issuing from a nozzle on the end of a 3″ pipe 250′ in length. The level of the supply water is 100′ above the nozzle. Loss of head due to friction*

$$h = \frac{0{\cdot}16LV^2}{2g} = \frac{mV^2}{2g}$$

Let $n = \frac{\text{Nozzle area}}{\text{Pipe area}}$ $\qquad n = \sqrt{\frac{KD}{2gL}}$. (See Example 11)

We also have $\frac{0{\cdot}16LV^2}{2g} = \frac{LV^2}{KD}$

$$K=\frac{2g}{0{\cdot}16D}=\frac{64{\cdot}4\times4}{0{\cdot}16}=1{,}610$$

$$n=\sqrt{\frac{1610}{4\times64{\cdot}4\times250}}=\sqrt{\frac{1}{40}}=\frac{1}{6{\cdot}33}$$

$$\frac{mV^2}{2g}=\frac{0{\cdot}16\times250V^2}{2g}=50$$

$$V^2=\frac{64{\cdot}4\times50}{0{\cdot}16\times250}=80{\cdot}5 \qquad V=8{\cdot}98 \text{ f.p.s. in pipe.}$$

$$P=\frac{WV}{ng}=\frac{62{\cdot}4AV^2}{ng}$$

$$P=\frac{62{\cdot}4\times7{\cdot}07\times80{\cdot}5\times6{\cdot}33}{32{\cdot}2\times144}=\mathbf{48{\cdot}5\ lb.}$$

13. *Find an expression for the maximum kinetic energy of a jet of water issuing from a nozzle on the end of a long pipe-line.*

Let the velocity in the main be V ft./sec.

„ „ through the nozzle be V_1. Let $n=\frac{V}{V_1}$.

The maximum power is transmitted when $\frac{1}{3}$ head is absorbed in pipe friction (see p. 704, Vol. I.).

$$V_1^2=2g\times\tfrac{1}{3}H \qquad V_1=\sqrt{\frac{4gH}{3}}$$

$$h=\frac{H}{3}=\frac{LV^2}{KD} \qquad V=\sqrt{\frac{KDH}{3L}} \qquad n=\sqrt{\frac{KD}{4gL}}$$

Kinetic energy of jet $=\frac{WV_1^2}{2g}=\frac{62{\cdot}4AV^3}{2gn^2}$

Max. kinetic energy $=18{\cdot}8\left(\frac{H^{\frac{3}{2}}K^{\frac{1}{2}}D^{\frac{5}{2}}}{L^{\frac{1}{2}}}\right)$

Where K is the constant in D'Arcy's formula.
H the head above the nozzle in feet.
D the diameter of the nozzle in feet.
L the length of the pipe in feet.

14. *The nozzle of a Pelton wheel is fitted with a conical spear rod for varying the area of the jet. The diameter of the pipe is* 18". *Length of pipe,* 3000'; *head of water above*

nozzle, 2000'. *Find the net area of the jet to give the maximum power, also its amount.*

The ratio of the area of the jet to the area of the pipe for maximum power is :

$$n=\sqrt{\frac{KD}{4gL}}=\sqrt{\frac{3750\times 1\cdot 5}{4\times 32\cdot 2\times 3000}}=\sqrt{\frac{1}{68\cdot 7}}=\frac{1}{8\cdot 29}$$

Area of pipe=254·5 sq. ins.

Area of jet (annular)= $\frac{254\cdot 5}{8\cdot 29}$ =**30·7 sq. ins.**

Max. kinetic energy (see Example 13)= $\frac{18\cdot 8H^{\frac{3}{2}}K^{\frac{1}{2}}D^{\frac{5}{2}}}{L^{\frac{1}{2}}}$

=5,150,000 foot-lb. per sec.

$$\text{H.P.}=\frac{5,150,000}{550}=\mathbf{9360.}$$

15. *In the last question the ratio of the length of pipe to the head is rarely so small as* 1·5. *Take the length as* 30,000' *and keep all the other data the same.*

$$n=\sqrt{\frac{1}{687}}=\frac{1}{26\cdot 2}$$

Area of jet= $\frac{254\cdot 5}{26\cdot 2}$ =**9·7 sq. ins.**

Max. kinetic energy= $\frac{5,150,000\times 54\cdot 8}{173\cdot 2}$

=1,630,000 foot-lb. per sec.

$$\text{H.P.}=\frac{1,630,000}{550}=\mathbf{2,960.}$$

For good governing the area of the jet must never exceed the values given when the spear rod is withdrawn to its full extent.

It is of interest to note that the max. horse-power varies inversely as the sq. root of the length of main when all the other quantities are kept the same.

$$\text{Thus } \frac{9360\sqrt{3000}}{\sqrt{30000}}=\mathbf{2,960.}$$

16. *A pipe of length* L *and diameter* D *discharges water through a nozzle of diameter* d ; *the total static head causing*

flow is H *and the loss of head due to friction is proportional to* $V^{1\cdot8}$, *where* V *is the velocity of the water in the pipe. Find the value of the ratio* $\frac{D}{d}$ *which corresponds to maximum power delivered by the jet.*

$$\text{Useful head} = H - \frac{LV^{1\cdot8}}{KD}$$

$$\text{Useful power P is proportional to } V\left(H - \frac{LV^{1\cdot8}}{KD}\right)$$

$$\frac{dP}{dV} = H - \frac{2\cdot8LV^{1\cdot8}}{KD}$$

Which is a maximum when

$$H = 2\cdot8\frac{LV^{1\cdot8}}{KD} = 2\cdot8h$$

That is when the friction head $h = \frac{H}{2\cdot8}$. Let $n = \frac{V}{v}$

$$\text{Kinetic energy of jet} = \frac{v^2}{2g} = H - \frac{LV^{1\cdot8}}{KD} = \frac{1\cdot8LV^{1\cdot8}}{KD}$$

$$\therefore \quad \frac{V^2}{2gn^2} = \frac{1\cdot8LV^{1\cdot8}}{KD}$$

$$\frac{1}{n^2} = \frac{1\cdot8\times2gL}{V^{0\cdot2}KD} = \frac{D^2}{d^2} \qquad \frac{D}{d} = \sqrt{\frac{3\cdot6gL}{V^{0\cdot2}KD}}$$

17. Pressure on a Series of Moving Vanes due to the Impact of a Jet of Water

W = lb. of water per sec.

$$cb = N = ab - ac \qquad N = V\cos\alpha - \frac{V}{n}\cos\phi$$

$$\text{Normal pressure on vane} = \frac{WV}{g}\left(\cos\alpha - \frac{\cos\phi}{n}\right)$$

Pressure component in the direction of motion of vanes :

$$P = \frac{WV\cos\phi}{g}\left(\cos\alpha - \frac{\cos\phi}{n}\right)$$

$$\text{Useful work} = \frac{PV}{n} \qquad \text{Energy supplied by jet} = \frac{WV^2}{2g}$$

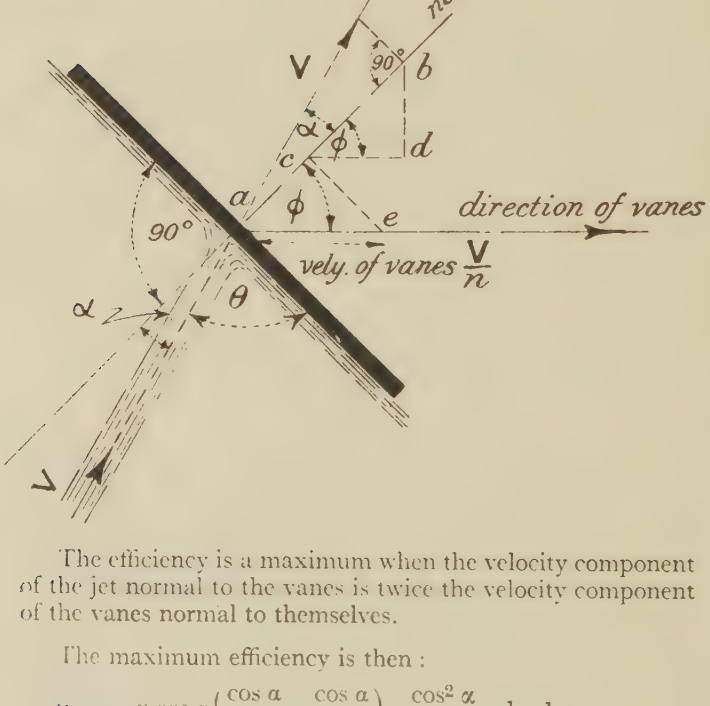
$$\text{Efficiency} \quad \eta = \frac{WV^2 \cos\phi\left(\cos\alpha - \frac{\cos\phi}{n}\right)2g}{ng \times WV^2}$$

$$= \frac{2\cos\phi}{n}\left(\cos\alpha - \frac{\cos\phi}{n}\right) = 2\cos\phi(n^{-1}\cos\alpha - n^{-2}\cos\phi)$$

$$\frac{d\eta}{dn} = 2\cos\phi(-n^{-2}\cos\alpha + 2n^{-3}\cos\phi)$$

This is a maximum when :

$$n^{-2}\cos\alpha = 2n^{-3}\cos\phi \qquad \cos\alpha = \frac{2\cos\phi}{n}$$



The efficiency is a maximum when the velocity component of the jet normal to the vanes is twice the velocity component of the vanes normal to themselves.

The maximum efficiency is then :

$$\eta_{max.} = n\cos\alpha\left(\frac{\cos\alpha}{n} - \frac{\cos\alpha}{2n}\right) = \frac{\cos^2\alpha}{2} = \tfrac{1}{2} \text{ when } \alpha = 0.$$

Pressure on a Single moving Vane

When the jet impinges on a single plate (not a series) the quantity of water reaching the plate is only $W\left(1-\frac{1}{n}\right)$

Then the efficiency becomes :

$$\eta=\frac{2\cos\phi}{n}\left(\cos\alpha-\frac{\cos\phi}{n}\right)\left(1-\frac{1}{n}\right)$$

and has a maximum value when $n=3$, and the efficiency is then about 30 per cent.

18. *A jet of water 3 inches in diameter strikes a flat plate at an angle of 60°. Discharge 1,000 gals. per minute. Find the pressure on the plate and thickness of the two discharge streams.*

If A is the area of the jet, A_1 and A_2 the areas of the discharge streams, then the force *normal* to the plate will be the change of momentum per second :

$$P=\frac{w}{g}\times Av\times v\sin 60^\circ=\frac{10000}{32{\cdot}2\times 60}\times\frac{\sqrt{3}}{2}\times\frac{10000\times 4\times 16}{60\times 62{\cdot}4\times\pi}$$
$$=\mathbf{245\ lb.}$$

Since there is no force acting *along* the plate, there is no change of momentum in that direction.

Hence $\frac{w}{g}A_1v^2-\frac{w}{g}A_2v^2=\frac{w}{g}Av^2\cos 60^\circ$

$A_1-A_2=A\cos 60^\circ$. Also $A_1+A_2=A$

since the velocity is unaltered.

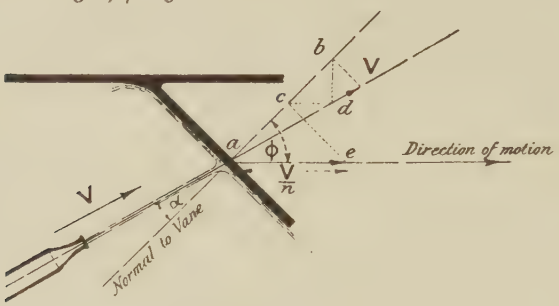
Hence $A_1=\frac{A}{2}(1+\cos 60^\circ)=\frac{3}{4}A$ $\quad A_2=\frac{A}{2}(1-\cos 60^\circ)=\frac{1}{4}A$

And the thicknesses are :

$t_1=\frac{3}{4}\times\frac{\pi\times 9}{4}\times\frac{1}{3}=\mathbf{1{\cdot}76\ ins.}$ $\quad t_2=\frac{1}{4}\times\frac{\pi\times 9}{4}\times\frac{1}{3}=\mathbf{0{\cdot}59\ in.}$

19. *In an experimental railway train jets of water play on a series of flat vanes beneath the vehicles as shown in the figure. Find* (i) *the pressure exerted on the vanes ;* (ii) *the velocity of the train for max. effy. ;* (iii) *the horse-power ;* (iv) *the efficiency. Total head of water,* 1,500′ ; *length of*

main, 3,000′; diameter of pipe, 4″; diameter of nozzle, 0·73″. $\alpha=30°$, $\phi=50°$.



$$\frac{\text{Area of pipe}}{\text{Area of jet}}=\frac{4^2}{0\cdot73^2}=30.$$

For max. horse-power in jet one-third of the head must be absorbed in friction.

$$V=8\cdot02\sqrt{\tfrac{2}{3}\times1500}=253\cdot5 \text{ ft./sec.}$$

Wt. of water per sec. $W=\dfrac{0\cdot418\times253\cdot5\times62\cdot4}{144}=46$ lb. p. sec.

The max. efficiency occurs when $n=\dfrac{2\cos\phi}{\cos\alpha}=\dfrac{2\times0\cdot643}{0\cdot866}=1\cdot485$

(see Example 15)

$$\text{Vely. of train}=\frac{253\cdot5}{1\cdot485}=171 \text{ ft./sec.}=\mathbf{117\ m.p.h.}$$

Pressure component in direction of motion

$$=P=\frac{46\times253\cdot5\times0\cdot643}{32\cdot2}\left(0\cdot866-\frac{0\cdot643}{1\cdot485}\right)=\mathbf{101\ pounds}$$

$$\text{Efficiency}=\frac{2\times0\cdot643}{1\cdot485}\left(0\cdot866+\frac{0\cdot643}{1\cdot485}\right)=\mathbf{0\cdot375}$$

or $\eta=\dfrac{\cos^2\alpha}{2}=\mathbf{0\cdot375}$ (See Example 12.)

$$\text{Horse-power}=\frac{WV^2\times0\cdot375}{2g\times550}=\frac{46\times253\cdot5^2\times0\cdot375}{64\cdot4\times550}=\mathbf{31.}$$

20. *A horizontal jet of water plays on a series of flat vanes moving in the same direction as the jet. Find the*

horse-power and the efficiency. Inclination of vanes, 45° ; *vel. of jet,* 40 *f.p.s.; diam. of nozzle,* 1·5″ ; *vel. of vanes,* 10 *f.p.s.*

Area of nozzle in sq. ft. = 0·0123

Water passing p.s. = 62·4 × 0·0123 × 40
= 30·6 lb.

Striking vel. = (40 − 10) = 30 f.p.s.

$$\text{Press. normal to plate} = \frac{WV_1}{g} = \frac{WV \sin\theta}{g}$$

$$\text{Pres. parallel to jet} = \frac{WV_2}{g} = \frac{WV_1 \sin\theta}{g} = \frac{WV \sin^2\theta}{g}$$

$$= \frac{30{\cdot}6 \times 30}{2 \times 32{\cdot}2} = 14{\cdot}3 \text{ lb.}$$

Work done per sec. = 14·3 × 10 = 143 ft.-lb.

Horse-power = $\frac{143}{550}$ = **0·26**

$$\text{Kinetic energy of jet} = \frac{WV^2}{2g} = \frac{30{\cdot}6 \times 40^2}{64{\cdot}4} = 760 \text{ ft.-lb.}$$

Efficiency = $\frac{143}{760}$ = **18·8 per cent.**

21. *A jet of water impinges axially on the apex of a fixed cone, the angle at the apex is* 90°. *The vel. of the water,* 150 *f.p.s.; diam. of jet,* 0·75 *in. Find the axial pressure.*

Area of jet 0·442 sq. in.

$$\text{Quantity of water delivered} = \frac{0{\cdot}442 \times 150}{144} = 0{\cdot}46 \text{ cusec.}$$

$$W = 0{\cdot}46 \times 62{\cdot}4 = 28{\cdot}7 \text{ lb./sec.}$$

$$P = \frac{WV(1 - \cos\theta)}{g} = \frac{28{\cdot}7 \times 150 \times 0{\cdot}293}{32{\cdot}2} = \mathbf{3{\cdot}92 \text{ lb.}}$$

22. *A jet of water plays into a cup the last wetted lips of which are at an angle of* 35° *with the axis of the jet, which coincides with that of the cup. Quantity of water delivered,* 1·2 *cusecs. Diam. of jet,* 0·9 *inch. Find the axial pressure.*

Area of jet = 0·636 sq. in.

$$\text{Vel. of jet} = \frac{144 \times 1{\cdot}2}{0{\cdot}636} = 272 \text{ f.p.s.}$$

$$W = 62{\cdot}4 \times 1{\cdot}2 = 74{\cdot}9 \text{ lb. p.s.}$$

The water is turned through an angle of $180°-35°=145°$, $\cos 145°=-0{\cdot}82$.

$$\text{Pressure } P=\frac{WV}{g}(1-\cos\theta)$$

$$P=\frac{74{\cdot}9\times 27^2}{32{\cdot}2}(1+0{\cdot}82)=\mathbf{1{,}150\ lb.}$$

(See p. 710, Vol. I.)

23. *A ship is propelled by means of a jet of water issuing from the stern. Find* (i) *the propelling force,* (ii) *the efficiency of the jet. Speed of ship,* 16 *knots; speed of jet relative to ship,* 40 *f.p.s.; sectional area of jet,* 15 *sq. feet.*

NOTE.—1 *knot*=1 *nautical mile* (6,080 *ft.*) *per hour.*

Let A=area of jet in sq. ft. S=speed of ship f.p.s.
S_j=speed of jet relative to ship f.p.s.
W=weight of one cubic foot of water=62·4 lb.
P=Propelling force in lb.
P=Change of momentum in water sent astern per second.

$$P=\frac{AS_jW}{g}(S_j-S) \quad . \quad . \quad . \quad . \quad . \quad (i)$$

Work done per second by jet in foot-lb. $=\dfrac{AS_jW}{g}(S_j-S)S \quad . \quad . \quad . \quad (ii)$

Kinetic energy of water sent astern, *i.e.* work wasted p.s.,

$$=\frac{AS_jW}{2g}(S_j-S)^2 \quad . \quad . \quad . \quad (iii)$$

Total energy supplied to water neglecting friction, etc.

$$=\text{ii}+\text{iii}=\frac{AS_jW}{2g}(S_j^2-S^2) \quad . \quad . \quad . \quad (iv)$$

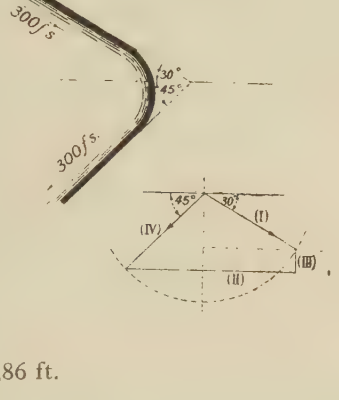
$$\text{Efficiency of jet}=\frac{\text{ii}}{\text{iv}}=\frac{2S}{S_j+S}$$

$$\text{Speed of ship} \qquad S=\frac{16\times 6080}{3600}=27 \text{ f.p.s.}$$

$$P=\frac{15\times 40\times 62{\cdot}4\times 13}{32{\cdot}2}=\mathbf{15{,}100\ lb.}$$

$$\text{Efficiency}=\frac{2\times 27}{40+27}=\mathbf{80{\cdot}6\ per\ cent.}$$

24. *A jet of water makes an angle of* 30° *with the horizontal and strikes a hollow cup without splash. The water leaves the cup at* 45° *to the horizontal. Find the pressure on the cup:* (i) *in the direction of the jet;* (ii) *horizontally;* (iii) *vertically;* (iv) *in the direction of the water on leaving. Diameter of jet,* 1·5 *ins.; pressure of water in the pipe close to the nozzle when the water is flowing,* 600 *lb. sq. in.*



Area of jet $= \frac{1{\cdot}767}{144}$ sq. ft.

$h = 600 \times 2{\cdot}31 = 1{,}386$ ft.

$V = 8{\cdot}02\sqrt{1386} = 300$ f.p.s.

$W = \frac{300 \times 1{\cdot}767 \times 62{\cdot}4}{144} = 230$ lb./sec.

(i) $\frac{230 \times 300}{32{\cdot}2}(1 + \cos 75°) =$ **2,720 lb.**

(ii) $\frac{230 \times 300}{32{\cdot}2}(\cos 30° + \cos 45°) =$ **3,370 lb.**

(iii) $\frac{230 \times 300}{32{\cdot}2}(\sin 45° - \sin 30°) =$ **440 lb.**

(iv) $\frac{230 \times 300}{32{\cdot}2}(1 + \cos 75°) =$ **2,720 lb.**

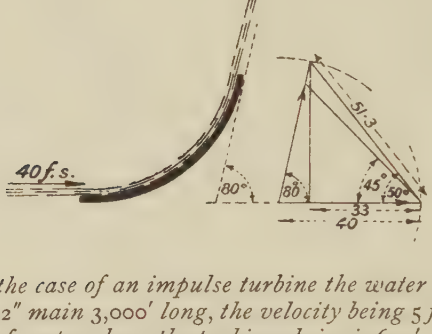
25. *Water impinges without shock on a smooth stationary curved vane which turns the water through an angle of* 80°. *Find the pressure on the vane,* (i) *in the direction of the jet,* (ii) *in the direction of maximum pressure,* (iii) *at an angle of* 45° *with the jet. Velocity of jet,* 40 *f.p.s.; quantity of water,* 0·37 *cusec.*

Pounds of water per sec. $= 0{\cdot}37 \times 62{\cdot}4 = 23{\cdot}1$.

(i) $P_1 = \frac{WV}{g}(1 - \cos\theta) = \frac{23{\cdot}1 \times 40}{32{\cdot}2}(1 - 0{\cdot}174) =$ **23·7 lb.**

(ii) $P_3=\frac{WV}{g}\sqrt{2(1-\cos\theta)}=\frac{23\cdot1\times40}{32\cdot2}\sqrt{2\times0\cdot826}=$ **36·9 lb.**

(iii) $P_2=36\cdot9\cos(50°-45°)=$ **36·7 lb.**



26. *In the case of an impulse turbine the water is supplied through a* 12″ *main* 3,000′ *long, the velocity being* 5 *f.p.s. The total head of water above the turbine sluice is* 600′. *The skin friction on the guide blades reduces the velocity of entry by* 8 *per cent. The velocity of the water at rejection is* $0\cdot14\sqrt{2gh}$. *Find the peripheral velocity of the turbine wheel. What is the hydraulic efficiency of the turbine and the whole plant ?*

$$h=\frac{3000\times5^2}{3330\times1}=22\cdot5'$$

Available head $=600-22\cdot5=577\cdot5'$

$V=8\cdot02\sqrt{577\cdot5}=192$ f.p.s.

Vely. on leaving guide blades $=192\times0\cdot92=176\cdot6$ f.p.s.

Tangential velocity $=192\sqrt{0\cdot92^2-0\cdot14^2}=174\cdot7$ f.p.s.

Velocity of wheel = **87 f.p.s.**

Hydraulic efficiency $=0\cdot91^2=$ **82·8 per cent.**

Efficiency of plant $=\frac{0\cdot828\times577}{600}=$ **80 per cent.**

This does not take into account friction of bearings, etc.

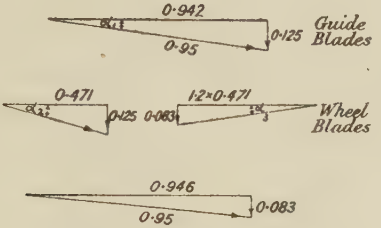
The hydraulic efficiency might have been arrived at thus :

$$0\cdot92\sqrt{2gh}=\sqrt{2g\times0\cdot846h}$$

$$0\cdot14\sqrt{2gh}=\sqrt{2g\times0\cdot019h}$$

$0\cdot846-0\cdot019=$ **82·7 per cent.**

27. *In an impulse turbine the velocity is reduced* 5 *per cent. in passing over the guide blades. The velocity of flow is* $\frac{1}{8}\sqrt{2gh}$. *The external diameter of the wheel (outward flow) is* 1·2 × *internal diameter. The wheel is flared out to reduce the velocity of flow at exit to* $\frac{1}{12}\sqrt{2gh}$. *Find the angle of:* (i) *the guide blades;* (ii) *the first tip of the wheel blades;* (iii) *the last tip. What is the hydraulic efficiency of the wheel?*



$$V_1 = 0{\cdot}95\sqrt{2gH} = \sqrt{2g \times 0{\cdot}902H}$$

$$V = \sqrt{0{\cdot}95^2 - 0{\cdot}125^2} = 0{\cdot}942\sqrt{2gH}$$

For guide blades:

$$\tan \alpha_1 = \frac{0{\cdot}125}{0{\cdot}942} = 0{\cdot}133 \qquad \alpha_1 = \mathbf{7{\cdot}5^\circ} \quad . \quad . \quad . \quad \text{(i)}$$

At entry to wheel blades:

$$\tan \alpha_2 = \frac{0{\cdot}125}{0{\cdot}471} = 0{\cdot}266 \qquad \alpha_2 = \mathbf{15^\circ} \quad . \quad . \quad . \quad \text{(ii)}$$

At discharge from wheel blades:

$$\tan \alpha_3 = \frac{0{\cdot}083}{0{\cdot}565} = 0{\cdot}147 \qquad \alpha_3 = \mathbf{8{\cdot}5^\circ} \quad . \quad . \quad . \quad \text{(iii)}$$

Hydraulic efficiency $= 0{\cdot}946^2 = \mathbf{0{\cdot}895.}$

The hydraulic efficiency can also be found thus:

9·8 per cent. of the head is absorbed in friction in guides
0·7 per cent. „ „ rejected in tail race
10·5 per cent. losses or **89·5 per cent. efficiency.**

28. *Determine from the data given below the angles of the tips of guide and moving blades of an impulse turbine.* (1) *Head of water,* 180′; (2) *percentage of head lost in friction,* 1·5 *per cent.;* (3) *percentage of head lost in tail race,* 5·0 *per cent. What is the theoretical efficiency of the wheel?*

Head lost in friction $=0{\cdot}015\times180=2{\cdot}7'$

,, ,, tail race $=0{\cdot}05\times180=9'$

Velocity at entry $=\sqrt{64{\cdot}4(180-2{\cdot}7)}=106{\cdot}9$ f.p.s.

,, ,, discharge $=\sqrt{64{\cdot}4\times9}=24{\cdot}1$ f.p.s.

θ_1 = guide blade angle $\quad \sin\theta_1=\dfrac{24{\cdot}1}{106{\cdot}9}=0{\cdot}225 \quad \theta_1=\mathbf{13^\circ}$

Tangential compt. of velocity of entering water

$$=\frac{24{\cdot}1}{\tan 13^\circ\ 1'}=104 \text{ f.p.s.}$$

Linear velocity of blades $=52$ f.p.s.

$$\tan\theta_2=\frac{24{\cdot}1}{52}=0{\cdot}463 \qquad \theta_2=\mathbf{22{\cdot}75^\circ}$$

Discharge angle $\theta_3 \quad \tan\theta_3=\dfrac{24{\cdot}1}{52{\cdot}0} \qquad \theta_3=\theta_2=\mathbf{22{\cdot}75^\circ}$

Work per lb. water $=\dfrac{104{\cdot}1^2}{64{\cdot}4}=168{\cdot}4$ ft.-lb.

Efficiency $=\dfrac{168{\cdot}4}{180}=\mathbf{93{\cdot}5\ per\ cent.}$

29. *Find the radius* R *of the inlet periphery of a turbine which develops* 80 *H.P. with an available head* H *of* 110 *feet. Overall efficiency,* 75 *per cent. Breadth of wheel,* $0{\cdot}25\times$ *radius. Velocity of flow,* $0{\cdot}125\sqrt{2gH}$. *There are* 24 *blades in the wheel, each* 0·1 *inch in thickness.*

$$Q=\left(2\pi R-\frac{24\times0{\cdot}1}{12}\right)0{\cdot}25R\times0{\cdot}125\sqrt{2gH}$$

$$Q=16{\cdot}5(R^2-0{\cdot}032R)$$

$$\text{H.P.}=0{\cdot}1135QH\times\text{efficiency}=80$$

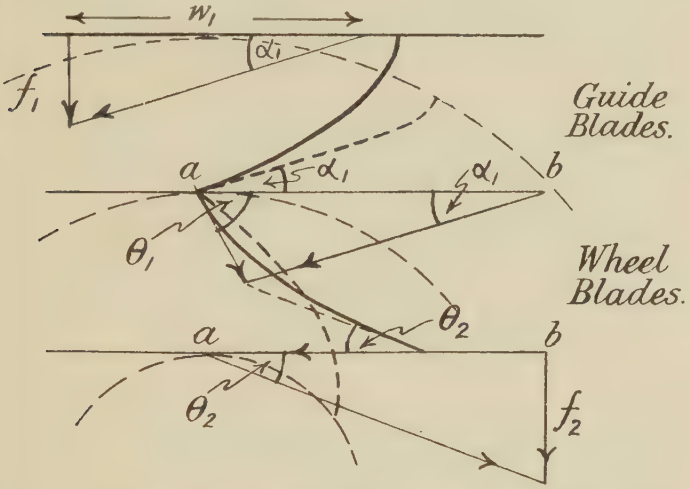
$$\frac{80}{0{\cdot}1135\times110\times16{\cdot}5\times0{\cdot}75}=R^2-0{\cdot}032R \qquad R=\mathbf{0{\cdot}735'}$$

If the thickness of blades be neglected, see p. 735, Vol. I.,

$$R=\sqrt{\frac{80\times4\times8}{0{\cdot}71\times0{\cdot}75\times110\times8\times10{\cdot}49}}=\mathbf{0{\cdot}72'}.$$

30. *In an inward flow pressure turbine* 6 *cusecs pass through the wheel when stationary. Radius of wheel* (R_1), 1′ ; *radius of inside* (R_2), 0·5′ ; *velocity of whirl at entry* (w_1), 18 *f.p.s.*, *velocity of flow at entry* (f_1), 6 *f.p.s.* ; *velocity of*

flow at exit (f_2), 8 f.p.s. (radial). Find the turning moment on the wheel, also the difference of pressure between the inlet and outlet when the wheel revolves at 200 r.p.m., and the same quantity of water passes.



Consider it as a turbine rod in the first instance, blades shown in full lines, the blades for the actual turbine are shown in broken lines.

$V_2 = V_1$ in turbine rod $\quad V_2 = \frac{V_1}{2}$ in wheel $\quad ab = V_1$

$V_1 = 2\pi R_1 N = 2 \times 3{\cdot}14 \times \frac{200}{60} = 20{\cdot}9$ f.p.s. $\quad V_2 = 10{\cdot}45$ f.p.s.

Difference of pressure head when rotating

(see p. 733, Vol. I.)

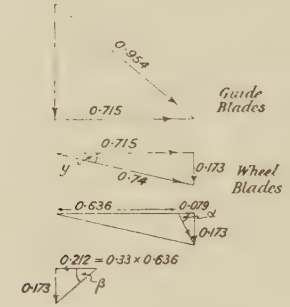
$$= \frac{V_1^2 - V_2^2}{2g} = \frac{20{\cdot}9^2 - \left(\frac{20{\cdot}9}{2}\right)^2}{2g} = \mathbf{5{\cdot}1\ feet.}$$

When rotating the velocity of whirl at exit is zero, but when standing it is $V_2 = -10{\cdot}45$ f.p.s.

Turning moment when standing $= \frac{6 \times 62{\cdot}4}{32{\cdot}2}(w_1 R_1 + V_2 R_2)$

$$= \frac{6 \times 62{\cdot}4}{32{\cdot}2}(18 \times 1 + 10{\cdot}45 \times 0{\cdot}5) = \mathbf{270\ lb.\text{-}feet.}$$

31. *Find the guide and wheel blade angles, also the speed of the wheel and the centrifugal head for an inward flow reaction turbine. Head of water,* 150′ ; *rejected in tail race,* 3 *per cent.; head wasted in friction,* 6 *per cent.; outer radius of wheel,* 1′ ; *inner,* 0·33′. *The horizontal component of the velocity of the water leaving the guide blades is* 75 *per cent. of the velocity utilised in the turbine.*



$V_r = \sqrt{0{\cdot}03} \times 2gH = 0{\cdot}173\sqrt{2gH}$
$V = \sqrt{0{\cdot}91} \times 2gH = 0{\cdot}954\sqrt{2gH}$ } 9 per cent. of head wasted
$V_h = 0{\cdot}954 \times 0{\cdot}75 = 0{\cdot}715\sqrt{2gH}$ } 91 per cent. utilised

$$\frac{V_h V_w}{g} = 0{\cdot}91H$$

$$V_w = 0{\cdot}636\sqrt{2gH} = 0{\cdot}636 \times 8{\cdot}02 \times 12{\cdot}25 = 62{\cdot}5 \text{ f.p.s.}$$

$$\tan y = \frac{0{\cdot}173}{0{\cdot}715} = 0{\cdot}242 \qquad y = 13{\cdot}5°$$

$$\tan \alpha = \frac{0{\cdot}173}{0{\cdot}079} = 2{\cdot}19 \qquad \alpha = 65{\cdot}5°$$

$$\tan \beta = \frac{0{\cdot}173}{0{\cdot}212} = 0{\cdot}82 \qquad \beta = \mathbf{39{\cdot}25°}$$

$$\text{Speed} = \frac{62{\cdot}5 \times 60}{2 \times 3{\cdot}14} = \mathbf{597 \text{ r.p.m.}}$$

$$\text{Centrifugal head} = \frac{V_e^2 - V_r^2}{2g} = \frac{62{\cdot}5^2 - 20{\cdot}8^2}{64{\cdot}4} = \mathbf{54 \text{ feet.}}$$

CHAPTER XX

PUMPS

1. *The suction pipe of a fly-wheel pump dips at an angle of* 10 *degrees with the horizontal. The water to be pumped is warm and boils when the pressure is* 12 *lb. per sq. inch below the atmosphere. Find the maximum length of suction pipe for satisfactory working. Diameter of suction pipe,* 4″; *pump-plunger,* 6″; *stroke of pump-plunger,* 10″; *length of connecting rod,* 4 *cranks; r.p.m.,* 64. *The plunger is below the crankshaft. Neglect friction.*

$$64=54{\cdot}5\sqrt{\frac{(12\times 2{\cdot}31-L\sin 10)16}{L\times 0{\cdot}42\times 1{\cdot}25\times 36}}$$

(See p. 748, Vol. I.)

$$\left(\frac{64}{54{\cdot}5}\right)^2\frac{36\times 1{\cdot}25\times 0{\cdot}42L}{16}=27{\cdot}7-0{\cdot}174L$$

$$1{\cdot}63L+0{\cdot}174L=27{\cdot}7 \qquad L=\mathbf{15{\cdot}4'}.$$

If the suction stroke commences at the "out" end the 1·25 in the expression above becomes 0·75, and the length L then becomes 24·2 ft.

2. *Find the speed at which cavitation will occur in the case of a reciprocating pump running under the following conditions. Suction,* 18 *ft.; friction,* 5 *ft.; diameter of suction pipe,* 6 *ins.; pump barrel,* 9 *ins.; length of suction pipe,* 72 *ft.; radius of crank,* 15 *ins.; length of connecting rod,* 50 *ins.*

$$N=54{\cdot}5\sqrt{\frac{(34-h_s-h_f)A_s}{LR\left(1\pm\frac{1}{n}\right)A_p}}$$

$$=54{\cdot}5\sqrt{\frac{11\times 6^2}{72\times 1{\cdot}25\times 1{\cdot}3\times 9^2}}=\mathbf{11{\cdot}1\ r.p.m.}\text{ at "in" end}$$

$$N=54{\cdot}5\sqrt{\frac{11\times 6^2}{72\times 1{\cdot}25\times 0{\cdot}7\times 9^2}}=\mathbf{15{\cdot}2\ r.p.m.}\text{ at "out" end.}$$

3. *What is the maximum quantity of water which can be pumped per minute by a reciprocating pump when running under the following conditions? Diameter of pump barrel,* 12 *ins.; suction pipe,* 8 *ins.; stroke of plunger,* 26 *ins.; length of connecting rod,* 40 *ins.; suction pipe,* 18 *ft.; height of suction,* 7 *ft.; friction in foot valve and pipe,* 4 *ft.; height of water barometer,* 33 *ft.; coefficient of discharge,* 0·96.

Cavitation occurs when:

$$N=54\cdot5\sqrt{\frac{(33-7-4)64}{18\times\frac{1\cdot2}{12}(1+\frac{13}{40})144}}=33\cdot5 \text{ r.p.m.}$$

Volume per stroke $=0\cdot96\times0\cdot785\times\frac{26}{12}=1\cdot63$ c. ft.

Total volume per min. $=1\cdot63\times33\cdot5=$ **54·6 c. ft.** (single acting).

If suction begins at the "out" end of the stroke the quantity $1+\frac{13}{40}$ becomes $1-\frac{13}{40}$ and $N=47$ r.p.m., and the volume of water becomes **76·6 c. ft.**

4. *Find the force in the piston rod of a double-acting pump when* 6″ *from the middle of the stroke. Radius of crank,* 9″; *r.p.m.* 70. *Long connecting rod. Water enters the pump under a positive head of* 18′ *and is discharged at a head of* 60′ *above the pump. The suction pipe is short; the delivery pipe is* 113′ *long. Area of piston* 18 *sq. ins. Area of main,* 10 *sq. ins.*

$$\left.\begin{array}{l}\text{Force on rod due to}\\ \text{pressure of water}\end{array}\right\}=\frac{62\cdot4\times18}{734}(60-18)=328 \text{ lb.}$$

The column of water in the delivery pipe is subject to acceleration in a similar manner to the piston, but its acceleration is greater in the ratio $(\frac{18}{10})$ of the areas of piston and main.

Force accelerating the water at end of stroke

$$=0\cdot00034(AL\times62\cdot4)\tfrac{6}{12}\times70^2\times\tfrac{18}{10}$$

Ditto when at 6 inches from middle of stroke

$$=0\cdot00034(\tfrac{18}{144}\times113\times62\cdot4)\tfrac{6}{12}\times4{,}900\times\tfrac{18}{10}=\textbf{1,322 lb.}$$

Total load on piston $=328\pm1{,}322=1{,}650$ lb. or -994 lb.

Cavitation will therefore occur during the second half of each delivery stroke.

5. *Find the diameter* d *of an accumulator ram to comply with the following conditions. Load on ram,* 100 *tons; horse-power transmitted by accumulator,* 50; *length of main,* 2000 *yds.; diam.* 4″; *loss by friction in main,* 2 *H.P.*

Let H=speed with which accumulator weights fall

$$= \frac{50 \times 550}{100 \times 2240} = 0{\cdot}123 \text{ f.p.s.}$$

$$\frac{\text{Velocity of flow in pipe (V)}}{\text{Velocity of accumulator weights (H)}} = \frac{\text{Area of ram}}{\text{Area of pipe}}$$

$$\frac{V}{H} = \frac{d^2}{4^2} \qquad V = \frac{d^2}{130}$$

Loss of head by friction h K in D'Arcy's formula=2,560

$$h = \frac{LV^2}{2560D} = \frac{6000 \times d^4 \times 3}{2560 \times 130^2} = \frac{d^4}{2400}$$

$$\text{H.P. lost in friction} = 0{\cdot}1135 \times \frac{0{\cdot}785 d^2}{144} \times \frac{d^2}{130} \times \frac{d^4}{2400} = 2$$

$$d^8 = 1{\cdot}01 \times 10^9 \qquad d = \mathbf{1{\cdot}335'}.$$

6. *An accumulator* (*Fig.* 665) *has a plunger* 20″ *diameter with a* 15′ *stroke. Load,* 90 *tons. Find the horse-power that can be exerted at a machine* 200′ *away for two minutes. Diam. of main,* 3″.

$$\text{Pressure } \frac{90 \times 2240}{314} = 642 \text{ lb. sq. in.} = 1{,}480' \text{ head}$$

$$Q = \frac{314 \times 15}{144 \times 2 \times 60} = 0{\cdot}273 \text{ cusec.}$$

$$V \text{ (pipe)} = \frac{0{\cdot}273 \times 16}{0{\cdot}785} = 5{\cdot}56 \text{ ft. p. sec.}$$

$$\text{H.P.} = \frac{0{\cdot}273 \times 62{\cdot}4}{550}\left(1480 - \frac{LV^2}{KD}\right)$$

$$\text{H.P.} = \frac{0{\cdot}273 \times 62{\cdot}4}{550}\left(1480 - \frac{200 \times 5{\cdot}56^2}{2400 \times 0{\cdot}25}\right) = \mathbf{45{\cdot}5}.$$

7. *A direct-acting lift is supplied with water from a reservoir* 110′ *above, of which* 19′ *are absorbed in overcoming the friction of the cup leather and the mechanism of the lift.*

Find the speed at which the lift will steadily raise a load of 1 *ton, also find the load it will lift when the speed is increased by* 25 *per cent. Diam. of plunger,* 12″ ; *length of supply pipe,* 180′ ; *diam. of supply pipe,* $4\frac{1}{2}$″. *The supply pipe is clean cast iron.*

Area of plunger = 113·1 sq. ins.

Press. required to support load $= \frac{2240}{113 \cdot 1} = 19 \cdot 8$ lb. sq. in. $= 45 \cdot 8'$

Head absorbed in pipe friction $= 110 - 19 - 45 \cdot 8 = 45 \cdot 2$ feet

$$45 \cdot 2 = \frac{LV^2}{KD} = \frac{180V^2 \times 12}{2620 \times 4 \cdot 5} \qquad V = 15 \cdot 7 \text{ f.p.s. in supply pipe}$$

Area of supply pipe 15·9 sq. ins.

Velocity of lift $= \frac{15 \cdot 7 \times 15 \cdot 9}{113 \cdot 1} =$ **2·2 f.p.s.**

When the speed is increased 25 per cent. we have :

$$h = \frac{180 \times (15 \cdot 7 \times 1 \cdot 25)^2 \times 12}{2620 \times 4 \cdot 5} = \begin{cases} 70 \cdot 5' \text{ head absorbed in pipe} \\ \text{friction} \end{cases}$$

110 − 19 − 70·5 = 20·5′ available = 8·9 lb. sq. inch

8·9 × 113·1 = **1,000 lb.** can be lifted at the higher speed.

In the above it has been assumed that the friction of the cup leather and mechanism does not alter with the speed.

8. *A three-throw plunger pump is working under the following conditions : Diameter of plungers,* 12″ ; *stroke,* 18″ ; *lift,* 450′ ; *r.p.m.,* 42 ; *volumetric efficiency,* 97 *per cent. ; length of main,* 900′ ; *diameter,* 8″. *The pump is driven by an electric motor having a useful horse-power of* 134. *Find* (i) *the delivery of the pump ;* (ii) *water horse-power ;* (iii) *efficiency ;* (iv) *horse-power expended in overcoming friction of water ;* (v) *friction horse-power of pump.*

(i) Vol. displaced by plungers per min.

$$= \frac{113 \cdot 1 \times 3 \times 42 \times 18}{144 \times 12} = 148 \text{ c. ft.} =$$ **923 gals. min.**

(ii) $\frac{9230 \times 450 \times 0 \cdot 97}{33000} =$ **122 water horse-power**

(iii) $\frac{122}{134} =$ **91 per cent.**

(iv) Velocity in main $= \frac{148 \times 0{\cdot}97}{60 \times 0{\cdot}349} = 6{\cdot}9$ f.p.s.

Loss of head by friction (D'Arcy) $h = \frac{900 \times 6{\cdot}9^2}{2900 \times 0{\cdot}67} = 22'$

Horse-power expended in overcoming friction $= \frac{0{\cdot}1135 \times 99 \times 0{\cdot}97 \times 22}{60} = \mathbf{4}$

(v) $134 - 122 - 4 = \mathbf{8}$.

9. *Water has to be delivered by a pump to a height of* 90′. *Velocity of flow in the pipe,* 10 *f.p.s.; length,* 280′. *The diameter of the pipe is* 14″. *Find* (i) *the gross head against which the pump will have to deliver.* (ii) *The useful horse-power of the pump.*

Loss of head by friction taking $K = \frac{1}{3500}$

$$h = \frac{280 \times 10^2 \times 12}{3500 \times 14} = 6{\cdot}9 \text{ feet.}$$

Kinetic energy at discharge $= \frac{10^2}{64{\cdot}4} = 1{\cdot}6$ feet.

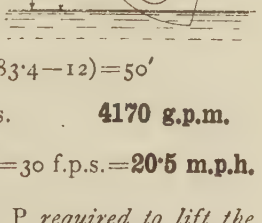
Total head wasted $= 8{\cdot}5'$.

(i) Gross head $= 98{\cdot}5'$ Area of main $= 1{\cdot}07$ sq. ft.

$Q = \mathbf{10{\cdot}7}$ **cusecs.**

(ii) Useful H.P. $= 10{\cdot}7 \times 0{\cdot}1135 \times 98{\cdot}5 = \mathbf{120}$.

10. *A locomotive scoop delivers water to the tender through a* 6″ *pipe. Assuming that* 0·3 *of the gross head is wasted in the scoop, find the quantity of water delivered at* 50 *m.p.h. At what speed will delivery start?*



$$h = \frac{V^2}{2g} = \frac{73{\cdot}3^2}{64{\cdot}4} = 83{\cdot}4'$$

Available head for delivery $= 0{\cdot}7(83{\cdot}4 - 12) = 50'$

$Q = \frac{0{\cdot}785}{4} \sqrt{64{\cdot}4 \times 50} = 11{\cdot}1$ cusecs. **4170 g.p.m.**

Delivery starts $V = \sqrt{2gh} = 8{\cdot}02\sqrt{14} = 30$ f.p.s. $= \mathbf{20{\cdot}5}$ **m.p.h.**

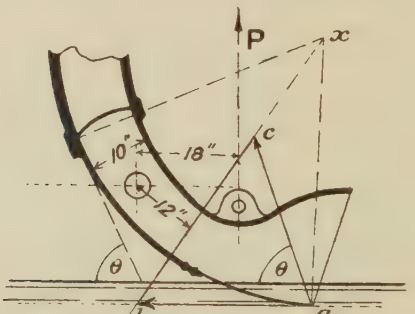
11. *Find the* (*approximate*) *force* P *required to lift the locomotive scoop when travelling at* 60 *m.p.h. assuming that it runs full.*

An exact solution is probably impossible. We shall assume that it is a case of a jet of water playing on a curved surface (see p. 710, Vol. I.). Tests show that this assumption is not far from the truth.

The point x is the centre of the scoop curve. The resultant pressure acts through this point. ac and ab represent 60 m.p.h. or 88 f.p.s. If $\theta = 70°$, bc to the same scale $= 100$ f.p.s. $= V_1$.

Quantity of water lifted per sec.

$$= \frac{0{\cdot}785 \times 10^2 \times 88}{144} = 48 \text{ cusecs} = 3000 \text{ lb. p. sec.}$$



$$P_1 = \frac{wV_1}{g} = \frac{3000 \times 100}{32{\cdot}2} = 9300 \text{ lb.}$$

Taking moments about the pivot we get

$$P = \frac{9300 \times 12}{18} = \mathbf{6200 \text{ lb.}}$$

12. *In a given instance the radial velocity of flow in a centrifugal pump was* 10 *f.p.s., and the velocity of whirl on leaving the impeller* 65 *f.p.s. Find the increase in pressure at the outer radius of the whirlpool chamber which is* 2·4 *times the radius of the impeller. In the pressure conversion one-third of the energy is dissipated in eddies.*

(See Chapter XX., Vol. I.)

Let the internal radius of the chamber be R_i and the external R_e.

By Bernoulli's theorem

$$\frac{p_e - p_i}{w} = \frac{V_i^2 - V_e^2}{2g} \qquad \delta p = \frac{wV_i^2}{g}\left(1 - \frac{V_e^2}{V_i^2}\right)$$

and if the walls of the whirlpool chamber are parallel,

$$\delta p = \frac{wV_i^2}{2g}\left(1 - \frac{R_i^2}{R_e^2}\right) \qquad V_i = \sqrt{65^2 + 10^2} = 65{\cdot}8 \text{ ft./sec.}$$

$$\delta p = \frac{0{\cdot}434 \times 65{\cdot}8^2}{64{\cdot}4}\left(1 - \frac{1}{2{\cdot}4^2}\right) = \mathbf{24 \text{ lb. sq. inch}}$$ with no losses

or **16 lb. sq. inch** when one-third is dissipated in eddies.

13. *A centrifugal pump delivers* 0·5 *cusec when running at* 800 *r.p.m. and* 3 *cusecs at* 1000 *r.p.m. The suction head is* 12′, *the resistance of the foot valve is equivalent to the friction on* 8·5′ *of pipe. Length of suction pipe* 20′, *diameter* 8″. *Find the r.p.m. of pump when cavitation is about to take place, also the corresponding delivery.*

Assume a straight line law connecting the speed and quantity of water delivered.

We have 0·5 cusec increase for 40 r.p.m. Pump begins to lift at 800−40=760 r.p.m.

N=760+80Q where Q is the quantity delivered

$$Q = \frac{N}{80} - 9{\cdot}5$$

We also have Q=area of suction pipe $\sqrt{2g(34 - h_s - h_f)}$

The velocity of flow in the suction pipe

$$V = \frac{Q}{A} = \frac{144Q}{50{\cdot}26} = 2{\cdot}87Q \qquad V^2 = 8{\cdot}21Q^2$$

Loss of head by friction $h_f = \frac{(20+8{\cdot}5)8{\cdot}21Q^2}{2850 \times 0{\cdot}67} = 0{\cdot}123Q^2$

$$Q = \frac{50{\cdot}3 \times 8{\cdot}02}{144}\sqrt{34 - 12 - 0{\cdot}123Q^2}$$

$$Q^2 = 7{\cdot}84(22 - 0{\cdot}123Q^2) = 87{\cdot}8 \qquad Q = \mathbf{9{\cdot}37 \text{ cusecs}}$$

$$N = 760 + 80 \times 9{\cdot}37 = \mathbf{1510 \text{ r.p.m.}}$$

when cavitation is about to take place.

14. *A small radial-bladed centrifugal pump which has no vortex chamber or other means of converting the kinetic energy of the water into pressure energy, is fitted with a vertical glass delivery pipe to enable it to be used as a revolution*

speed indicator. The external diameter of the impeller is 4″. *The supply water is taken from a tank* 6″ *below the centre of the impeller. At what height will the water stand in the glass pipe when the impeller rotates at* 1200 *r.p.m.?*

$\frac{1200}{60}=20$ r.p.s. $\quad V=\frac{3{\cdot}14\times4\times20}{12}=20{\cdot}9$ f.p.s. $=8{\cdot}02\sqrt{h}$

$\left(\frac{20{\cdot}9}{8{\cdot}02}\right)^2=h=6{\cdot}77'$ above water level in tank

$-0{\cdot}50'$

6·27′ above centre of pump.

This arrangement was used nearly fifty years ago by Mr. Stroudley, L.B. and S.C.R., for indicating the speed of his locomotives.

15. *Find* V *and* η *for a centrifugal pump in which* $V_r=\frac{1}{4}\sqrt{2gH}$, $\theta=15°$, (i) *with no volute;* (ii) *with a volute.*

$\theta=15°$, cosec $15°=3{\cdot}86$, cosec^2 $15°=14{\cdot}9$, cot $15°=3{\cdot}73$.

No volute—

$$V=\sqrt{2gH+V_r^2\operatorname{cosec}^2\theta}=\sqrt{2gH+\frac{2gH}{16}\times14{\cdot}9}=\mathbf{1{\cdot}39\sqrt{2gH}}$$

$$\eta=\frac{2gH}{2V(V-V_r\cot\theta)}=\frac{2gH}{2\times1{\cdot}39(1{\cdot}39-\frac{1}{4}\times3{\cdot}73)2gH}=\mathbf{0{\cdot}785.}$$

With volute—

$$V_v=\frac{V-V_r\cot\theta}{2}=\frac{V-3{\cdot}73V_r}{2}$$

$$V=\sqrt{2gH+V_r^2\operatorname{cosec}^2\theta-2V_v^2}$$

$$V=\sqrt{2gH+14{\cdot}9V_r^2-2\left(\frac{V-3{\cdot}73V_r}{2}\right)^2}=\mathbf{1{\cdot}36\sqrt{2gH}}$$

$$\eta=\frac{2gH}{2V(V-V_r\cot\theta)}=\frac{2gH}{2\times1{\cdot}36(1{\cdot}36-0{\cdot}93)2gH}=\mathbf{0{\cdot}856.}$$

16. *A centrifugal pump is required to work against a head of* 200′. *Compare the horse-power required for a pump with a single runner and a* 4*-stage pump.* $\theta=15°$; $V_r=\frac{1}{8}\sqrt{2gh}$.

Cotan $15°=3{\cdot}73$. Cosec $15°=3{\cdot}85$.

SINGLE RUNNER H = 200′.

$V = \sqrt{2gH + V_r^2 \operatorname{cosec}^2 \theta}$

$V = \sqrt{64{\cdot}4 \times 200 + 200 \times 14{\cdot}9} = 126$ f.p.s.

D = say 18″

Circumference = 4·71′

$$\text{R.p.m.} = \frac{126 \times 60}{4{\cdot}71} = 1600.$$

Width of waterway on edge of runner say 2″ = $\frac{1}{8}$ foot.

Area of waterway neglecting vanes $\frac{4{\cdot}71}{6} = 0{\cdot}78$ sq. ft.

$$V_r = \tfrac{1}{8}\sqrt{2g \times 200} = 14{\cdot}1 \text{ ft./sec.}$$

$$\eta = \frac{V^2 - V_r^2 \operatorname{cosec}^2 \theta}{2V(V - V_r \cot \theta)}$$

$$= \frac{15860 - 3000}{252(126 - 52{\cdot}6)} = 0{\cdot}7$$

Gross H.P. neglecting friction on impeller

$$= \frac{0{\cdot}1135 \times 11 \times 200}{0{\cdot}7} = 356$$

Friction on impeller—

$$\text{H.P.} = \frac{f V^3 R^2}{223} = \frac{0{\cdot}006 \times 126^3 \times 3^2}{223 \times 4^2} = 30$$

Gross H.P. including friction on impeller = **386**.

FOUR RUNNERS H = 50′ each.

$V = \sqrt{64{\cdot}4 \times 50 + 50 \times 14{\cdot}9} = 63$ f.p.s.

$$D = \frac{18 \times 63}{126} = 9''$$

Circ. = 2·36′

$$\text{R.p.m.} = \frac{63 \times 60}{2{\cdot}36} = 1600$$

$V_r = \frac{1}{8}\sqrt{2g \times 50} = 7{\cdot}05$ f.p.s.

Width of waterway

$$= \frac{11}{2{\cdot}36 \times 7{\cdot}05} = 0{\cdot}66' \text{ total}$$

$$\text{or } \frac{0{\cdot}66}{4} = \tfrac{1}{8} \text{ foot each.}$$

$$\eta = \frac{3969 - 750}{126(63 - 26{\cdot}3)} = 0{\cdot}7$$

Gross H.P. = 356 neglecting friction on impeller.

Friction on impeller—

$$\text{H.P.} = \frac{0{\cdot}006 \times 63^3 \times 3^2}{223 \times 8^2} \times 4 = 3{\cdot}8$$

Gross H.P. = **360**.

17. *A radial-vaned centrifugal pump runs at* 600 *r.p.m. but delivers no water. Diam. of runner,* 18″. *Find the height* H *to which the water will rise in the delivery pipe.*

If the velocity of flow through the runner V_r *is* $\frac{1}{5}\sqrt{2gH}$. *find the head, the quantity delivered, and the hydraulic efficiency. Width of waterway at periphery of runner* = $\frac{r}{4}$, *thickness of vanes* 0·5″, *number of vanes* 9.

$$H = \frac{V^2}{2g} = \left(\frac{600 \times 3{\cdot}14 \times 1{\cdot}5}{60}\right)^2 \frac{1}{64{\cdot}4} = \mathbf{34{\cdot}4'}$$

$V = 47{\cdot}1$ f.p.s.

$2gH = V^2 - V_r^2 \operatorname{cosec}^2 \theta = V^2 - V_r^2 = 47{\cdot}1^2 - \frac{1}{25} 2gH$

$$H = \frac{25}{26} \times \frac{47{\cdot}1^2}{64{\cdot}4} = \mathbf{33{\cdot}2'}.$$

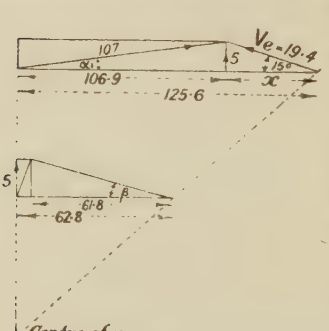
Then $V_r = \frac{1}{5}\sqrt{64{\cdot}4 \times 33{\cdot}2} = 9{\cdot}25$ f.p.s.

Area of passages in runner $= \left(2\pi r - \frac{9 \times 0{\cdot}5}{12}\right)\frac{0{\cdot}75}{4} = 0{\cdot}81$ sq. ft.

$Q = 9{\cdot}25 \times 0{\cdot}81 = \mathbf{7{\cdot}5\ cusecs}$

$$\eta = \frac{2gH}{2V(V - V_r \cot\theta)} = \frac{64{\cdot}4 \times 33{\cdot}2}{2 \times 47{\cdot}1^2} = \mathbf{48\ per\ cent.}$$

18. *In the case of a centrifugal pump with no volute, determine the angles of* (i) *the vanes at inlet;* (ii) *the guide blades at outlet; also* (iii) *the lift of the pump;* (iv) *the velocity of the water leaving the impeller.*



Diameter of impeller — outer, 2′; *inner,* 1′; *angle of vanes at outlet* (θ), 15°; *velocity of flow through impeller,* 5 *f.p.s.; vely. of whirl in "eye,"* 1 *f.p.s.; r.p.m.,* 1200.

Velocity of periphery of runner $= 20 \times 6{\cdot}28 = 125{\cdot}6$ f.p.s.

$$x = \frac{5}{\tan 15^\circ} = 18{\cdot}7 \text{ f.p.s.}$$

Velocity of whirl $= 125{\cdot}6 - 18{\cdot}7 = 106{\cdot}9$ f.p.s.

(iv) Relative velocity $V_e = \sqrt{18{\cdot}7^2 + 5^2} = \mathbf{19{\cdot}4\ f.p.s.}$

$\tan\alpha = \frac{5}{106{\cdot}9} = 0{\cdot}0465$ (ii) $\alpha = \mathbf{2^\circ\ 40'}$ (guide blade angle)

$\tan\beta = \frac{5}{61{\cdot}8} = 0{\cdot}081$ $\beta = \mathbf{4^\circ\ 40'}$ . . . . (i)

$$H = \frac{V^2 - V_r^2 \operatorname{cosec}^2\theta}{2g} = \frac{125{\cdot}6^2 - 5^2\left(\frac{5}{19{\cdot}4}\right)^2}{64{\cdot}4} = \mathbf{239'} \quad \text{(iii)}$$

The angles α and β in the above are unusually small—α is usually 6° to 10°, and β 12° to 15°.

19. *In a centrifugal pump the delivery is* $\left(\frac{N}{90} - 9\right)$ *cusecs, where* N *is the revs. per min.*

The pump works against a suction of 6′, *the diam. of the suction pipe is* 8″ *and the length* 12′. *The friction of the foot valve is equivalent to the resistance of* 10′ *of pipe. Find the speed at which cavitation takes place. Height of water barometer,* 34′.

Area of suction pipe $=0{\cdot}35$ sq. ft.

$$h=\frac{LV^2}{KD}=\frac{22\left(\frac{Q}{0{\cdot}35}\right)^2 12}{2850\times 8}=0{\cdot}095Q^2$$

$$\frac{V^2}{2g}=\frac{Q^2}{64{\cdot}4\times 0{\cdot}35^2}=0{\cdot}127Q^2$$

Quantity entering pump—

$$Q=A\sqrt{2g\left(34-\text{suction}-\text{friction}-\frac{V^2}{2g}\right)}$$

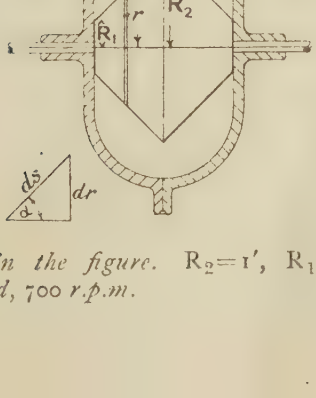
$$Q=0{\cdot}35\times 8{\cdot}02\sqrt{34-6-0{\cdot}095Q^2-0{\cdot}127Q^2}$$

$$Q^2=7{\cdot}9(28-0{\cdot}222Q^2) \qquad Q=9 \text{ cusecs}$$

$$9=\frac{N}{90}-9 \qquad N=\mathbf{1620\ r.p.m.}$$

when cavitation occurs in the eye of the pump.

20. *Find the horse-power required to rotate in water the*



body shown in the figure. $R_2=1'$, $R_1=0{\cdot}25'$, $\alpha=45°$, $f=0{\cdot}006$; *speed,* 700 *r.p.m.*

Let the velocity at periphery $= V_2$ f.p.s.; at surface of ring $= v$.

$$v = \frac{V_2 r}{R_2}$$

$$\left.\begin{matrix}\text{Skin resistance on}\\ \text{surface of ring}\end{matrix}\right\} = fv^2 2\pi r \,.\, ds = \frac{fV_2^2}{R_2^2} 2\pi r^3 \frac{dr}{\sin\alpha}$$

$$\left.\begin{matrix}\text{Moment of resistance for both}\\ \text{sides of runner}\end{matrix}\right\} = \frac{4fV_2^2\pi}{R_2^2 \sin\alpha}\int_{R_1}^{R_2} r^4 dr$$

$$M = \frac{4fV_2^2\pi}{5R_2^2 \sin\alpha}(R_2^5 - R_1^5)$$

$$\text{Horse-power} = \frac{MV_2}{550R} = \frac{fV_2^3}{219 \sin\alpha}\left(\frac{R_2^5 - R_1^5}{R_2^3}\right)$$

$$\text{H.P.} = \frac{0{\cdot}006 \times 73{\cdot}3^3}{219 \times 0{\cdot}707}\left(\frac{1 - \frac{1}{1024}}{1}\right) = \mathbf{15{\cdot}3.}$$

21. *Find the horse-power required to overcome the fluid friction on a centrifugal pump runner* 15″ *diameter, diameter of "eye"* $7\frac{1}{2}$″, *when lifting water against a head of* 170′.

$$V = 1{\cdot}24\sqrt{2gH} \qquad f = 0{\cdot}0065$$

$$V = 1{\cdot}24 \times 8{\cdot}02\sqrt{170} = 130 \text{ f.p.s.}$$

$$\text{H.P. wasted in friction} = \frac{fV^3}{219}\left(\frac{R_e^5 - R_i^5}{R_e^3}\right) = \mathbf{25.}$$

The useful horse-power of the pump will be about 240. Thus the disc friction amounts to about 10 per cent. of the whole power.

22. *An absorption dynamometer consists of a series of plain discs mounted on a shaft rotating in water and contained in a suitably balanced stationary casing mounted on the main shaft. Find the smallest number of discs required to absorb* 500 *horse-power at* 800 *r.p.m. Diameter of discs,* 3 *feet; coefficient of friction,* 0·007.

$$\text{Peripheral velocity} = \frac{3{\cdot}14 \times 3 \times 800}{60} = 126 \text{ f.p.s.}$$

$$\text{Horse-power} = \frac{0{\cdot}007 \times 126^3 \times 1{\cdot}5^2}{216} = 146$$

(See p. 770, Vol. I.)

Number of discs required, $\frac{500}{146} = \mathbf{say\ 4.}$

23. *Cooling water is supplied to a trough rim wheel of a motor when running at* 600 *r.p.m. Internal diameter of wheel,* 18″ ; *quantity of water supplied per min.,* 5 *gallons. Find the horse-power absorbed in imparting rotation to the water.*

(See p. 762, Vol. I.)

Angular momentum generated per second $= \frac{W}{g} V_w R$

$$= \frac{50}{60 \times 32{\cdot}2} \times \frac{2\pi \times 3 \times 600}{4 \times 60} \times \frac{3}{4} = 0{\cdot}91 \text{ lb-feet.}$$

$$\text{H.P.} = 0{\cdot}91 \frac{\omega}{550} = \frac{0{\cdot}91 \times 10 \times 2\pi}{550} = \mathbf{0{\cdot}1.}$$

24. *The following figures refer to a vertical condenser : Steam per hour,* 37,000 *lb. ; cooling surface,* 4,500 *sq. ft. ; tubes* 1 *inch external diameter,* 18 *s.w.g. ; length between plates,* 12′ 6″.

Circulating water is supplied at the rate of 48 *lb. per lb. steam, and makes three passes through the condenser before discharge. Estimate the power required to drive the circulating pump. Efficiency,* 75 *per cent.*

$$\text{Water per second} = \frac{37000 \times 48}{3600 \times 62{\cdot}5} = 7{\cdot}9 \text{ cusecs}$$

Let area available for flow of water be A.

$$\text{Hydraulic mean depth} = \frac{\text{area A}}{\text{perimeter}}$$

$$\therefore \frac{d}{4} = \frac{A}{\frac{1}{3} \times \frac{4500}{12{\cdot}5}} \qquad \text{Inner diameter of tubes} = 0{\cdot}904''$$

$$\therefore A = \frac{0{\cdot}904}{12 \times 4} \times \frac{4500 \times 2}{3 \times 25} = 2{\cdot}26 \text{ sq. feet}$$

$$\text{Velocity of flow} = \frac{7{\cdot}9}{2{\cdot}26} = 3{\cdot}5 \text{ f.p.s.}$$

Head lost in one passage through tubes

$$= \frac{1}{2} \cdot \frac{v^2}{2g} + \frac{1}{100} \times \frac{\frac{25}{2} \times 48}{0{\cdot}904} \cdot \frac{v^2}{2g} + \frac{v^2}{2g} = \frac{v^2}{2g} \times 8{\cdot}14.$$

Total head lost in condenser $= 3 \frac{v^2}{2g} \times 8{\cdot}14 = 4{\cdot}65$ feet.

Water horse-power $= 4{\cdot}65 \times \frac{37000 \times 48}{3600 \times 550} = 4{\cdot}17$.

Horse-power to drive circulating pump

$= 3 \times \frac{100}{75} \times 4{\cdot}17 =$ **16·7 H.P.**

APPENDIX

POWER OF NUMBERS

Calculated by L. A. Barry, B.Sc.

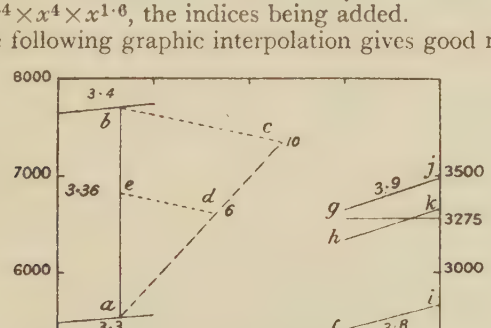
Unless one is constantly using logarithms one is very liable to make errors when getting fractional roots and powers of numbers: a table such as this is a useful check. In many instances the exact number and power can be taken direct from the tables; for intermediate numbers and powers a graphic interpolation gives approximate results.

By very simple calculations the table may be utilised for numbers and powers beyond its range, a few examples of such are given below.

Even for those familiar with logarithms it is believed that the use of the tables will save a great deal of time and labour.

It is useful to remember that, say, $x^{7\cdot3}=x^4\times x^{3\cdot3}$, and $x^{9\cdot6}=x^4\times x^4\times x^{1\cdot6}$, the indices being added.

The following graphic interpolation gives good results:



Required $27\cdot365^{3\cdot36}$. Set off on squared paper the value $27\cdot3^{3\cdot3}=5487$, $27\cdot4^{3\cdot3}=5554$ also $27\cdot3^{3\cdot4}=7638$, $27\cdot4^{3\cdot4}$

$=.7733$ from the tables. Join. Draw a vertical *ab* at 27·365, then *a* gives $27{\cdot}365^{3{\cdot}3}$ and *b* gives $27{\cdot}365^{3{\cdot}4}$, but we want $27{\cdot}365^{3{\cdot}36}$, that is 0·6 of the difference, so divide *ab* at *e* so that $ae = 0{\cdot}6\ ab$. This is most conveniently done by drawing *ac* at any angle, set off $ac = 10$ units and *ad* 6 units, join *cb*, draw *de* parallel. The point *e* on the vertical scale is 6800, which is the (approx.) value of $27{\cdot}365^{3{\cdot}36}$. The exact value as found by logarithms is 6746.

The graphical method gives results slightly too high, owing to our assuming the quantities to follow straight line laws.

Required $3663^{0{\cdot}256}$. This may be written $3663^{\frac{1}{3{\cdot}9}}$. Look up the 3·9 power column for 3663, we find the corresponding number is 8·2 which is $3663^{0{\cdot}256}$.

Required $3275^{0{\cdot}258}$ or $3275^{\frac{1}{3{\cdot}876}}$. Set off from the tables $8{\cdot}0^{3{\cdot}8} = 2702$, $8{\cdot}1^{3{\cdot}8} = 2833$, also $8{\cdot}0^{3{\cdot}9} = 3327$, $8{\cdot}1^{3{\cdot}9} = 3492$. Join. Draw a horizontal at 3275. Divide *fg* and *ij* so that $fh = 0{\cdot}76\,fg$ and $ik = 0{\cdot}76\,ij$. Join. This line cuts the 3275 line at 8·07, which is $3275^{\frac{1}{3{\cdot}876}}$ or $3275^{0{\cdot}258}$, which agrees with the exact value as found by logs, viz. 8·07.

Required $189^{1{\cdot}7} = 18{\cdot}9^{1{\cdot}7} \times 10^{1{\cdot}7}$. Look up $18{\cdot}9^{1{\cdot}7} = 147{\cdot}9$, also $10^{1{\cdot}7} = 50{\cdot}12$. Then $189^{1{\cdot}7} = 147{\cdot}9 \times 50{\cdot}12 = 7414$.

Required $1890^{1{\cdot}7} = 18{\cdot}9^{1{\cdot}7} \times 10^{1{\cdot}7} \times 10^{1{\cdot}7} = 147{\cdot}9 \times 50{\cdot}12 \times 50{\cdot}12 = 371600$.

Required $1{\cdot}89^{1{\cdot}7} = 18{\cdot}9^{1{\cdot}7} \div 10^{1{\cdot}7} = \dfrac{147{\cdot}9}{50{\cdot}12} = 2{\cdot}951$.

Required $0{\cdot}189^{1{\cdot}7} = 18{\cdot}9^{1{\cdot}7} \div (10^{1{\cdot}7} \times 10^{1{\cdot}7}) = \dfrac{147{\cdot}9}{50{\cdot}12 \times 50{\cdot}12} = 0{\cdot}0589$.

Required $18{\cdot}9^{\frac{1}{1{\cdot}7}}$. Look up tables for $19{\cdot}27^{\frac{1}{1{\cdot}7}} = 5{\cdot}7$, $18{\cdot}70^{\frac{1}{1{\cdot}7}} = 5{\cdot}6$, difference 0·1 for $0{\cdot}57^{\frac{1}{1{\cdot}7}}$. Difference required $19{\cdot}27 - 18{\cdot}9 = 0{\cdot}37$, $\dfrac{0{\cdot}37}{0{\cdot}57} = 0{\cdot}65$, hence $18{\cdot}9^{\frac{1}{1{\cdot}7}} = 5{\cdot}7 - 0{\cdot}065 = 5{\cdot}635$.

Required $18{\cdot}9^5 = 18{\cdot}9^4 \times 18{\cdot}9$. Look up $18{\cdot}9^4 = 127600$.

Then $18{\cdot}9^5 = 18{\cdot}9 \times 127{,}600 = 2{,}411{,}000$; or $18{\cdot}9^3 \times 18{\cdot}9^2$, $6751 \times 3572 = 2{,}411{,}000$.

Required $18{\cdot}9^{1{\cdot}3} = 18{\cdot}9^3 \times 18{\cdot}9^{1{\cdot}3}$. Look up $18{\cdot}9^3 = 6751$ and $18{\cdot}9^{1{\cdot}3} = 45{\cdot}65$ Then $18{\cdot}9^{4{\cdot}3} = 6751 \times 45{\cdot}65 = 308{,}200$

Required $18{\cdot}9^{0{\cdot}7} = 18{\cdot}9^{1{\cdot}7} \div 18{\cdot}9$. Look up $18{\cdot}9^{1{\cdot}7} = 147{\cdot}9$.

Then $18{\cdot}9^{0{\cdot}7} = \frac{147{\cdot}9}{18{\cdot}9} = 7{\cdot}826$.

Power

N	1·1	1·2	1·3	1·4	1·5	1·6	1·7	1·8	1·9	2·0	2·1	2·2	2·3	2·4	2·5
1·1	1·111	1·121	1·132	1·143	1·154	1·165	1·176	1·187	1·199	1·210	1·222	1·233	1·245	1·257	1·269
1·2	1·222	1·245	1·267	1·291	1·315	1·339	1·363	1·388	1·414	1·440	1·466	1·493	1·521	1·549	1·577
1·3	1·335	1·370	1·406	1·444	1·482	1·522	1·562	1·604	1·646	1·690	1·735	1·781	1·828	1·877	1·927
1·4	1·448	1·497	1·549	1·602	1·656	1·713	1·772	1·832	1·895	1·960	2·027	2·096	2·168	2·242	2·319
1·5	1·562	1·627	1·694	1·764	1·837	1·913	1·992	2·075	2·161	2·250	2·343	2·440	2·541	2·646	2·756
1·6	1·677	1·758	1·842	1·931	2·024	2·121	2·223	2·330	2·442	2·560	2·683	2·812	2·948	3·089	3·238
1·7	1·793	1·890	1·993	2·102	2·217	2·337	2·465	2·599	2·741	2·890	3·047	3·214	3·389	3·573	3·768
1·8	1·909	2·025	2·147	2·277	2·415	2·561	2·716	2·881	3·055	3·240	3·436	3·644	3·865	4·099	4·347
1·9	2·026	2·160	2·303	2·456	2·619	2·793	2·978	3·175	3·386	3·610	3·849	4·104	4·377	4·667	4·976
2·0	2·144	2·297	2·462	2·639	2·828	3·031	3·249	3·482	3·732	4·000	4·287	4·595	4·925	5·278	5·657
2·1	2·262	2·436	2·624	2·826	3·043	3·278	3·530	3·802	4·095	4·410	4·750	5·115	5·509	5·934	6·391
2·2	2·381	2·576	2·787	3·016	3·263	3·531	3·820	4·134	4·473	4·840	5·237	5·667	6·132	6·635	7·179
2·3	2·500	2·717	2·953	3·209	3·488	3·791	4·120	4·478	4·867	5·290	5·749	6·249	6·792	7·382	8·023
2·4	2·620	2·859	3·121	3·406	3·718	4·058	4·430	4·835	5·277	5·760	6·287	6·862	7·490	8·175	8·923
2·5	2·740	3·003	3·291	3·607	3·953	4·332	4·748	5·203	5·703	6·250	6·850	7·507	8·227	9·017	9·882
2·6	2·861	3·148	3·463	3·810	4·192	4·613	5·075	5·584	6·144	6·760	7·438	8·184	9·004	9·907	10·90
2·7	2·982	3·293	3·637	4·017	4·437	4·900	5·412	5·977	6·601	7·290	8·051	8·892	9·821	10·85	11·98
2·8	3·104	3·440	3·813	4·227	4·685	5·193	5·757	6·381	7·073	7·840	8·690	9·633	10·68	11·84	13·12
2·9	3·226	3·588	3·991	4·440	4·939	5·493	6·111	6·797	7·561	8·410	9·355	10·41	11·57	12·88	14·32
3·0	3·348	3·737	4·171	4·656	5·196	5·800	6·473	7·225	8·064	9·000	10·05	11·21	12·51	13·97	15·59
3·1	3·471	3·887	4·353	4·874	5·458	6·112	6·844	7·664	8·582	9·610	10·76	12·05	13·49	15·11	16·92
3·2	3·595	4·038	4·536	5·096	5·724	6·430	7·224	8·115	9·016	10·24	11·50	12·92	14·52	16·31	18·32
3·3	3·718	4·190	4·721	5·320	5·995	6·755	7·612	8·577	9·664	10·89	12·27	13·83	15·58	17·56	19·78
3·4	3·843	4·343	4·908	5·547	6·269	7·085	8·008	9·050	10·23	11·56	13·06	14·77	16·69	18·86	21·32
3·5	3·967	4·497	5·097	5·777	6·548	7·422	8·412	9·535	10·81	12·25	13·88	15·74	17·84	20·22	22·92
3·6	4 092	4·651	5·287	6·009	6·831	7·764	8·825	10·03	11·40	12·96	14·73	16·74	19·03	21·63	24·59
3·7	4·217	4·807	5·479	6·244	7·117	8·112	9·246	10·54	12·01	13·69	15·60	17·78	20·27	23·10	26·33
3·8	4·343	4·963	5·672	6·482	7·408	8·466	9·675	11·06	12·64	14·44	16·50	18·86	21·55	24·63	28·15
3·9	4·469	5·120	5·867	6·722	7·702	8·825	10·11	11·59	13·27	15·21	17·43	19·97	22·88	26·22	30·04
4·0	4·595	5·278	6·063	6·964	8·000	9·190	10·56	12·13	13·93	16·00	18·38	21·11	24·25	27·86	32·00

N	2·6	2·7	2·8	2·9	3·0	3·1	3·2	3·3	3·4	3·5	3·6	3·7	3·8	3·9	4·0
1·1	1·281	1·293	1·306	1·318	1·331	1·344	1·357	1·370	1·383	1·396	1·409	1·423	1·436	1·450	1·464
1·2	1·606	1·636	1·666	1·697	1·728	1·760	1·792	1·825	1·859	1·893	1·928	1·963	1·999	2·036	2·074
1·3	1·978	2·031	2·085	2·140	2·197	2·255	2·315	2·377	2·440	2·505	2·572	2·640	2·710	2·782	2·856
1·4	2·398	2·481	2·565	2·653	2·744	2·838	2·935	3·035	3·139	3·247	3·358	3·473	3·592	3·714	3·842
1·5	2·870	2·988	3·112	3·241	3·375	3·515	3·661	3·812	3·969	4·134	4·305	4·483	4·668	4·861	5·062
1·6	3·394	3·557	3·729	3·908	4·096	4·293	4·500	4·716	4·943	5·181	5·430	5·692	5·966	6·253	6·554
1·7	3·973	4·190	4·418	4·659	4·913	5·181	5·463	5·761	6·075	6·406	6·755	7·123	7·511	7·920	8·352
1·8	4·610	4·889	5·185	5·499	5·832	6·185	6·560	6·957	7·378	7·824	8·298	8·801	9·333	9·898	10·50
1·9	5·306	5·658	6·033	6·433	6·859	7·314	7·799	8·215	8·867	9·455	10·08	10·75	11·46	12·22	13·03
2·0	6·063	6·498	6·964	7·464	8·000	8·574	9·190	9·849	10·56	11·31	12·13	13·00	13·93	14·93	16·00
2·1	6·883	7·413	7·984	8·599	9·261	9·974	10·74	11·57	12·46	13·42	14·45	15·57	16·77	18·06	19·45
2·2	7·768	8·405	9·095	9·841	10·65	11·52	12·47	13·49	14·60	15·79	17·09	18·49	20·01	21·65	23·43
2·3	8·720	9·477	10·30	11·19	12·17	13·22	14·37	15·62	16·98	18·45	20·05	21·80	22·11	25·75	27·98
2·4	9·740	10·63	11·60	12·67	13·82	15·09	16·47	17·98	19·62	21·42	23·38	25·51	27·85	30·40	33·18
2·5	10·83	11·87	13·01	14·26	15·62	17·12	18·77	20·57	22·54	24·71	27·08	29·67	32·52	35·64	39·06
2·6	11·99	13·20	14·52	15·97	17·58	19·34	21·28	23·41	25·76	28·34	31·18	34·31	37·75	41·53	45·70
2·7	13·23	14·61	16·14	17·82	19·68	21·74	24·01	26·52	29·28	32·34	35·72	39·45	43·57	48·12	53·14
2·8	14·54	16·12	17·87	19·80	21·95	24·33	26·97	29·90	33·14	36·73	40·72	45·13	50·03	55·45	61·47
2·9	15·93	17·72	19·71	21·93	24·39	27·13	30·18	33·57	37·34	41·53	46·20	51·39	58·49	63·58	70·73
3·0	17·40	19·42	21·67	24·19	27·00	30·14	33·63	37·54	41·90	46·77	52·20	58·26	65·02	72·57	81·00
3·1	18·95	21·22	23·76	26·60	29·79	33·36	37·36	41·83	46·84	52·45	58·74	65·77	73·65	82·47	92·35
3·2	20·58	23·12	25·97	29·17	32·77	36·81	41·35	46·45	52·18	58·62	65·85	73·97	83·09	93·34	104·9
3·3	22·29	25·12	28·30	31·89	35·94	40·49	45·63	51·42	57·94	65·28	73·56	82·89	93·40	105·2	118·6
3·4	24·09	27·23	30·77	34·78	39·30	44·42	50·20	56·74	64·13	72·47	81·91	92·57	104·6	118·2	133·6
3·5	25·98	29·44	33·37	37·83	42·87	48·60	55·08	62·43	70·77	80·21	90·92	103·1	116·8	132·4	150·1
3·6	27·95	31·77	36·11	41·05	46·66	53·03	60·28	68·52	77·88	88·52	100·6	114·4	130·0	147·8	168·0
3·7	30·01	34·21	38·99	44·44	50·65	57·73	65·80	75·00	85·48	97·43	111·1	126·6	144·3	164·4	187·4
3·8	32·17	36·76	42·01	48·01	54·87	62·71	71·67	81·90	93·60	107·0	122·2	139·7	159·7	182·5	208·5
3·9	34·42	39·43	45·18	51·77	59·32	67·97	77·88	89·23	102·2	117·1	134·2	153·8	176·2	201·9	231·3
4·0	36·76	42·22	48·50	55·72	64·00	73·52	84·45	97·01	111·4	128·0	147·0	168·9	194·0	222·9	256·0

Power

N	1·1	1·2	1·3	1·4	1·5	1·6	1·7	1·8	1·9	2·0	2·1	2·2	2·3	2·4	2·5
4·1	4·721	5·437	6·261	7·209	8·302	9·560	11·01	12·68	14·60	16·81	19·36	22·29	25·67	29·56	34·04
4·2	4·848	5·596	6·460	7·457	8·507	9·936	11·47	13·24	15·28	17·64	20·36	23·50	27·13	31·32	36·15
4·3	4·975	5·757	6·661	7·706	8·917	10·32	11·94	13·81	15·98	18·49	21·39	24·75	28·64	33·14	38·34
4·4	5·103	5·918	6·863	7·959	9·230	10·70	12·41	14·40	16·69	19·36	22·45	26·04	30·20	35·02	40·61
4·5	5·230	6·079	7·066	8·213	9·546	11·10	12·90	14·99	17·42	20·25	23·54	27·36	31·80	36·96	42·96
4·6	5·358	6·242	7·271	8·470	9·866	11·49	13·39	15·59	18·17	21·16	24·65	28·71	33·45	38·96	45·38
4·7	5·487	6·405	7·477	8·728	10·19	11·89	13·89	16·21	18·92	22·09	25·79	30·10	35·14	41·02	47·89
4·8	5·615	6·569	7·684	8·990	10·52	12·30	14·39	16·84	19·70	23·04	26·95	31·53	36·89	43·15	50·48
4·9	5·744	6·733	7·893	9·253	10·85	12·71	14·91	17·47	20·48	24·01	28·15	32·99	38·68	45·34	53·15
5·0	5·873	6·899	8·103	9·518	11·18	13·13	15·43	18·12	21·28	25·00	29·37	34·49	40·52	47·59	55·90
5·1	6·002	7·065	8·315	9·786	11·52	13·56	15·95	18·78	22·10	26·01	30·61	36·03	42·40	49·91	58·74
5·2	6·132	7·231	8·527	10·06	11·86	13·98	16·49	19·44	22·93	27·04	31·89	37·60	44·34	52·29	61·66
5·3	6·262	7·398	8·741	10·33	12·20	14·42	17·03	20·12	23·78	28·09	33·19	39·21	46·33	54·73	64·67
5·4	6·392	7·566	8·956	10·60	12·55	14·85	17·58	20·81	24·63	29·16	34·52	40·86	48·36	57·25	67·76
5·5	6·522	7·735	9·172	10·88	12·90	15·30	18·14	21·51	25·51	30·25	35·87	42·54	50·45	59·82	70·94
5·6	6·653	7·904	9·390	11·15	13·25	15·74	18·70	22·22	26·40	31·36	37·26	44·26	52·58	62·47	74·21
5·7	6·784	8·073	9·608	11·43	13·61	16·20	19·27	22·94	27·30	32 49	38·67	46·02	54·77	65·18	77·57
5·8	6·915	8·244	9·828	11·72	13·97	16·65	19·85	23·67	28·22	33·64	40·10	47·81	57·00	67·96	81·02
5·9	7·046	8·414	10·05	12·00	14·33	17·11	20·44	24·41	29·15	34·81	41·57	49·64	59·29	70·80	84·55
6·0	7·177	8·586	10·27	12·29	14·70	17·58	21·03	25·16	30·09	36·00	43·06	51·51	61·62	73·72	88·18
6·1	7·309	8·758	10·49	12·57	15·07	18·05	21·63	25·92	31·05	37·21	44·59	53·42	64·01	76·70	91·90
6·2	7·441	8·930	10·72	12·86	15·44	18·53	22·24	26·69	32·03	38·44	46·13	55·37	66·45	79·75	95·71
6·3	7·573	9·104	10·94	13·15	15·81	19·01	22·85	27·47	33·02	39·69	47·71	57·35	68·94	82·87	99·62
6·4	7·705	9·277	11·17	13·45	16·19	19·49	23·47	28·26	34·02	40·96	49·31	59·37	71·48	86·07	103·6
6·5	7·838	9·451	11·40	13·74	16·57	19·98	24·10	29·06	35·04	42·25	50·95	61·43	74·08	89·33	107·7
6·6	7·971	9·626	11·63	14·04	16·96	20·48	24·73	29·87	36·07	43·56	52·61	63·53	76·73	92·66	111·9
6·7	8·104	9·801	11·85	14·34	17·34	20·98	25·37	30·69	37·11	44·89	54·29	65·67	79·43	96·07	116·2
6·8	8·237	9·977	12·09	14·64	17·73	21·48	26·02	31·51	38·17	46·24	56·01	67·85	82·18	99·55	120·6
6·9	8·370	10·15	12·32	14·94	18·12	21·99	26·67	32·35	39·25	47·61	57·75	70·06	84·99	103·1	125·1
7·0	8·504	10·33	12·55	15·25	18·52	22·50	27·33	33·20	40·34	49·00	59·53	72·31	87·85	106·7	129·6

Power

N	2·6	2·7	2·8	2·9	3·0	3·1	3·2	3·3	3·4	3·5	3·6	3·7	3·8	3·9	4·0
4·1	39·20	45·14	51·98	59·85	68·92	79·37	91·39	105·2	121·2	139·6	160·7	185·1	213·1	245·4	282·6
4·2	41·73	48·17	55·60	64·18	74·09	85·52	98·72	114·0	131·5	151·8	175·3	202·3	233·5	269·6	311·2
4·3	44·36	51·33	59·39	68·72	79·51	91·99	106·4	123·2	142·5	164·9	190·8	220·7	255·4	295·5	341·9
4·4	47·10	54·62	63·34	73·45	85·18	98·79	114·6	132·9	154·1	178·7	207·2	240·3	278·7	323·2	374·8
4·5	49·93	58·03	67·59	78·40	91·13	105·9	123·1	143·1	166·3	193·3	224·7	261·1	303·5	352·8	410·1
4·6	52·87	61·58	71·73	83·56	97·34	113·4	132·1	153·9	179·2	208·8	243·2	283·3	330·0	384·4	447·7
4·7	55·91	65·26	76·19	88·94	103·8	121·2	141·5	165·2	192·8	225·1	262·8	306·7	358·1	418·0	488·0
4·8	59·05	69·08	80·81	94·54	110·6	129·4	151·3	177·1	207·1	242·3	283·4	331·6	387·9	453·8	530·8
4·9	62·30	73·03	85·61	100·4	117·6	137·9	161·7	189·5	222·2	260·4	305·3	357·9	419·5	491·8	576·5
5·0	65·66	77·13	90·60	106·4	125·0	146·8	172·5	202·6	238·0	279·5	328·3	385·6	453·0	532·1	625·0
5·1	69·13	81·37	95·76	112·7	132·7	156·1	183·7	216·3	254·5	299·6	352·6	415·0	488·4	574·8	676·5
5·2	72·71	85·74	101·1	119·2	140·6	165·8	195·5	230·6	271·9	320·6	378·1	445·9	525·8	620·0	731·2
5·3	76·40	90·27	106·7	126·0	148·9	175·9	207·8	245·5	290·1	342·7	404·9	478·4	565·3	667·8	789·0
5·4	80·21	94·99	112·4	133·0	157·5	186·4	220·6	261·2	309·1	365·9	433·1	512·7	606·9	718·4	850·3
5·5	84·13	99·77	118·3	140·3	166·4	197·3	234·0	277·5	329·0	390·2	462·7	548·7	650·7	771·6	915·1
5·6	88·16	104·7	124·4	147·8	175·6	208·6	247·9	294·5	349·8	415·6	493·7	586·5	696·8	827·8	983·4
5·7	92·32	109·9	130·8	155·6	185·2	220·4	262·3	312·2	371·5	442·1	526·2	626·2	745·3	887·0	105 6
5·8	96·59	115·1	137·3	163·7	195·1	232·6	277·3	330·6	394·1	469·9	560·2	667·9	796·2	949 2	113 2
5·9	101·0	120·6	144·0	172·0	205·4	245·3	292·9	349·8	417·7	498·9	595·8	711·5	849·6	101 5	121 2
6·0	105·5	126·2	150·9	180·6	216·0	258·4	309·1	369·7	442·3	529·1	632·9	757·1	905·7	108 3	129 6
6·1	110·1	131·9	158·1	189·4	227·0	272·0	325·9	390·5	467·9	560·6	671·7	804·9	964·4	115 6	138 5
6·2	114·9	137·9	165·5	198·6	238·3	286·0	343·3	412·0	494·5	593·4	712·2	854·8	102 6	123 1	147 8
6·3	119·8	144·0	173·0	208·0	250·0	300·6	361·3	434·3	522·1	627·6	754·4	906·9	109 0	131 0	157 5
6·4	124·8	150·2	180·8	217·7	262·1	315·6	380·0	457·5	550·8	663·2	798·5	961·3	115 7	139 3	167 8
6·5	129·9	156·6	188·9	227·7	274·6	331·2	399·3	481·5	580·6	700·2	844·3	101 8	122 8	148 0	178 5
6·6	135·1	163·2	197·1	238·1	287·5	347·2	419·3	506·4	611·6	738·6	892·0	107 7	130 1	157 1	189 7
6·7	140·5	170·0	205 6	248·7	300·8	363·8	440·0	532·2	643·7	778·5	941·6	113 9	137 7	166 6	201 5
6·8	146·1	176·9	214·3	259·6	314·4	380·9	461·3	558·8	676·9	819·9	993·2	120 3	145 7	176 5	213 8
6·9	151·7	184·0	223·2	270·8	328·5	398·5	483·4	586·4	711·4	862·9	104 7	127 0	154 0	186 9	226 7
7·0	157·5	191·3	232·4	282·3	343·0	416·7	506·2	614·9	747·0	907·5	110 2	133 9	162 7	197 6	240 1

Power

N	1·1	1·2	1·3	1·4	1·5	1·6	1·7	1·8	1·9	2·0	2·1	2·2	2·3	2·4	2·5
7·1	8·637	10·51	12·78	15·55	18·92	23·02	28·00	34·06	41·44	50·41	61·33	74·60	90·76	110·4	134·3
7·2	8·771	10·69	13·02	15·86	19·32	23·54	28·67	34·93	42·55	51·84	63·15	76·94	93·73	114·2	139·1
7·3	8·905	10·86	13·25	16·17	19·72	24·06	29·35	35·81	43·68	53·29	65·01	79·31	96·75	118·0	144·0
7·4	9·040	11·04	13·49	16·48	20·13	24·59	30·04	36·70	44·83	54·76	66·89	81·72	99·82	121·9	149·0
7·5	9·174	11·22	13·73	16·79	20·54	25·12	30·73	37·59	45·98	56·25	68·81	84·17	103·0	125·9	154·0
7·6	9·309	11·40	13·97	17·11	20·95	25·66	31·43	38·50	47·16	57·76	70·75	86·65	106·1	130·0	159·2
7·7	9·444	11·58	14·20	17·42	21·37	26·21	32·14	39·42	48·34	59·29	72·72	89·18	109·4	134·1	164·5
7·8	9·579	11·76	14·45	17·74	21·78	26·75	32·85	40·34	49·54	60·84	74·71	91·75	112·7	138·4	169·9
7·9	9·714	11·94	14·69	18·06	22·20	27·30	33·57	41·28	50·76	62·41	76·74	94·36	116·0	142·7	175·4
8·0	9·849	12·13	14·93	18·38	22·63	27·86	34·30	42·22	51·98	64·00	78·79	97·01	119·4	147·0	181·0
8·1	9·985	12·31	15·17	18·70	23·05	28·42	35·03	43·18	53·23	65·61	80·88	99·69	122·9	151·5	186·7
8·2	10·12	12·49	15·42	19·03	23·48	28·98	35·77	44·14	54·48	67·24	82·99	102·4	126·4	156·0	192·5
8·3	10·26	12·67	15·66	19·35	23·91	29·55	36·51	45·12	55·75	68·89	85·13	105·2	130·0	160·6	198·5
8·4	10·39	12·86	15·91	19·68	24·35	30·12	37·26	46·10	57·03	70·56	87·29	108·0	133·6	165·3	204·5
8·5	10·53	13·04	16·15	20·01	24·78	30·70	38·02	47·09	58·33	72·25	89·49	110·8	137·3	170·1	210·6
8·6	10·66	13·23	16·40	20·34	25·22	31·27	38·78	48·09	59·64	73·96	91·72	113·7	141·0	174·9	216·9
8·7	10·80	13·41	16·65	20·67	25·66	31·86	39·55	49·11	60·97	75·69	93·97	116·7	144·8	179·8	223·3
8·8	10·94	13·60	16·90	21·00	26·11	32·45	40·33	50·13	62·30	77·44	96·25	119·6	148·7	184·8	229·7
8·9	11·07	13·78	17·15	21·34	26·55	33·04	41·11	51·16	63·66	79·21	98·56	122·6	152·6	189·9	236·3
9·0	11·21	13·97	17·40	21·67	27·00	33·63	41·90	52·20	65·02	81·00	100·9	125·7	156·6	195·1	243·0
9·1	11·35	14·15	17·65	22·01	27·45	34·23	42·69	53·24	66·40	82·81	103·3	128·8	160·6	200·3	249·8
9·2	11·49	14·34	17·90	22·35	27·91	34·84	43·50	54·30	67·79	84·64	105·7	131·9	164·7	205·6	256·7
9·3	11·62	14·53	18·16	22·69	28·36	35·45	44·30	55·37	69·20	86·49	108·1	135·1	168·9	211·0	263·8
9·4	11·76	14·71	18·41	23·03	28·82	36·06	45·11	56·45	70·62	88·36	110·6	138·3	173·1	216·5	270·9
9·5	11·90	14·90	18·67	23·38	29·28	36·67	45·93	57·53	72·06	90·25	113·0	141·6	177·3	222·1	278·2
9·6	12·04	15·09	18·92	23·72	29·74	37·29	46·76	58·63	73·50	92·16	115·5	144·9	181·6	227·7	285·5
9·7	12·17	15·28	19·18	24·07	30·21	37·92	47·59	59·73	74·97	94·09	118·1	148·2	186·0	233·5	293·0
9·8	12·31	15·47	19·44	24·42	30·68	38·54	48·43	60·84	76·44	96·04	120·7	151·6	190·5	239·3	300·7
9·9	12·45	15·66	19·69	24·77	31·15	39·18	49·27	61·96	77·93	98·01	123·3	155·0	195·0	245·2	308·4
10·0	12·59	15·85	19·95	25·12	31·62	39·81	50·12	63·10	79·43	100·0	125·9	158·5	199·5	251·2	316·2

Power

N	2·6	2·7	2·8	2·9	3·0	3·1	3·2	3·3	3·4	3·5	3·6	3·7	3·8	3·9	4·0
7·1	163·4	198·8	241·8	294·2	357·9	435·4	529·7	644·4	783·9	953·7	1160	1411	1717	2089	2541
7·2	169·5	206·4	251·5	306·4	373·2	454·7	553·9	674·8	822·1	100 2	1220	1486	1811	2206	2689
7·3	175·6	214·3	261·4	318·9	389·0	474·6	578·9	706·3	861·6	105 1	1282	1564	1908	2328	2840
7·4	182·0	222·3	271·5	331·7	405·2	495·0	604·7	738·7	902·4	110 2	1347	1645	2009	2455	2999
7·5	188·4	230·5	281·9	344·9	421·9	516·0	631·2	772·1	944·5	115 5	1413	1729	2115	2587	3164
7·6	195·0	238·9	292·6	358·4	439·0	537·7	658·6	806·7	988·0	121 0	1482	1816	2224	2724	3336
7·7	201·8	247·5	303·5	372·2	456·5	559·9	686·7	842·2	103 3	126 7	1554	1906	2337	2866	3515
7·8	208·7	256·2	314·7	386·4	474·6	582·8	715·7	878·8	107 9	132 5	1628	1999	2454	3014	3702
7·9	215·7	265·2	326·1	401·0	493·0	606·2	745·4	916·6	112 7	138 6	1704	2095	2576	3168	3895
8·0	222·9	274·4	337·8	415·9	512·0	630·3	776·0	955·4	117 6	144 8	1783	2195	2702	3327	4096
8·1	230·2	283·7	349·8	431·1	531·4	655·1	807·5	995·4	122 7	151 3	1864	2298	2833	3492	4305
8·2	237·6	293·3	362·0	446·7	551·4	680·5	839·9	103 7	127 9	157 9	1949	2405	2968	3663	4521
8·3	245·2	303·0	374·5	462·7	571·8	706·5	873·1	108 0	133 3	164 7	2036	2515	3108	3841	4746
8·4	253·0	313·0	387·2	479·1	592·7	733·3	907·2	112 2	138 9	171 8	2125	2629	3253	4024	4979
8·5	260·9	323·2	400·3	495·8	614·1	760·7	942·2	116 7	144 6	179 0	2218	2747	3402	4214	5220
8·6	269·0	333·5	413·6	512·9	636·1	788·8	978·1	121 3	150 4	186 5	2313	2868	3557	4411	5470
8·7	277·2	344·1	427·2	530·4	658·5	817·5	101 5	126 0	156 4	194 2	2411	2994	3717	4614	5729
8·8	285·5	354·9	441·1	548·3	681·5	847·0	105 3	130 9	162 6	202 2	2513	3123	3882	4825	5997
8·9	294·0	365·9	455·3	566·5	705·0	877·2	109 2	135 8	169 0	210 3	2617	3256	4052	5042	6274
9·0	302·7	377·1	469·8	585·2	729·0	908·1	113 1	140 9	175 6	218 7	2724	3394	4228	5267	6561
9·1	311·5	388·5	484·5	604·3	753·6	939·8	117 2	146 2	182 3	227 3	2835	3536	4409	5499	6857
9·2	320·5	400·2	499·6	623·7	778·7	972·2	121 4	151 5	189 2	236 2	2949	3681	4596	5738	7164
9·3	329·7	412·0	514·9	643·6	804·4	100 5	125 6	157 0	196 3	245 3	3066	3832	4789	5985	7481
9·4	338·9	424·1	530·6	663·9	830·6	103 9	130 0	162 7	203 5	254 7	3186	3986	4988	6240	7807
9·5	348·4	436·4	546·5	684·5	857·4	107 4	134 5	168 5	211 0	264 3	3310	4146	5192	6503	8145
9·6	358·0	448·9	562·8	705·6	884·7	110 9	139 1	174 4	218 6	274 1	3437	4309	5403	6774	8493
9·7	367·8	461·6	579·4	727·2	912·7	114 5	143 8	180 4	226 5	284 3	3568	4478	5620	7054	8853
9·8	377·7	474·6	596·3	749·1	941·2	118 3	148 6	186 7	234 5	294 6	3702	4651	5843	7341	9224
9·9	387·8	487·8	613·4	771·5	970·3	122 0	153 5	193 0	242 8	301 1	3840	4829	6073	7638	9606
10·0	398·1	501·2	631·0	794·3	100 0	125 9	158 5	199 5	251 2	316 2	3981	5012	6310	7943	10000

Power

N	1·1	1·2	1·3	1·4	1·5	1·6	1·7	1·8	1·9	2·0	2·1	2·2	2·3	2·4	2·5
10·1	12·73	16·04	20·21	25·47	32·10	40·45	50·97	64·24	80·95	102·0	128·6	162·0	204·1	257·3	324·2
10·2	12·87	16·23	20·47	25·82	32·58	41·09	51·83	65·39	82·48	104·0	131·2	165·5	208·8	263·4	332·3
10·3	13·01	16·42	20·73	26·18	33·06	41·74	52·70	66·54	84·02	106·1	134·0	169·1	213·6	269·7	340·5
10·4	13·14	16·61	21·00	26·54	33·54	42·39	53·57	67·71	85·58	108·2	136·7	172·8	218·4	276·0	348·8
10·5	13·28	16·80	21·26	26·89	34·02	43·04	54·45	68·89	87·15	110·2	139·5	176·4	223·2	282·4	357·3
10·6	13·42	17·00	21·52	27·25	34·51	43·70	55·34	70·07	88·73	112·4	142·3	180·2	228·1	288·9	365·8
10·7	13·56	17·19	21·79	27·61	35·00	44·36	56·23	71·27	90·33	114·5	145·1	183·9	233·1	295·5	374·5
10·8	13·70	17·38	22·05	27·98	35·49	45·03	57·12	72·47	91·92	116·6	148·0	187·7	238·2	302·1	383·3
10·9	13·84	17·58	22·32	28·34	35·99	45·70	58·03	73·68	93·56	118·8	150·9	191·6	243·3	308·9	392·3
11·0	13·98	17·77	22·58	28·70	36·48	46·37	58·93	74·90	95·20	121·0	153·8	195·5	248·4	315·7	401·3
11·1	14·12	17·96	22·85	29·07	36·98	47·05	59·85	76·13	96·85	123·2	156·7	199·4	253·7	322·7	410·5
11·2	14·26	18·16	23·12	29·44	37·48	47·73	60·77	77·37	98·52	125·4	159·7	203·4	258·9	329·7	419·8
11·3	14·40	18·35	23·39	29·81	37·99	48·41	61·69	78·62	100·2	127·7	162·7	207·4	264·3	336·8	429·2
11·4	14·54	18·55	23·66	30·18	38·49	49·10	62·62	79·88	101·9	130·0	165·8	211·4	269·7	344·0	438·8
11·5	14·68	18·74	23·93	30·55	39·00	49·79	63·56	81·14	103·6	132·2	168·8	215·5	275·2	351·3	448·5
11·6	14·82	18·94	24·20	30·92	39·51	50·48	64·50	82·42	105·3	134·6	171·9	219·7	280·7	358·7	458·3
11·7	14·96	19·13	24·47	31·29	40·02	51·18	65·45	83·70	107·0	136·9	175·1	223·9	286·3	366·1	468·2
11·8	15·10	19·33	24·74	31·67	40·53	51·88	66·40	84·99	108·8	139·2	178·2	228·1	292·0	373·7	478·3
11·9	15·24	19·53	25·02	32·05	41·05	52·59	67·36	86·29	110·5	141·6	181·4	232·4	297·7	381·3	488·5
12·0	15·39	19·72	25·29	32·42	41·57	53·30	68·33	87·60	112·3	144·0	184·6	236·7	303·5	389·1	498·8
12·1	15·53	19·92	25·56	32·80	42·09	54·01	69·30	88·92	114·1	146·4	187·9	241·1	309·3	396·9	509·3
12·2	15·67	20·12	25·84	33·18	42·61	54·72	70·28	90·25	115·9	148·8	191·1	245·5	315·2	404·8	519·9
12·3	15·81	20·32	26·11	33·56	43·14	55·44	71·26	91·59	117·7	151·3	194·4	249·9	321·2	412·8	530·6
12·4	15·95	20·52	26·39	33·95	43·66	56·17	72·25	92·93	119·5	153·8	197·8	254·4	327·2	420·9	541·4
12·5	16·09	20·72	26·67	34·33	44·19	56·89	73·24	94·28	121·4	156·2	201·1	258·9	333·3	429·1	552·4
12·6	16·23	20·91	26·95	34·72	44·73	57·62	74·24	95·65	123·2	158·8	204·5	263·5	339·5	437·4	563·5
12·7	16·38	21·11	27·22	35·10	45·26	58·36	75·24	97·02	125·1	161·3	208·0	268·1	345·7	445·8	574·8
12·8	16·52	21·31	27·50	35·49	45·79	59·09	76·25	98·40	127·0	163·8	211·4	272·8	352·0	454·3	586·2
12·9	16·66	21·51	27·78	35·88	46·33	59·83	77·27	99·78	128·9	166·4	214·9	277·5	358·4	462·8	597·7
13·0	16·80	21·71	28·06	36·27	46·87	60·58	78·29	101·2	130·8	169·0	218·4	282·3	364·8	471·5	609·3

N	2·6	2·7	2·8	2·9	3·0	3·1	3·2	3·3	3·4	3·5	3·6	3·7	3·8	3·9	4·0
10·1	408·5	514·8	648·8	817·6	1030	1298	1636	2062	2598	3274	4126	5200	6553	8258	1041
10·2	419·1	528·7	666·9	841·3	1061	1339	1689	2130	2687	3389	4275	5393	6803	8581	1082
10·3	429·9	542·8	685·4	865·4	1093	1380	1742	2200	2777	3507	4428	5591	7060	8914	1126
10·4	440·8	557·2	704·2	890·0	1125	1422	1797	2271	2870	3628	4585	5795	7324	9256	1170
10·5	452·0	571·8	723·3	915·1	1158	1464	1853	2344	2965	3751	4745	6003	7595	9608	1216
10·6	463·2	586·6	742·8	940·6	1191	1508	1910	2418	3062	3878	4910	6218	7873	9970	1262
10·7	474·7	601·6	762·6	966·5	1225	1553	1968	2494	3162	4007	5079	6438	8159	1034	1311
10·8	486·3	616·9	782·7	992·7	1260	1598	2027	2572	3263	4140	5252	6663	8453	1072	1360
10·9	498·1	632·5	803·1	102 0	1295	1644	2088	2652	3367	4276	5429	6894	8754	1112	1412
11·0	510·1	648·3	823·9	104 7	1331	1692	2150	2733	3473	4414	5611	7131	9063	1152	1464
11·1	522·2	664·3	845·1	107 5	1368	1740	2213	2816	3582	4556	5796	7374	9381	1193	1518
11·2	534·5	680·6	866·6	110 3	1405	1789	2278	2900	3693	4702	5987	7623	9706	1236	1574
11·3	547·0	697·1	888·4	113 2	1443	1839	2343	2986	3806	4850	6181	7878	1004	1279	1630
11·4	559·7	713·9	910·6	116 2	1482	1890	2410	3075	3922	5002	6381	8139	1038	1324	1689
11·5	572·6	730·9	933·2	119 1	1521	1942	2479	3164	4040	5158	6584	8406	1073	1370	1749
11·6	585·6	748·2	956·1	122 2	1561	1994	2548	3256	4161	5316	6793	8680	1109	1417	1811
11·7	598·8	765·8	979·3	125 2	1602	2048	2619	3350	4284	5478	7006	8960	1146	1465	1874
11·8	612·2	783·6	100 3	128 4	1643	2103	2692	3445	4410	5644	7224	9246	1183	1515	1939
11·9	625·8	801·6	102 7	131 5	1685	2159	2765	3542	4538	5813	7447	9539	1222	1565	2005
12·0	639·5	820·0	105 1	134 8	1728	2215	2840	3642	4669	5986	7675	9839	1262	1617	2074
12·1	653·5	838·5	107 5	138 1	1772	2273	2917	3743	4803	6162	7907	1015	1302	1671	2144
12·2	667·6	857·4	110 1	141 4	1816	2332	2995	3846	4939	6342	8145	1046	1343	1725	2215
12·3	682·0	876·5	112 7	144 8	1861	2392	3074	3951	5078	6526	8388	1078	1386	1781	2289
12·4	696·5	895·9	115 2	148 2	1907	2452	3155	4058	5220	6714	8636	1111	1429	1838	2364
12·5	711·2	915·5	117 9	151 7	1953	2514	3237	4167	5364	6905	8889	1144	1473	1870	2441
12·6	726·0	935·4	120 5	155 3	2000	2577	3320	4278	5511	7101	9148	1179	1518	1956	2520
12·7	741·1	955·6	123 2	158 9	2048	2641	3405	4391	5662	7300	9412	1214	1565	2018	2601
12·8	756·4	976·0	125 9	162 5	2097	2706	3492	4506	5815	7503	9682	1249	1612	2080	2684
12·9	771·9	996·8	128 7	166 2	2147	2772	3580	4623	5970	7710	9957	1286	1661	2144	2769
13·0	787·5	101 8	131 5	170 0	2197	2839	3670	4743	6129	7921	1024	1323	1710	2210	2856

Add one cypher

N	1·1	1·2	1·3	1·4	1·5	1·6	1·7	1·8	1·9	2·0	2·1	2·2	2·3	2·4	2·5
13·1	16·94	21·91	28·34	36·66	47·41	61·32	79·32	102·6	132·7	171·6	222·0	287·1	371·3	480·2	621·1
13·2	17·09	22·12	28·63	37·05	47·96	62·08	80·35	104·0	134·6	174·2	225·5	291·9	377·9	489·1	633·0
13·3	17·23	22·32	28·91	37·44	48·50	62·83	81·39	105·4	136·6	176·9	229·1	296·8	384·5	498·0	645·1
13·4	17·37	22·52	29·19	37·84	49·05	63·59	82·43	106·9	138·5	179·6	232·8	301·7	391·1	507·1	657·3
13·5	17·51	22·72	29·47	38·24	49·60	64·35	83·48	108·3	140·5	182·2	236·4	306·7	397·9	516·2	669·6
13·6	17·66	22·92	29·76	38·63	50·15	65·11	84·53	109·7	142·5	185·0	240·1	311·7	404·7	525·4	682·1
13·7	17·80	23·12	30·04	39·03	50·71	65·88	85·59	111·2	144·5	187·7	243·8	316·8	411·6	534·7	694·7
13·8	17·94	23·33	30·33	39·43	51·26	66·65	86·66	112·7	146·5	190·4	247·6	321·9	418·5	544·1	707·5
13·9	18·08	23·53	30·61	39·83	51·82	67·43	87·73	114·1	148·5	193·2	251·4	327·1	425·5	553·7	720·3
14·0	18·23	23·73	30·90	40·23	52·38	68·20	88·80	115·6	150·5	196·0	255·2	332·3	432·6	563·3	733·4
14·1	18·37	23·94	31·19	40·64	52·95	68·98	89·88	117·1	152·6	198·8	259·0	337·5	439·7	573·0	746·5
14·2	18·51	24·14	31·48	41·04	53·51	69·77	90·97	118·6	154·6	201·6	262·9	342·8	447·0	582·8	759·8
14·3	18·66	24·34	31·76	41·44	54·08	70·56	92·06	120·1	156·7	204·5	266·8	348·1	454·2	592·7	773·3
14·4	18·80	24·55	32·05	41·85	54·64	71·35	93·16	121·6	158·8	207·4	270·7	353·5	461·6	602·7	786·9
14·5	18·95	24·75	32·34	42·26	55·21	72·14	94·26	123·2	160·9	210·2	274·7	358·9	469·0	612·8	800·6
14·6	19·09	24·96	32·63	42·67	55·79	72·94	95·37	124·7	163·0	213·2	278·7	364·4	476·4	622·9	814·5
14·7	19·23	25·16	32·92	43·08	56·36	73·74	96·48	126·2	165·2	216·1	282·7	369·9	484·0	633·2	828·5
14·8	19·38	25·37	33·22	43·49	56·94	74·55	97·60	127·8	167·3	219·0	286·8	375·5	491·6	643·6	842·7
14·9	19·52	25·58	33·51	43·90	57·51	75·35	98·72	129·3	169·5	222·0	290·9	381·1	499·3	654·1	857·0
15·0	19·67	25·78	33·80	44·31	58·09	76·16	99·85	130·9	171·6	225·0	295·0	386·7	507·0	664·7	871·4
15·1	19·81	25·99	34·09	44·73	58·68	76·98	101·0	132·5	173·8	228·0	299·1	392·4	514·8	675·4	886·0
15·2	19·95	26·19	34·39	45·14	59·26	77·79	102·1	134·1	176·0	231·0	303·3	398·2	522·7	686·2	900·8
15·3	20·10	26·40	34·68	45·56	59·85	78·62	103·3	135·7	178·2	234·1	307·5	403·9	530·6	697·0	915·6
15·4	20·24	26·61	34·98	45·98	60·43	79·44	104·4	137·3	180·4	237·2	311·7	409·8	538·6	708·0	930·7
15·5	20·39	26·82	35·27	46·39	61·02	80·27	105·6	138·9	182·7	240·2	316·0	415·7	546·7	719·1	945·9
15·6	20·53	27·02	35·57	46·81	61·62	81·10	106·7	140·5	184·9	243·4	320·3	421·6	554·9	730·3	961·2
15·7	20·68	27·23	35·86	47·23	62·21	81·93	107·9	142·1	187·2	246·5	324·6	427·5	563·1	741·6	976·7
15·8	20·82	27·44	36·16	47·66	62·80	82·77	109·1	143·7	189·4	249·6	329·0	433·6	571·4	753·0	992·3
15·9	20·97	27·65	36·46	48·08	63·40	83·61	110·2	145·4	191·7	252·8	333·4	439·6	579·7	764·5	100 8
16·0	21·11	27·86	36·76	48·50	64·00	84·45	111·4	147·0	194·0	256·0	337·8	445·7	588·1	776·0	102 4

N	2·6	2·7	2·8	2·9	3·0	3·1	3·2	3·3	3·4	3·5	3·6	3·7	3·8	3·9	4·0
13·1	803·3	1039	1344	1738	2248	2908	3761	4864	6291	8137	1052	1361	1760	2277	2945
13·2	819·4	1061	1373	1777	2300	2977	3853	4988	6456	8356	1082	1400	1812	2329	3036
13·3	835·6	1082	1402	1816	2353	3047	3948	5113	6624	8589	1111	1440	1865	2416	3129
13·4	852·1	1105	1432	1856	2406	3119	4043	5241	6794	8808	1142	1480	1919	2487	3224
13·5	868·7	1127	1462	1897	2460	3192	4141	5372	6968	9040	1173	1521	1974	2560	3322
13·6	885·5	1150	1492	1938	2515	3266	4240	5504	7146	9277	1204	1563	2030	2635	3421
13·7	902·6	1173	1523	1979	2571	3341	4340	5639	7326	9518	1237	1606	2087	2712	3523
13·8	919·8	1196	1555	2021	2628	3417	4442	5776	7509	9763	1269	1650	2146	2790	3627
13·9	937·2	1219	1587	2064	2686	3494	4546	5915	7696	1001	1303	1695	2205	2869	3733
14·0	954·8	1243	1619	2108	2744	3573	4652	6057	7886	1027	1337	1740	2266	2951	3842
14·1	972·7	1267	1651	2151	2803	3652	4759	6200	8079	1053	1371	1787	2328	3034	3953
14·2	990·7	1292	1684	2196	2863	3733	4868	6347	8275	1079	1407	1834	2392	3118	4066
14·3	100·9	1316	1718	2241	2924	3815	4978	6495	8475	1106	1443	1883	2456	3205	4182
14·4	102·7	1341	1752	2287	2986	3899	5090	6647	8678	1133	1479	1932	2522	3293	4300
14·5	104·6	1367	1786	2333	3049	3983	5204	6800	8885	1161	1517	1982	2589	3383	4421
14·6	106·5	1392	1820	2380	3112	4069	5320	6956	9095	1189	1555	2033	2658	3475	4544
14·7	108·4	1418	1856	2428	3177	4156	5438	7115	9308	1218	1593	2085	2728	3569	4669
14·8	110·3	1444	1891	2476	3242	4244	5557	7276	9526	1247	1633	2138	2799	3665	4797
14·9	112·3	1471	1927	2525	3308	4334	5678	7439	9746	1277	1673	2192	2871	3762	4929
15·0	114·2	1498	1964	2574	3375	4425	5801	7605	9970	1307	1714	2247	2945	3861	5062
15·1	116·2	1525	2000	2624	3443	4517	5926	7774	1020	1338	1755	2303	3021	3963	5199
15·2	118·2	1552	2038	2675	3512	4610	6052	7945	1043	1369	1797	2360	3097	4066	5338
15·3	120·3	1580	2076	2726	3582	4705	6180	8119	1066	1401	1840	2417	3176	4172	5480
15·4	122·3	1608	2114	2778	3652	4801	6311	8295	1090	1433	1884	2746	3255	4279	5624
15·5	124·4	1636	2152	2831	3724	4898	6443	8474	1115	1466	1928	2536	3336	4388	5772
15·6	126·5	1665	2192	2884	3796	4997	6577	8656	1139	1499	1974	2598	3419	4500	5922
15·7	128·6	1694	2231	2938	3870	5097	6712	8840	1164	1533	2019	2660	3503	4613	6076
15·8	130·8	1723	2271	2993	3944	5198	6850	9028	1190	1568	2066	2723	3588	4729	6232
15·9	132·9	1753	2312	3048	4020	5301	6990	9217	1215	1603	2114	2787	3675	4847	6391
16·0	135·1	1783	2353	3104	4096	5405	7132	9410	1242	1638	2162	2853	3764	4967	6554

Add one cypher

Power

N	1·1	1·2	1·3	1·4	1·5	1·6	1·7	1·8	1·9	2·0	2·1	2·2	2·3	2·4	2·5
16·1	21·26	28·07	37·06	48·93	64·60	85·29	112·6	148·7	196·3	259·2	342·2	451·9	596·6	787·7	1040
16·2	21·40	28·28	37·36	49·35	65·20	86·14	113·8	150·4	198·6	262·4	346·7	458·1	605·2	799·5	1056
16·3	21·55	28·49	37·66	49·78	65·81	87·00	115·0	152·0	201·0	265·7	351·2	464·3	613·8	811·4	1073
16·4	21·69	28·70	37·96	50·21	66·41	87·85	116·2	153·7	203·3	269·0	355·8	470·6	622·5	823·4	1089
16·5	21·84	28·91	38·26	50·64	67·02	88·71	117·4	155·4	205·7	272·2	360·3	476·9	631·3	835·5	1106
16·6	21·98	29·12	38·56	51·07	67·63	89·57	118·6	157·1	208·1	275·6	364·9	483·3	640·1	847·7	1123
16·7	22·13	29·33	38·86	51·50	68·25	90·54	119·8	158·8	210·5	278·9	369·6	489·8	649·0	860·0	1140
16·8	22·28	29·54	39·17	51·93	68·86	91·30	121·1	160·5	212·9	282·2	374·2	496·2	658·0	872·5	1157
16·9	22·42	29·75	39·47	52·37	69·48	92·18	122·3	162·3	215·3	285·6	378·9	502·7	667·0	885·0	1174
17·0	22·57	29·96	39·77	52·80	70·09	93·05	123·5	164·0	217·7	289·0	383·7	509·3	676·1	897·6	1192
17·1	22·71	30·17	40·08	53·23	70·71	93·93	124·8	165·7	220·1	292·4	388·4	515·9	685·3	910·3	1209
17·2	22·86	30·38	40·38	53·67	71·33	94·81	126·0	167·5	222·6	295·8	393·2	522·6	694·6	923·1	1227
17·3	23·01	30·60	40·69	54·11	71·96	95·69	127·3	169·2	225·1	299·3	398·0	529·3	703·9	936·1	1245
17·4	23·15	30·81	40·99	54·55	72·58	96·58	128·5	171·0	227·5	302·8	402·9	536·1	713·3	949·1	1263
17·5	23·30	31·02	41·30	54·99	73·21	97·47	129·8	172·8	230·0	306·2	407·7	542·9	722·7	962·3	1281
17·6	23·45	31·23	41·61	55·43	73·84	98·36	131·0	174·6	232·5	309·8	412·6	549·7	732·3	975·5	1300
17·7	23·59	31·45	41·91	55·87	74·47	99·26	132·3	176·3	235·0	313·3	417·6	556·6	741·9	988·9	1318
17·8	23·74	31·66	42·22	56·31	75·10	100·2	133·6	178·1	237·6	316·8	422·6	563·5	751·6	100 2	1337
17·9	23·89	31·87	42·53	56·75	75·73	101·1	134·9	179·9	240·1	320·4	427·6	570·5	761·3	101 6	1356
18·0	24·03	32·09	42·84	57·20	76·37	102·0	136·1	181·8	242·7	324·0	432·6	577·6	771·1	103 0	1375
18·1	24·18	32·30	43·15	57·64	77·00	102·9	137·4	183·6	245·2	327·6	437·6	584·6	781·0	104 3	1394
18·2	24·33	32·52	43·46	58·09	77·64	103·8	138·7	185·4	247·8	331·2	442·7	591·8	791·0	105 7	1413
18·3	24·47	32·73	43·77	58·54	78·28	104·7	140·0	187·2	250·4	334·9	447·9	599·0	801·0	107 1	1433
18·4	24·62	32·94	44·08	58·99	78·93	105·6	141·3	189·1	253·0	338·6	453·0	606·2	811·1	108 5	1452
18·5	24·77	33·16	44·39	59·43	79·57	106·5	142·6	190·9	255·6	342·2	458·2	613·4	821·3	110 0	1472
18·6	24·92	33·37	44·71	59·89	80·22	107·5	143·9	192·8	258·3	346·0	463·4	620·8	831·5	111 4	1492
18·7	25·06	33·59	45·02	60·34	80·87	108·4	145·3	194·7	260·9	349·7	468·7	628·1	841·9	112 8	1512
18·8	25·21	33·81	45·33	60·79	81·51	109·3	146·6	196·6	263·6	353·4	473·9	635·5	852·2	114 3	1532
18·9	25·36	34·02	45·65	61·24	82·17	110·2	147·9	198·4	266·2	357·2	479·3	643·0	862·7	115 7	1553
19·0	25·51	34·24	45·96	61·70	82·82	111·2	149·2	200·3	268·9	361·0	484·6	650·5	873·2	117 2	1574

N	2·6	2·7	2·8	2·9	3·0	3·1	3·2	3·3	3·4	3·5	3·6	3·7	3·8	3·9	4·0
16·1	1373	1813	2394	3161	4173	5510	7275	9606	1268	1675	2211	2919	3854	5089	6718
16·2	1396	1844	2436	3218	4252	5617	7421	9804	1295	1711	2261	2987	3946	5213	6887
16·3	1418	1875	2478	3276	4331	5725	7568	1001	1323	1748	2311	3056	4039	5340	7058
16·4	1441	1906	2521	3335	4411	5835	7718	1021	1350	1786	2363	3126	4134	5469	7233
16·5	1464	1937	2564	3394	4492	5946	7870	1042	1379	1825	2415	3197	4231	5600	7412
16·6	1487	1969	2608	3454	4574	6058	8023	1063	1407	1864	2468	3269	4329	5734	7593
16·7	1510	2001	2652	3515	4657	6158	8179	1084	1436	1903	2522	3342	4429	5869	7778
16·8	1534	2034	2697	3576	4742	6287	8337	1105	1466	1943	2577	3417	4531	6008	7966
16·9	1558	2067	2742	3638	4827	6404	8496	1127	1496	1984	2633	3493	4634	6148	8157
17·0	1582	2100	2788	3701	4913	6522	8658	1149	1526	2026	2689	3570	4739	6291	8352
17·1	1606	2133	2834	3764	5000	6642	8822	1172	1557	2068	2747	3648	4846	6437	8550
17·2	1631	2167	2881	3829	5088	6763	8989	1195	1588	2110	2805	3728	4955	6585	8752
17·3	1655	2202	2928	3893	5178	6886	9157	1218	1619	2154	2864	3809	5065	6736	8957
17·4	1680	2236	2975	3959	5268	7010	9327	1241	1651	2197	2924	3891	5177	6889	9166
17·5	1706	2271	3023	4025	5359	7135	9500	1265	1684	2242	2985	3974	5291	7044	9379
17·6	1731	2306	3072	4092	5452	7263	9675	1289	1717	2287	3047	4059	5407	7203	9595
17·7	1757	2342	3121	4160	5545	7391	9852	1313	1750	2333	3110	4145	5525	7364	9815
17·8	1783	2378	3171	4229	5640	7521	1003	1338	1784	2379	3173	4232	5644	7527	1004
17·9	1809	2414	3221	4298	5735	7653	1021	1363	1818	2427	3238	4321	5766	7694	1027
18·0	1835	2450	3272	4368	5832	7787	1040	1388	1853	2474	3304	4411	5889	7863	1050
18·1	1862	2487	3323	4439	5930	7921	1058	1414	1888	2523	3370	4502	6014	8034	1073
18·2	1889	2525	3374	4510	6029	8058	1077	1440	1924	2572	3438	4595	6141	8209	1097
18·3	1916	2562	3427	4583	6128	8196	1096	1466	1960	2622	3506	4689	6271	8386	1122
18·4	1943	2600	3479	4656	6230	8336	1115	1492	1997	2672	3576	4784	6402	8566	1146
18·5	1971	2639	3533	4729	6332	8477	1135	1519	2034	2723	3646	4881	6535	8749	1171
18·6	1999	2677	3586	4804	6435	8620	1155	1547	2072	2775	3717	4980	6670	8935	1197
18·7	2027	2716	3640	4879	6539	8764	1175	1574	2110	2828	3790	5079	6808	9124	1223
18·8	2055	2756	3695	4955	6645	8910	1195	1602	2149	2881	3863	5181	6947	9316	1249
18·9	2084	2795	3751	5032	6751	9058	1215	1631	2188	2935	3938	5283	7089	9510	1276
19·0	2112	2836	3806	5110	6859	9207	1236	1659	2227	2990	4013	5388	7232	9708	1303
							Add one cypher								Two cyphers

Power

N	1·1	1·2	1·3	1·4	1·5	1·6	1·7	1·8	1·9	2·0	2·1	2·2	2·3	2·4	2·5
19·1	25·65	34·45	46·27	62·15	83·47	112·1	150·6	202·2	271·6	364·8	490·0	658·1	883·8	1187	1594
19·2	25·89	34·67	46·59	62·61	84·13	113·1	151·9	204·1	274·3	368·6	495·4	665·7	894·5	1202	1615
19·3	25·95	34·89	46·91	63·06	84·79	114·0	153·3	206·1	277·0	372·5	500·8	673·3	905·3	1217	1636
19·4	26·10	35·10	47·22	63·52	85·45	114·9	154·6	208·0	279·8	376·4	506·3	681·0	916·1	1232	1658
19·5	26·24	35·32	47·54	63·98	86·11	115·9	156·0	209·9	282·5	380·2	511·8	688·8	927·0	1248	1679
19·6	26·39	35·54	47·86	64·44	86·77	116·8	157·3	211·9	285·3	384·2	517·3	696·6	938·0	1263	1701
19·7	26·54	35·76	48·17	64·90	87·44	117·8	158·7	213·8	288·1	388·1	522·9	704·4	949·0	1279	1723
19·8	26·69	35·97	48·49	65·36	88·10	118·8	160·1	215·8	290·8	392·0	528·4	712·3	960·1	1294	1744
19·9	26·84	36·19	48·81	65·83	88·77	119·7	161·5	217·7	293·6	396·0	534·1	720·2	971·3	1310	1767
20·0	26·99	36·41	49·13	66·29	89·44	120·7	162·8	219·7	296·5	400·0	539·7	728·2	982·6	1326	1789
20·1	27·13	36·63	49·45	66·75	90·11	121·7	164·2	221·7	299·3	404·0	545·4	736·3	993·9	1342	1811
20·2	27·28	36·85	49·77	67·22	90·79	122·6	165·6	223·7	302·1	408·0	551·1	744·3	100·5	1358	1834
20·3	27·43	37·07	50·09	67·69	91·46	123·6	167·0	225·7	305·0	412·1	556·9	752·5	101·7	1374	1857
20·4	27·58	37·29	50·41	68·15	92·14	124·6	168·4	227·7	307·8	416·2	562·6	760·7	102·8	1390	1880
20·5	27·73	37·51	50·73	68·62	92·82	125·5	169·8	229·7	310·7	420·2	568·4	768·9	104·0	1407	1903
20·6	27·88	37·73	51·05	69·09	93·50	126·5	171·2	231·7	313·6	424·4	574·3	777·2	105·2	1423	1926
20·7	28·03	37·95	51·38	69·56	94·18	127·5	172·6	233·7	316·5	428·5	580·1	785·5	106·3	1440	1950
20·8	28·18	38·17	51·70	70·03	94·86	128·5	174·1	235·8	319·4	432·6	586·0	793·9	107·5	1457	1973
20·9	28·32	38·39	52·02	70·50	95·55	129·5	175·5	237·8	322·3	436·8	592·0	802·3	108·7	1474	1997
21·0	28·47	38·61	52·35	70·98	96·23	130·5	176·9	239·9	325·2	441·0	597·9	810·7	109·9	1490	2021
21·1	28·62	38·83	52·67	71·45	96·92	131·5	178·4	241·9	328·2	445·2	603·9	819·3	111·1	1508	2045
21·2	28·77	39·05	53·00	71·92	97·61	132·5	179·8	244·0	331·2	449·4	610·0	827·8	112·3	1525	2069
21·3	28·92	39·27	53·32	72·40	98·30	133·5	181·2	246·1	334·1	453·7	616·0	836·4	113·6	1542	2094
21·4	29·07	39·49	53·65	72·88	99·00	134·5	182·7	248·2	337·1	458·0	622·1	845·1	114·8	1560	2119
21·5	29·22	39·71	53·97	73·35	99·69	135·5	184·1	250·3	340·1	462·2	628·2	853·8	116·0	1577	2143
21·6	29·37	39·93	54·30	73·83	100·4	136·5	185·6	252·4	343·1	466·6	634·4	862·6	117·3	1595	2168
21·7	29·52	40·16	54·63	74·31	101·1	137·5	187·1	254·5	346·2	470·9	640·6	871·4	118·5	1613	2194
21·8	29·67	40·38	54·95	74·79	101·8	138·5	188·5	256·6	349·2	475·2	646·8	880·2	119·8	1630	2219
21·9	29·82	40·60	55·28	75·27	102·5	139·5	190·0	258·7	352·2	479·6	653·0	889·2	121·1	1648	2244
22·0	29·97	40·82	55·61	75·75	103·2	140·6	191·5	260·8	355·3	484·0	659·3	898·1	122·3	1667	2270

N	2·6	2·7	2·8	2·9	3·0	3·1	3·2	3·3	3·4	3·5	3·6	3·7	3·8	3·9	4·0
19·1	2141	2876	3863	5188	6968	9358	1257	1688	2267	3045	4090	5493	7378	9909	1331
19·2	2171	2917	3920	5267	7078	9511	1278	1717	2308	3101	4168	5600	7526	1011	1359
19·3	2200	2958	3977	5347	7190	9666	1300	1747	2349	3158	4246	5709	7676	1032	1387
19·4	2230	3000	4035	5428	7301	9822	1321	1777	2391	3216	4326	5819	7828	1053	1416
19·5	2260	3042	4094	5509	7415	9979	1343	1808	2433	3274	4407	5931	7982	1074	1446
19·6	2290	3084	4153	5592	7530	1014	1365	1838	2476	3333	4489	6044	8139	1096	1476
19·7	2321	3126	4212	5675	7645	1030	1388	1870	2519	3393	4572	6159	8298	1118	1506
19·8	2351	3170	4272	5759	7762	1046	1410	1901	2562	3454	4656	6276	8459	1140	1537
19·9	2382	3213	4333	5844	7881	1063	1433	1933	2607	3515	4741	6394	8623	1163	1568
20·0	2414	3257	4394	5929	8000	1079	1456	1965	2652	3578	4827	6513	8788	1186	1600
20·1	2445	3301	4456	6015	8121	1096	1480	1998	2697	3641	4915	6635	8957	1209	1632
20·2	2477	3345	4518	6103	8242	1113	1504	2031	2743	3704	5003	6758	9127	1233	1665
20·3	2509	3390	4581	6191	8365	1130	1528	2064	2789	3769	5093	6882	9300	1257	1698
20·4	2541	3436	4645	6280	8490	1148	1552	2098	2836	3834	5184	7009	9475	1281	1732
20·5	2574	3481	4709	6369	8615	1165	1576	2132	2884	3901	5276	7137	9653	1306	1766
20·6	2606	3527	4773	6460	8742	1183	1601	2167	2932	3968	5369	7266	9833	1331	1801
20·7	2640	3574	4839	6551	8870	1201	1626	2201	2981	4035	5464	7398	1002	1356	1836
20·8	2673	3621	4904	6643	8999	1219	1651	2237	3030	4104	5559	7531	1020	1382	1872
20·9	2706	3668	4971	6736	9129	1237	1677	2272	3080	4174	5656	7665	1039	1408	1908
21·0	2740	3715	5037	6830	9261	1256	1703	2308	3130	4244	5754	7802	1058	1434	1945
21·1	2774	3763	5105	6925	9394	1274	1729	2345	3181	4315	5853	7940	1077	1461	1982
21·2	2809	3812	5173	7021	9528	1293	1755	2382	3233	4387	5954	8081	1097	1488	2020
21·3	2843	3860	5242	7117	9664	1312	1782	2419	3285	4460	6056	8223	1116	1516	2058
21·4	2878	3909	5311	7214	9800	1331	1809	2457	3337	4534	6159	8366	1137	1544	2097
21·5	2913	3959	5381	7313	9938	1351	1836	2495	3391	4608	6263	8512	1157	1572	2137
21·6	2948	4009	5451	7412	1008	1370	1863	2533	3445	4684	6368	8659	1177	1601	2177
21·7	2984	4059	5522	7512	1022	1390	1891	2572	3499	4760	6475	8808	1198	1630	2217
21·8	3020	4110	5593	7612	1036	1410	1919	2612	3554	4837	6583	8960	1219	1660	2259
21·9	3056	4161	5666	7714	1050	1430	1947	2651	3610	4915	6693	9113	1241	1689	2300
22·0	3092	4213	5738	7817	1065	1450	1976	2692	3666	4994	6803	9268	1262	1720	2343
					Add one cypher								Add two cyphers		

Power

	1·1	1·2	1·3	1·4	1·5	1·6	1·7	1·8	1·9	2·0	2·1	2·2	2·3	2·4	2·5
22·6	30·42	41·05	55·94	76·23	103·9	141·6	193·0	263·0	358·4	488 4	665·6	907·1	1236	1685	2296
22·7	30·57	41·27	56·27	76·72	104·6	142·6	194·4	265·1	361·5	492·8	672·0	916·2	1249	1703	2322
22·8	30·72	41 49	56·60	77·20	105·3	143·6	195·9	267·3	364·6	497·3	678·3	925·3	1262	1722	2348
22·9	30·87	41·72	56·93	77·69	106·0	144·7	197·4	269·4	367·7	501·8	684·7	934·4	1275	1740	2375
23·0	30·72	41 94	57·26	78·17	106·7	145·7	198·9	271·6	370·8	506·2	691·2	943·6	1288	1759	2401
23·1	30·87	42·16	57·59	78·66	107·4	146·7	200·4	273·8	373·9	510·8	697·6	952·9	1302	1778	2428
23·2	31·02	42·39	57·92	79·15	108·2	147·8	202·0	276·0	377·1	515·3	704·0	962·2	1315	1797	2455
23·3	31·17	42·61	58·25	79·64	108·9	148·8	203·5	278·2	380·3	519·8	710·7	971·5	1328	1816	2482
23·4	31·32	42 84	58·58	80·13	109·6	149·9	205·0	280·4	383·4	524·4	717·2	980·9	1342	1835	2509
23·5	31·47	43·06	58·92	80·62	110·3	150·9	206·5	282·6	386·6	529·0	723·8	990·4	1355	1854	2537
23·6	31·62	43·28	59·25	81·11	111·0	152·0	208·0	284·8	389·8	533·6	730·4	999·9	1369	1874	2565
23·7	31·77	43·51	59·58	81·60	111·7	153·0	209·6	287·0	393·0	538·2	737·1	100 9	1382	1893	2593
23·8	31·92	43·74	59·92	82·09	112·5	154·1	211·1	289·2	396·3	542·9	743·8	101 9	1396	1913	2621
23·9	32·07	43·96	60·25	82·59	113·2	155·1	212·7	291·5	399·5	547·6	750·5	102 9	1410	1932	2649
24·0	32·22	44·19	60·59	83·08	113·9	156·2	214·2	293·7	402·7	552·2	757·3	103 8	1424	1952	2677
24·1	32·37	44·41	60·92	83·58	114·6	157·3	215·8	296·0	406·0	557·0	764·0	104 8	1438	1972	2706
24·2	32·53	44·64	61·26	84·07	115·4	158·3	217·3	298·2	409·3	561·7	770·9	105 8	1452	1992	2734
24·3	32·68	44·86	61·60	84·57	116·1	159·4	218·9	300·5	412·6	566·4	777·7	106 8	1466	2013	2763
24·4	32·83	45·09	61·93	85·07	116·8	160·5	220·4	302·8	415·9	571·2	784·6	107 8	1480	2033	2793
24·5	32·98	45·32	62·27	85·56	117·6	161·6	222·0	305·1	419·2	576·0	791·5	108 8	1494	2054	2822
24·6	33·13	45·54	62·61	86·06	118·3	162·6	223·6	307·3	422·5	580·8	798·4	109 8	1509	2074	2851
24·7	33·28	45·77	62·94	86·56	119·0	163·7	225·2	309·6	425·8	585·6	805·4	110 8	1523	2095	2881
24·8	33·43	46·00	63·28	87·07	119·8	164·8	226·7	312·0	429·2	590·5	812·4	111 8	1538	2116	2911
24·9	33·58	46·22	63·62	87·57	120·5	165·9	228·3	314·3	432·6	595·4	819·4	112 8	1552	2137	2941
25·0	33·74	46·45	63·96	88·07	121·3	167·0	229·9	316·6	435·9	600·2	826·5	113 8	1567	2158	2971
25·1	33·89	46·68	64·30	88·57	122·0	168·1	231·5	318·9	439·3	605·2	833·6	114 8	1582	2179	3001
25·2	34·04	46·91	64·64	89·08	122·8	169·2	233·1	321·3	442·7	610·1	840·7	115 9	1597	2200	3032
25·3	34·19	47·13	64·98	89·58	123·5	170·3	234·7	323·6	446·1	615·0	847·9	116 9	1612	2222	3063
25·4	34·34	47·36	65·32	90·09	124·3	171·4	236·3	326·0	449·6	620·0	855·1	117 9	1627	2243	3094
25·5	34·49	47·59	65·67	90·60	125·0	172·5	238·0	328·3	453·0	625·0	862·3	119 0	1642	2265	3125

N	2·6	2·7	2·8	2·9	3·0	3·1	3·2	3·3	3·4	3·5	3·6	3·7	3·8	3·9	4·0
22·1	3129	4264	5812	7920	1079	1471	2005	2732	3723	5074	6915	9424	1284	1750	2385
22·2	3166	4317	5886	8025	1094	1492	2034	2773	3781	5155	7029	9583	1307	1781	2429
22·3	3203	4369	5960	8130	1109	1513	2063	2815	3839	5237	7143	9744	1329	1813	2473
22·4	3241	4423	6035	8236	1124	1534	2093	2856	3898	5319	7259	9906	1352	1845	2518
22·5	3279	4476	6111	8343	1139	1555	2123	2899	3958	5403	7377	1007	1375	1877	2563
22·6	3317	4530	6187	8451	1154	1577	2154	2941	4018	5488	7495	1024	1398	1910	2609
22·7	3355	4584	6264	8560	1170	1598	2184	2985	4078	5573	7615	1041	1422	1943	2655
22·8	3393	4639	6342	8670	1185	1620	2215	3028	4140	5659	7737	1058	1446	1977	2702
22·9	3432	4694	6420	8781	1201	1642	2246	3072	4202	5747	7860	1075	1470	2011	2750
23·0	3471	4750	6499	8892	1217	1665	2278	3117	4265	5835	7984	1092	1495	2045	2798
23·1	3511	4806	6578	9005	1233	1687	2310	3162	4328	5924	8110	1110	1520	2080	2847
23·2	3550	4862	6658	9118	1249	1710	2342	3207	4392	6015	8237	1128	1545	2115	2897
23·3	3590	4919	6739	9233	1265	1733	2374	3253	4457	6106	8365	1146	1570	2151	2947
23·4	3630	4976	6820	9348	1281	1756	2407	3299	4522	6198	8495	1164	1596	2187	2998
23·5	3671	5034	6902	9464	1298	1780	2440	3346	4588	6291	8627	1183	1622	2224	3050
23·6	3712	5092	6985	9582	1314	1803	2474	3393	4655	6385	8760	1202	1648	2261	3102
23·7	3753	5150	7068	9700	1331	1827	2507	3441	4722	6481	8894	1221	1675	2299	3155
23·8	3794	5209	7152	9819	1348	1851	2541	3489	4790	6577	9030	1240	1702	2337	3209
23·9	3836	5268	7236	9939	1365	1875	2576	3538	4859	6674	9167	1259	1729	2375	3263
24·0	3877	5328	7321	1006	1382	1900	2610	3587	4929	6772	9306	1279	1757	2414	3318
24·1	3920	5388	7407	1018	1400	1924	2645	3636	4999	6872	9446	1299	1785	2454	3373
24·2	3962	5449	7493	1031	1417	1949	2680	3686	5070	6972	9588	1319	1813	2494	3430
24·3	4005	5510	7580	1043	1435	1974	2716	3737	5141	7073	9732	1339	1842	2534	3487
24·4	4048	5571	7668	1055	1453	1999	2752	3788	5213	7176	9877	1359	1871	2575	3545
24·5	4091	5633	7756	1068	1471	2025	2788	3839	5286	7279	1002	1380	1900	2617	3603
24·6	4135	5695	7845	1081	1489	2051	2825	3891	5360	7384	1017	1401	1930	2659	3662
24·7	4178	5758	7935	1094	1507	2077	2862	3944	5435	7489	1032	1422	1960	2701	3722
24·8	4223	5821	8025	1106	1525	2103	2899	3997	5510	7596	1047	1444	1990	2744	3783
24·9	4267	5885	8116	1119	1544	2129	2937	4050	5586	7704	1062	1465	2021	2787	3844
25·0	4312	5949	8208	1132	1562	2156	2974	4104	5662	7812	1078	1487	2052	2831	3906

Add one cypher

Add two cyphers

N	1·1	1·2	1·3	1·4	1·5	1·6	1·7	1·8	1·9	2·0	2·1	2·2	2·3	2·4	2·5
25·1	34·65	47·82	66·00	90·11	125·8	173·6	239·6	330·7	456·4	630·0	869·6	1200	1657	2287	3156
25·2	34·80	48·05	66·35	91·61	126·5	174·7	241·2	333·1	459·9	635·0	876·9	1211	1672	2309	3188
25·3	34·95	48·28	66·69	92·12	127·3	175·8	242·8	335·4	463·4	640·1	884·2	1221	1687	2331	3220
25·4	35·10	48·51	67·03	92·63	128·0	176·9	244·5	337·8	466·9	645·2	891·6	1232	1703	2353	3252
25·5	35·25	48·74	67·38	93·14	128·8	178·0	246·1	340·2	470·4	650·2	898·9	1243	1718	2375	3284
25·6	35·40	48·97	67·72	93·66	129·5	179·1	247·7	342·6	473·9	655·4	906·4	1254	1734	2398	3316
25·7	35·56	49·19	68·06	94·17	130·3	180·3	249·4	345·0	477·4	660·5	913·8	1264	1749	2420	3348
25·8	35·71	49·42	68·41	94·68	131·0	181·4	251·0	347·5	480·9	665·6	921·3	1275	1765	2443	3381
25·9	35·86	49·65	68·75	95·20	131·8	182·5	252·7	349·9	484·5	670·8	928·8	1286	1781	2466	3414
26·0	36·01	49·88	69·10	95·71	132·6	183·6	254·4	352·3	488·0	676·0	936·4	1297	1797	2488	3447
26·1	36·17	50·12	69·44	96·23	133·3	184·8	256·0	354·8	491·6	681·2	943·9	1308	1812	2512	3480
26·2	36·32	50·35	69·79	96·74	134·1	185·9	257·7	357·2	495·2	686·4	951·6	1319	1828	2535	3514
26·3	36·47	50·58	70·14	97·26	134·9	187·0	259·4	359·7	498·8	691·7	959·2	1330	1845	2558	3547
26·4	36·62	50·81	70·48	97·78	135·6	188·2	261·1	362·1	502·4	697·0	966·9	1341	1861	2581	3581
26·5	36·78	51·04	70·83	98·30	136·4	189·3	262·7	364·6	506·0	702·2	974·6	1353	1877	2605	3615
26·6	36·93	51·27	71·18	98·82	137·2	190·5	264·4	367·1	509·7	707·6	982·3	1364	1893	2629	3649
26·7	37·08	51·50	71·53	99·34	138·0	191·6	266·1	369·6	513·3	712·9	990·1	1375	1910	2652	3684
26·8	37·23	51·73	71·87	99·86	138·7	192·8	267·8	372·1	517·0	718·2	997·9	1386	1926	2676	3718
26·9	37·39	51·96	72·22	100·4	139·5	193·9	269·5	374·6	520·6	723·6	100 6	1398	1943	2700	3753
27·0	37·54	52·20	72·57	100·9	140·3	195·1	271·2	377·1	524·3	729 0	101 4	1409	1959	2724	3788
27·1	37·69	52·43	72·92	101·4	141·1	196·2	272·9	379·6	528·0	734·4	102 1	1421	1976	2749	3823
27·2	37·85	52·66	73·27	102·0	141·9	197·4	274·6	382·1	531·7	739·8	102 9	1432	1993	2773	3859
27·3	38·00	52·89	73·62	102·5	142·6	198·5	276·4	384·7	535·4	745·3	103 7	1444	2010	2798	3894
27·4	38·15	53·13	73·97	103·0	143·4	199·7	278·1	387·2	539·2	750·8	104 5	1456	2027	2822	3930
27·5	38·31	53·36	74·32	103·5	144·2	200·9	279·8	389·8	542·9	756·2	105 3	1467	2044	2847	3966
27·6	38·46	53·59	74·68	104·1	145·0	202·0	281·5	392·3	546·7	761·8	106 1	1479	2061	2872	4002
27·7	38·61	53·82	75·03	104·6	145·8	203·2	283·3	394·9	550·4	767·3	107 0	1491	2078	2897	4038
27·8	38·77	54·06	75·38	105·1	146·6	204·4	285·0	397·4	554·2	772·8	107 8	1503	2096	2922	4075
27·9	38·92	54·29	75·73	105·6	147·4	205·6	286·8	400·0	558·0	778·4	108 6	1515	2113	2947	4112
28·0	39·07	54·52	76·09	106·2	148·2	206·8	288·5	402·6	561·8	784·0	109 4	1527	2130	2973	4149

N	2·6	2·7	2·8	2·9	3·0	3·1	3·2	3·3	3·4	3·5	3·6	3·7	3·8	3·9	4·0
25·1	4357	6013	8300	1146	1581	2183	3013	4158	5740	7922	1094	1509	2083	2876	3969
25·2	4402	6078	8393	1159	1600	2210	3051	4213	5818	8033	1109	1532	2115	2921	4033
25·3	4447	6144	8487	1172	1619	2237	3090	4269	5897	8146	1125	1554	2147	2966	4097
25·4	4493	6209	8581	1186	1639	2265	3129	4325	5976	8259	1141	1577	2180	3012	4162
25·5	4539	6276	8676	1199	1658	2292	3169	4381	6057	8373	1158	1600	2212	3058	4228
25·6	4586	6342	8771	1213	1678	2320	3209	4438	6138	8489	1174	1624	2245	3106	4295
25·7	4633	6409	8868	1227	1697	2349	3249	4496	6220	8605	1191	1647	2279	3153	4362
25·8	4680	6477	8965	1241	1717	2377	3290	4554	6302	8723	1207	1671	2313	3201	4431
25·9	4727	6545	9062	1255	1737	2406	3331	4612	6386	8842	1224	1695	2347	3250	4500
26·0	4775	6613	9161	1269	1758	2435	3372	4671	6470	8962	1241	1719	2382	3299	4570
26·1	4822	6682	9260	1283	1778	2464	3414	4731	6555	9083	1259	1744	2417	3349	4640
26·2	4871	6752	9359	1297	1798	2493	3456	4791	6641	9206	1276	1769	2452	3399	4712
26·3	4919	6822	9460	1312	1819	2523	3498	4851	6727	9329	1294	1794	2488	3450	4784
26·4	4968	6892	9561	1326	1840	2553	3541	4912	6815	9454	1312	1819	2524	3502	4858
26·5	5017	6962	9662	1341	1861	2583	3584	4974	6903	9580	1329	1845	2561	3554	4932
26·6	5066	7034	9765	1356	1882	2613	3628	5036	6992	9707	1348	1871	2597	3606	5006
26·7	5116	7105	9868	1371	1903	2644	3671	5099	7082	9835	1366	1897	2635	3659	5082
26·8	5166	7177	9972	1385	1925	2674	3716	5162	7172	9965	1384	1924	2672	3713	5159
26·9	5216	7250	1008	1400	1947	2705	3760	5226	7264	1010	1403	1950	2711	3767	5236
27·0	5267	7323	1018	1416	1968	2737	3805	5291	7356	1023	1422	1977	2749	3822	5314
27·1	5318	7396	1029	1431	1990	2768	3850	5355	7449	1036	1441	2004	2788	3878	5394
27·2	5369	7470	1039	1446	2012	2800	3896	5421	7543	1050	1460	2032	2827	3934	5474
27·3	5420	7545	1050	1462	2035	2832	3942	5487	7638	1063	1480	2060	2867	3991	5555
27·4	5472	7620	1061	1477	2057	2864	3988	5554	7733	1077	1499	2088	2907	4048	5636
27·5	5524	7695	1072	1493	2080	2897	4035	5621	7829	1091	1519	2116	2948	4106	5719
27·6	5577	7771	1083	1509	2102	2930	4082	5689	7927	1105	1539	2145	2989	4164	5803
27·7	5629	7847	1094	1525	2125	2963	4130	5757	8025	1119	1559	2174	3030	4223	5887
27·8	5682	7924	1105	1541	2148	2996	4178	5826	8124	1133	1580	2203	3072	4283	5973
27·9	5735	8001	1116	1557	2172	3030	4226	5895	8223	1147	1600	2232	3114	4344	6059
28·0	5789	8078	1127	1573	2195	3063	4275	5965	8324	1162	1621	2262	3156	4405	6147

Add one cypher

Add two cyphers

Power

N	1·1	1·2	1·3	1·4	1·5	1·6	1·7	1·8	1·9	2·0	2·1	2·2	2·3	2·4	2·5
28·1	39·23	54·76	76·44	106·7	149·0	207·9	290·3	405·2	565·6	789·6	1102	1539	2148	2998	4186
28·2	39·38	54·99	76·79	107·2	149·8	209·1	292·0	407·8	569·5	795·2	1111	1551	2166	3024	4223
28·3	39·53	55·23	77·15	107·8	150·5	210·3	293·8	410·4	573·3	800·9	1119	1563	2183	3050	4261
28·4	39·69	55·46	77·50	108·3	151·3	211·5	295·6	413·0	577·2	806·6	1127	1575	2201	3076	4297
28·5	39·84	55·69	77·86	108·8	152·1	212·7	297·3	415·6	581·0	812·2	1135	1587	2219	3102	4336
28·6	39·99	55·93	78·21	109·4	153·0	213·9	299·1	418·3	584·9	818·0	1144	1600	2237	3128	4374
28·7	40·15	56·16	78·57	109·9	153·8	215·1	300·9	420·9	588·8	823·7	1152	1612	2255	3154	4413
28·8	40·30	56·40	78·92	110·4	154·6	216·3	302·7	423·6	592·7	829·4	1161	1624	2273	3181	4451
28·9	40·46	56·63	79·28	111·0	155·4	217·5	304·5	426·2	596·6	835·2	1169	1637	2291	3207	4490
29·0	40·61	56·87	79·64	111·5	156 2	218·7	306·3	428·9	600·6	841·0	1178	1649	2309	3234	4529
29·1	40·76	57·10	79·99	112·1	157·0	219·9	308·0	431·5	604·5	846·8	1186	1662	2328	3261	4568
29·2	40·92	57·34	80·35	112·6	157·8	221·1	309·8	434·2	608·5	852·6	1195	1674	2346	3288	4607
29·3	41·07	57·58	80·71	113·1	158·6	222·3	311·7	436·9	612·4	858·5	1203	1687	2365	3315	4647
29·4	41·23	57·81	81·07	113·7	159·4	223·5	313·5	439·6	616·4	864·4	1212	1700	2383	3342	4687
29·5	41·38	58·05	81·43	114·2	160·2	224·8	315·3	442·3	620·4	870·2	1221	1712	2402	3370	4727
29·6	41·54	58·28	81·79	114·8	161·0	226·0	317·1	445·0	624·4	876·2	1229	1725	2421	3397	4767
29·7	41·69	58·52	82·15	115·3	161·9	227·2	318·9	447·7	628·4	882·1	1238	1738	2440	3425	4807
29·8	41·84	58·76	82·51	115·9	162·7	228·4	320·8	450·4	632·4	888·0	1247	1751	2459	3452	4848
29·9	42·00	58·99	82·87	116·4	163·5	229·7	322·6	453·1	636·5	894·0	1256	1764	2478	3480	4889
30·0	42·15	59·23	83·23	116·9	164·3	230·9	324·4	455·8	640·5	900·0	1265	1777	2497	3508	4929

Power

N	2·6	2·7	2·8	2·9	3·0	3·1	3·2	3·3	3·4	3·5	3·6	3·7	3·8	3·9	4·0
28·1	5843	8157	1139	1589	2219	3097	4324	6036	8426	1176	1642	2292	3200	4466	6235
28·2	5897	8235	1150	1606	2243	3132	4373	6107	8528	1191	1663	2322	3243	4529	6324
28·3	5952	8314	1161	1622	2267	3166	4423	6179	8631	1206	1684	2353	3287	4592	6414
28·4	6007	8394	1173	1639	2291	3201	4473	6251	8735	1221	1706	2384	3331	4655	6505
28·5	6062	8474	1185	1656	2315	3236	4524	6324	8840	1236	1728	2415	3376	4719	6598
28·6	6117	8554	1196	1673	2339	3271	4575	6397	8946	1251	1750	2447	3421	4784	6691
28·7	6173	8635	1208	1690	2364	3307	4626	6472	9053	1266	1772	2478	3467	4850	6785
28·8	6229	8717	1220	1707	2389	3343	4678	6546	9161	1282	1794	2510	3513	4916	6880
28·9	6285	8799	1232	1724	2414	3379	4730	6622	9269	1298	1816	2543	3560	4983	6976
29·0	6342	8881	1244	1742	2439	3415	4783	6698	9379	1313	1839	2576	3607	5051	7073
29·1	6399	8964	1256	1759	2464	3452	4836	6774	9489	1329	1862	2609	2654	5119	7171
29·2	6456	9048	1268	1777	2490	3489	4889	6851	9601	1345	1885	2642	3702	5188	7270
29·3	6514	9132	1280	1794	2515	3526	4943	6929	9713	1362	1909	2676	3751	5258	7370
29·4	6572	9216	1292	1812	2541	3563	4997	7007	9826	1378	1932	2709	37\ 9	5328	7471
29·5	6630	9301	1305	1830	2567	3601	5052	7086	9940	1394	1956	2744	3849	5399	7573
29·6	6689	9386	1317	1848	2593	3639	5107	7166	1006	1411	1980	2778	3899	5471	7677
29·7	6748	9472	1330	1866	2620	3677	5162	7246	1017	1428	2004	2813	3949	5543	7781
29·8	6807	9558	1342	1885	2646	3716	5218	7327	1029	1445	2029	2848	4000	5616	7886
29·9	6867	9645	1355	1903	2673	3755	5274	7408	1041	1462	2053	2884	4051	5690	7993
30·0	6927	9733	1368	1922	2700	3794	5331	7490	1052	1479	2078	2920	4103	5765	8100
			Add one cypher						Add two cyphers						

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